An approach based on α-cuts and max-min technique to linear fractional programming with fuzzy coefficients

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Abstract

This paper presents an efficient and straightforward method with less computational complexities to address the linear fractional programming with fuzzy coefficients (FLFPP). To construct the approach, the concept of α-cut is used to tackle the fuzzy numbers in addition to rank them. Accordingly, the fuzzy problem is changed into a bi-objective linear fractional programming problem (BOLFPP) by the use of interval arithmetic. Afterwards, an equivalent BOLFPP is defined in terms of the membership functions of the objectives, which is transformed into a bi-objective linear programming problem (BOLPP) applying suitable non-linear variable transformations. Max-min theory is utilized to alter the BOLPP into a linear programming problem (LPP). It is proven that the optimal solution of the LPP is an ϵ-optimal solution for the fuzzy problem. Four numerical examples are given to illustrate the method and comparisons are made to show the efficiency.

Keywords: Efficient solution, ϵ-optimal solution, bi-objective programming, membership function.

1   Introduction

The linear fractional programming problem (LFPP) represents an optimization problem that can be used in mathematical modeling or other applications. In [33], applications of the LFPP were demonstrated in economy, business, engineering, management, and etc. [31] addressed a solid transportation problem with interval cost by the use of fractional goal programming method.[40] developed a framework of bi-level multi-objective linear fractional programming problem to optimize water consumption structure. [1] investigated the fractional-order tumour-immune-vitamin model trough fixed point results. [16] presented an application of the LFPP with fuzzy nature in industry sector. [10] proposed the best ever method dealing with the LFPP in which the fractional problem is transformed into a LPP by the use of variable transformation technique. [19] showed that any fractional programming problem (FPP) can be replaced by a series of non-fractional problems. Based on this principle, many approaches have been developed [7, 20, 26]. [4] designed a non-iterative method to obtain the global optimal solution of the sum of linear fractional programming problem (S-LFPP) by the use of variable transformation. [23] constructed an iterative algorithm for the large scale S-LFPP using a branch and bound technique.

The notion of fuzzy sets has played a significant rule in optimization for different disciplines such as engineering, business, and management [5, 13, 14, 22, 30, 32, 41]. Specifically, one can use fuzzy numbers when there exists an ambiguity to specify coefficients. In the LFPP, we deal with the fuzzy linear fractional programming problem (FLFPP) if the coefficients are fuzzy numbers. One way of addressing the FLFPP is to use fuzzy ranking approaches. In this manner, a fuzzy number is changed into fixed number(s). Therefore, multiple LFPPs may be considered instead of the main fuzzy problem. Although these kinds of approaches are easy and straightforward, representing a fuzzy number with fixed numbers may not be as comprehensive as we expected generally. On the other hand, using the concept of α-cut has been considered by many researchers as an efficient and comprehensive approach dealing with fuzzy numbers [2, 15, 38, 39]. In general, when the concept of α-cut is used, maximizing of the FLFPP is changed

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into a BOLFPP of the form $\underset{X \in S}{\text{Maximize}} \left\{ F^L(X), F^U(X) \right\}$. [24] developed a method treating this bi-objective problem in which only $F^U(X)$ is used. Ignoring $F^L(X)$ can be considered as a drawback of their approach. In order to overcome this shortcoming, convex combinations of the solutions of problems $\underset{X \in S}{\text{Maximize}} \ F^L(X)$ and $\underset{X \in S}{\text{Maximize}} \ F^U(X)$ were suggested by Stanojevi and Stanojevi [34]. However, their method increases the computational expenses since there is no rule to recognize which combination gives the best result. The methodology of [24] was developed by Chinnadurai and Muthukumar [12] to address the LFPP with positive fuzzy coefficients and positive fuzzy decision variables.[6] presented an approach to deal with the LFPP with interval coefficients. In their method, the original problem is transformed into a LPP using suitable variable transformations. In the literature, there are several methods to deal with the multi objective linear fractional programming problem (MOLFPP). These approaches can be also employed to tackle $\underset{X \in S}{\text{Maximize}} \ \left\{ F^L(X), F^U(X) \right\}$ where $F^L(X)$ and $F^U(X)$ are linear fractional functions and S is a polyhedral set. [9] developed a method to address MOLFPP. In their method, the multi objective problem is transformed into a multi objective linear programming problem (MOLPP). Subsequently, the membership functions are specified after identifying the fuzzy aspiration levels of the linear objectives. Finally, the MOLPP is changed into a LPP using max-min technique. Their method was designed such that it has not been possible to prove that the outcome is efficient, which is a drawback. Motivated by [9], Veeramani and Sumathi[36], and De and Deb [18] introduced approaches to deal with LFPP with fuzzy coefficients and MOLFPP, respectively. [28] transformed the MOLFPP into a LPP using a fuzzy goal programming approach in addition to suitable variable transformations. [35] introduced an approach to tackle the MOLFPP where the membership functions of the objectives are defined and then linearized by using the first order Taylor series about the individual optimal solutions. For some examples, [8] reported that the results of using the first order Taylor series proposed by Toksari are to some extent more accurate than the results of the fuzzy goal programming used by Pal et al. Nayak and Ojha [27] introduced a method dealing with the MOLFPP with fuzzy coefficients where the fuzzy problem is altered into an interval valued LFPP using the concept of α-cut. In their method, the fuzzy problem is reduced into the MOLFPP. Afterwards, they reach a MOLPP employing the first order Taylor series. Finally, weighted sum technique is utilized to reach a LPP. In general, there exists a drawback to the methods which use first order Taylor expansion since this expansion reduces the accuracy. As we mentioned above, to the best of our known, there is a drawback to the existing approaches which are used to address the FLFPP. In this paper, we aim to present an efficient and easy approach with less computational cost and complexities to cope with the FLFPP. To reach this aim, the concept of α-cut is used to tackle the fuzzy numbers. As the consequence, a BOLFPP is obtained which is changed into a BOLPP using suitable non-linear variable transformations. Finally, the BOLPP is altered into a LPP employing the max-min technique. It is proven that the unique optimal solution of the LPP is an ϵ-optimal solution for the fuzzy problem. Numerical examples are solved to illustrate the proposed approach in addition to make comparison with different methods.

This article is organized in 5 sections. Following the introduction, in section 2, some basic notions and preliminaries are given for convenience. In section3, the main outcome of this survey is released. In section 4, some illustrative examples are solved and comparisons are made to evaluate the approach. Finally, section 5 concludes the paper.

2   Preliminaries

2.1   Fuzzy numbers and intervals

Definition 2.1. [37] *Let $\tilde{A}$ be a normalized fuzzy set. A triangular fuzzy number $\tilde{A}$ is defined as:*

$$\mu_{\tilde{A}}(x) = \mu_{\tilde{A}}(x, a, b, c) = \begin{cases} (x-a)/(b-a), & x \in [a, b) \\ (c-x)/(c-b), & x \in [b, c] \\ \qquad\qquad 0, & otherwise. \end{cases}$$
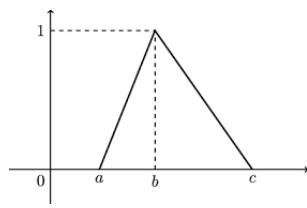

Figure 1: Triangular fuzzy number

Definition 2.2. [37] *Let $\tilde{A}$ be a fuzzy set in X and $\alpha \in [0,1]$. The α-cut of the fuzzy set $\tilde{A}$ is the crisp set $\tilde{A}_\alpha$ given by:*

$$[\tilde{A}]_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}.$$

Let $\tilde{A}$ be a triangular fuzzy number with the membership function $\mu_{\tilde{A}}(x; a, b, c)$, then $[\tilde{A}]_\alpha = [a + \alpha(b-a), c - \alpha(c-b)]$.

Definition 2.3. *(Ranking of fuzzy numbers) Let $\tilde{A}, \tilde{B}, \tilde{C}$ be fuzzy numbers with α-cuts $[\tilde{A}]_\alpha = [a_\alpha{}^-, a_\alpha{}^+], [\tilde{B}]_\alpha = [b_\alpha{}^-, b_\alpha{}^+], [\tilde{C}]_\alpha = [c_\alpha{}^-, c_\alpha{}^+]$. According to [21], possibility and necessity theories can be used to rank fuzzy numbers based on their α-cuts as follows:*
Method 1. We say $\tilde{A}$ is smaller than $\tilde{B}$ and denoted by $\tilde{A} \leq \tilde{B}$ if and only if $a_\alpha{}^- \leq b_\alpha{}^-$, and $a_\alpha{}^+ \leq b_\alpha{}^+$ for $\alpha \in (0,1]$. Moreover, from [42], for $k_1, k_2 \geq 0$, we say $k_1\tilde{A} + k_2\tilde{B} \leq \tilde{C}$ if and only if $k_1 a_\alpha{}^- + k_2 b_\alpha{}^- \leq c_\alpha{}^-$, and $k_1 a_\alpha{}^+ + k_2 b_\alpha{}^+ \leq c_\alpha{}^+$.
Method 2. We say $\tilde{A}$ is smaller than $\tilde{B}$ and denoted by $\tilde{A} \leq \tilde{B}$ if and only if $a_\alpha{}^+ \leq b_\alpha{}^+$ for $\alpha \in (0.5, 1]$. Furthermore, for $k_1, k_2 \geq 0$, we say $k_1\tilde{A} + k_2\tilde{B} \leq \tilde{C}$ if and only if $k_1 a_\alpha{}^+ + k_2 b_\alpha{}^+ \leq c_\alpha{}^+$.

Remark 2.4. *In spite of method 1, method 2 can be applied to rank any two fuzzy numbers. However, method 2 is weaker since only the upper bounds of the intervals are utilized. Therefore, in this paper, we use method 1 as long as this method works successfully. Otherwise, method 2 is examined.*

Definition 2.5. [25] *Assume that $A = [A^L, A^U], B = [B^L, B^U]$ and $k \geq 0$ is a scalar. Therefore, addition, multiplication, and division on the intervals are defined as follows:*

$$A + B = [A^L + B^L, A^U + B^U], -A = [-A^U, -A^L], kA = [kA^L, kA^U],$$
$$AB = [\min\{A^L B^L, A^L B^U, A^U B^L, A^U B^U\}, \max\{A^L B^L, A^L B^U, A^U B^L, A^U B^U\}],$$
$$A/B = [\min\{A^L/B^L, A^L/B^U, A^U/B^L, A^U/B^U\}, \max\{A^L/B^L, A^L/B^U, A^U/B^L, A^U/B^U\}].$$

2.2 Linear fractional programming problem

Consider the general form of the LFPP as follows:

$$\text{Maximize } \frac{C^T X + \alpha}{D^T X + \beta} \tag{1}$$

$$s.t \;\; AX \leq b, \;\; D^T X + \beta > 0, \;\; X \geq 0.$$

The (1) is changed into the following linear programming problem by the use of variable transformations $t = \frac{1}{D^T X + \beta}, Y = tX$.

$$\text{Maximize } C^T Y + \alpha t \tag{2}$$
$$s.t \;\; AY - bt \geq 0, \;\; D^T Y + \beta t = 1, \;\; Y, t \geq 0.$$

Theorem 2.6. [10] *Let(Y^*, t^*) be the optimal solution of (2), then the optimal solution of (1) is: $X^* = \frac{Y^*}{t^*}$.*

2.3 Multi objective programming problem

Let us consider the general form of the multi objective programming (MOPP) as follows:

$$\text{Maximize } \{F_1(X), ..., F_k(X)\} \;\; s.t \;\; X \in S. \tag{3}$$

Definition 2.7. [3] *For (3), a solution $X^* \in S$ is called efficient if and only if $\nexists X \in S$ such that $F_j(X_*) \leq F_j(X), j = 1, ..., k$, and $\exists l \in \{1, ..., k\}$ such that $F_j(X^*) < F_j(X)$.*

Max-min approach is a classical method which is used to scalarize the MOPP as follows:

$$\text{Maximize } \beta \;\;\; s.t \;\; X \in S, \;\; \beta \leq F_i(x) \; for \;\; i = 1, ..., k. \tag{4}$$

Definition 2.8. *Consider the single objective problem $\underset{X \in S}{\text{Maximize}} \; G(X)$. The point $X^* \in S$ is called an ϵ-optimal solution if $G(X) \leq G(X^*) + \epsilon, \forall X \in S$.*

In this article, the word "Maximize" is used when we aim to maximize an optimization problem, and the abbreviation "max" is used when we are going to determine the maximum value of an specific set. In addition, for convenience, a triangular fuzzy number given by definition 1 is denoted by (a, b, c).

3   Main results

In this section, we alter the LFPP with fuzzy coefficients into a LPP. Moreover, it is proven that the solution resulted by the LPP is an ϵ-optimal solution for the fuzzy problem. To design our method, variable transformations, max-min technique in addition to α-cut are utilized. Consider the general form of the LFPP with fuzzy coefficients as follows:

$$\text{Maximize} \quad \frac{\tilde{C}^T X + \tilde{d}}{\tilde{P}^T X + \tilde{q}} \quad s.t \quad \tilde{A}X \le \tilde{b}, \quad X \ge 0, \tag{5}$$

where $X = (X_1, , X_n)$, $\tilde{A}$ is an $m \times n$ matrix with fuzzy element $\tilde{a}_{ij}$, and $\tilde{b}$ is an $m \times 1$ matrix with fuzzy element $\tilde{b}_i$, $i = 1, , m, j = 1, , n$. By the use of α-cut, (5) is changed into:

$$\text{Maximize} \quad \frac{\left[\underline{C}^T, \overline{C}^T\right] X + \left[\underline{d}, \overline{d}\right]}{\left[\underline{P}^T, \overline{P}^T\right] X + \left[\underline{q}, \overline{q}\right]}, \quad s.t \quad \left[\underline{A}, \overline{A}\right] X \le \left[\underline{b}, \overline{b}\right], \quad X \ge 0. \tag{6}$$

Using operations on intervals and ranking of fuzzy numbers, (6) is altered into:

$$\text{Maximize} \quad \bar{F}(X) = \left[\underline{F}(X), \overline{F}(X)\right] \tag{7}$$

$$s.t \quad S = \{\underline{A}X \le \underline{b}, \ \overline{A}X \le \overline{b}, \ \overline{P}^T X + \overline{q}, \ \underline{P}^T X + \underline{q} > 0, \ X \ge 0\},$$

where $\underline{F}(X) = \frac{\underline{C}^T + \underline{d}}{\overline{P}^T X + \overline{q}}$ if $\underset{X \in S}{\text{Minimize}} \ \underline{C}^T X + \underline{d} \ge 0$. Otherwise, $\underline{F}(X) = \frac{\underline{C}^T + \underline{d}}{\underline{P}^T X + \underline{q}}$. And,

$\overline{F}(X) = \frac{\overline{C}^T + \overline{d}}{\underline{P}^T X + \underline{q}}$ if $\underset{X \in S}{\text{Maximize}} \ \overline{C}^T X + \overline{d} \ge 0$. Otherwise, $\overline{F}(X) = \frac{\overline{C}^T X + \overline{d}}{\overline{P}^T X + \overline{q}}$.

It is additionally assumed that S is a regular set i.e. a non-empty and bounded feasible region.

Remark 3.1. *Without loss of generality, we assume that:* $\underline{F}(X) = \frac{\underline{C}^T + \underline{d}}{\overline{P}^T X + \overline{q}}$, $\overline{F}(X) = \frac{\overline{C}^T + \overline{d}}{\underline{P}^T X + \underline{q}}$ *in the rest of the paper.*

According to [11], (7) can be represented as:

$$\underset{X \in S}{\text{Maximize}} \quad \{\underline{F}(X), \overline{F}(X)\} = \left\{\frac{\underline{C}^T X + \underline{d}}{\overline{P}^T X + \overline{q}}, \frac{\overline{C}^T X + \overline{d}}{\underline{P}^T X + \underline{q}}\right\}. \tag{8}$$

To change (8) into a problem by non-negative numerators and positive denominators, the membership functions of the objectives are defined, and an equivalent bi-objective problem is considered in terms of the membership functions. In fact, these non-negativities conditions help us to prove that this method yields an efficient solution. For this purpose, let:

$$\underset{X \in S}{\text{Maximize}} \ \underline{F}(X) = \underline{F}^{\max}, \ \underset{X \in S}{\text{Minimize}} \ \underline{F}(X) = \underline{F}^{\min}, \ \underset{X \in S}{\text{Maximize}} \ \overline{F}(X) = \overline{F}^{\max}, \ \underset{X \in S}{\text{Minimize}} \ \overline{F}(X) = \overline{F}^{\min}.$$

Therefore, the membership functions related to the objective functions $\underline{F}(X)$, $\overline{F}(X)$ are: $\underline{\mu}(x) = \frac{E^T X + f}{\overline{P}^T X + \overline{q}}$, and $\overline{\mu}(x) = \frac{G^T X + h}{\underline{P}^T + \underline{q}}$, respectively, where

$$E = \left(\frac{\underline{C}}{\underline{F}^{\max} - \underline{F}^{\min}} - \underline{F}^{\min} \ \overline{P}\right), \quad f = \left(\frac{\underline{d}}{\underline{F}^{\max} - \underline{F}^{\min}} - \underline{F}^{\min} \overline{q}\right),$$

$$G = \left(\frac{\overline{C}}{\overline{F}^{\max} - \overline{F}^{\min}} - \overline{F}^{\min} \ \underline{P}\right), \quad h = \left(\frac{\overline{d}}{\overline{F}^{\max} - \overline{F}^{\min}} - \overline{F}^{\min} \ \underline{q}\right), \quad \forall X \in S.$$

Since $\mu(X)$, $\bar{\mu}(X) \in [0, 1]$, $\underline{P}^T X + \underline{q}$, $\overline{P}^T X + \overline{q} > 0$, then $E^T X + f$, $G^T X + h \geq 0, \forall X \in S$.

The equivalent of (8) in terms of the membership functions is as follows:

$$\underset{X \in S}{\text{Maximize}} \left\{ \underline{\mu}(X) = \frac{E^T X + f}{\overline{P}^T X + \overline{q}}, \ \bar{\mu}(X) = \frac{G^T X + h}{\underline{P}^T X + \underline{q}} \right\}. \tag{9}$$

By setting:

$$\lambda = \min \left\{ \frac{1}{\overline{P}^T X + \overline{q}}, \ \frac{1}{\underline{P}^T X + \underline{q}} \right\}, \ \lambda X = Y, \ \forall X \in S, \tag{10}$$

the (9) is transformed into:

$$\text{Maximize} \ \{ E^T Y + \lambda f, \ G^T Y + \lambda h \} \tag{11}$$

$s.t \ \ \Omega = \{ \underline{A}Y - \lambda \underline{b} \leq 0, \ \overline{A}Y - \lambda \overline{b} \leq 0, \ \overline{P}^T Y + \lambda \overline{q}, \ \underline{P}^T Y + \lambda \underline{q} \leq 1, \ Y, \lambda \geq 0 \}$,

where Ω is assumed to be a regular set.

Proposition 3.2. *In (11), variable λ cannot be zero.*

Proof. Let $\left(\hat{Y}, 0 \right) \in \Omega$, then $\underline{A}\hat{Y} \leq 0, \overline{A}\hat{Y} \leq 0$. Therefore, $\hat{X} \in S$ results in $\underline{A}\left(\hat{X} + \beta\hat{Y} \right) = \underline{A}\hat{X} + \beta\underline{A}\hat{Y} \leq \underline{A}\hat{X} \leq 0$, $\overline{A}\left(\hat{X} + \beta\hat{Y} \right) = \overline{A}\hat{X} + \beta\overline{A}\hat{Y} \leq 0, \ \forall \beta \geq 0$; this means $\hat{X} + \beta\hat{Y}$ is feasible point of S, $\forall \beta \geq 0$. Thus, S must be unbounded. This is a contradiction to the fact that S is a regular set. $\square$

Proposition 3.3. *If $\left(\overline{Y}, \overline{\lambda} \right) \in \Omega$, then $\frac{\overline{Y}}{\overline{\lambda}} \in S$.*

Proof. Since $\left(\overline{Y}, \overline{\lambda} \right) \in \Omega$, then $\overline{Y} \geq 0, \ \overline{\lambda} > 0, \ \underline{A}\overline{Y} - \overline{\lambda}\underline{b}, \ \overline{A}\overline{Y} - \overline{\lambda}\overline{b} \leq 0$. Thus,

$$\frac{\overline{Y}}{\overline{\lambda}} \geq 0, \ \underline{A}\left(\frac{\overline{Y}}{\overline{\lambda}} - \underline{b} \right) = \frac{1}{\overline{\lambda}}\left(\underline{A}\overline{Y} - \overline{\lambda}\underline{b} \right) \leq 0, \ \overline{A}\left(\frac{\overline{Y}}{\overline{\lambda}} \right) - \overline{b} = \frac{1}{\overline{\lambda}}\left(\overline{A}\overline{Y} - \overline{\lambda}\overline{b} \right) \leq 0.$$

$\square$

Let us assume $\beta \leq E^T Y + \lambda f, \ \beta \leq G^T Y + \lambda h, \forall (Y, \lambda) \in \Omega$. Then,(11) is changed into:

$$\text{Maximize} \ \ \beta \tag{12}$$

$s.t \ \ \mho = \left\{ \underline{A}Y - \lambda \underline{b} \leq 0, \ \overline{A}Y - \lambda \overline{b} \leq 0, \overline{P}^T Y + \lambda \overline{q} \leq 1, \ \underline{P}^T Y + \lambda \underline{q} \leq 1, \beta \leq E^T Y + \lambda f, \ \beta \leq G^T Y + \lambda h, \ Y, \lambda, \beta \geq 0 \right\}$,

where $\mho$ is a regular set.

Lemma 3.4. *The optimal solution of (12) is unique.*

Proof. Let $(Y^*, \lambda^*, \beta^*)$ be the optimal solution and is not unique; this means constraint $\beta \geq 0$ is active at the optimum i.e. $\beta^* = 0$. In the other word, if $(Y, \lambda, \beta) \in \mho$, then $\beta = 0$. Therefore, either $E^T Y + \lambda f = 0$ or $G^T y + \lambda h = 0, \forall (Y, \lambda, 0) \in \mho$. Without loss of generality, let $E^T Y + \lambda f = 0, \forall (Y, \lambda, 0) \in \mho$. Since $\lambda > 0$, then $E^T X + f = 0, \forall X \in S$; this means $\underline{\mu}(X) = 0, \forall X \in S$. As the consequence, (9) is reduced into a single objective LFPP. This is a contradiction. $\square$

Theorem 3.5. *Let $(Y^*, \lambda^*, \beta^*)$ be the optimal solution of (12). Then, $X^* = \frac{Y^*}{\lambda^*}$ is an efficient solution for (9).*

Proof. Let $X^* = \frac{Y^*}{\lambda^*}$ not be an efficient solution for (9), then $\exists \bar{X} \in S$ such that:

$$\frac{E^T X^* + f}{\bar{P}^T X^* + \bar{q}} \leq \frac{E^T \bar{X} + f}{\bar{P}^T \bar{X} + \bar{q}}, \frac{G^T X^* + h}{\underline{P}^T X^* + \underline{q}} < \frac{G^T \bar{X} + h}{\underline{P}^T \bar{X} + \underline{q}} \text{ or } \frac{E^T X^* + f}{\bar{P}^T X^* + \bar{q}} < \frac{E^T \bar{X} + f}{\bar{P}^T \bar{X} + \bar{q}}, \frac{G^T X^* + h}{\underline{P}^T X^* + \underline{q}} \leq \frac{G^T \bar{X} + h}{\underline{P}^T \bar{X} + \underline{q}} \quad (13)$$

Without loss of generality, let:

$$\frac{E^T X^* + f}{\bar{P}^T X^* + \bar{q}} \leq \frac{E^T \bar{X} + f}{\bar{P}^T \bar{X} + \bar{q}}, \quad \frac{G^T X^* + h}{\underline{P}^T X^* + \underline{q}} < \frac{G^T \bar{X} + h}{\underline{P}^T \bar{X} + \underline{q}}, \quad (14)$$

$(Y^*, \lambda^*, \beta^*) \in \mho \Rightarrow$

$$\lambda^* \leq \lambda_1^* = \frac{1}{\bar{P}^T X^* + \bar{q}}, \ \lambda^* \leq \lambda_2^* = \frac{1}{\underline{P}^T X^* + \underline{q}}, 0 \leq \beta^*, \ \beta^* \leq E^T Y^* + \lambda^* f, \ \beta^* \leq G^T Y^* + \lambda^* h. \quad (15)$$

Let us define:

$$\bar{\theta} = \max \left\{ \bar{\lambda}_1 = \frac{1}{\bar{P}^T \bar{X} + \bar{q}}, \ \bar{\lambda}_2 = \frac{1}{\underline{P}^T \bar{X} + \underline{q}} \right\}, \ \bar{\lambda} = \bar{\theta} - \epsilon, \quad (16)$$

where

$$\bar{\theta} - \bar{\lambda}_1 \leq \epsilon \leq \bar{\theta} - \lambda^* \left(\frac{E^T X^* + f}{E^T \bar{X} + f} \right), \ \bar{\theta} - \bar{\lambda}_2 \leq \epsilon \leq \bar{\theta} - \lambda^* \left(\frac{G^T X^* + h}{G^T \bar{X} + h} \right), \quad (17)$$

We aim to show that (17) is well defined. In the other word, there must exist ϵ satisfying (17.) To ensure this, two below conditions must hold true.

(I) $E^T \bar{X} + f, \ G^T \bar{X} + h > 0$,

(II) $\bar{\theta} - \bar{\lambda}_1 \leq \bar{\theta} - \lambda^* \left(\frac{E^T X^* + f}{E^T \bar{X} + f} \right), \ \bar{\theta} - \bar{\lambda}_2 \leq \bar{\theta} - \lambda^* \left(\frac{G^T X^* + h}{G^T \bar{X} + h} \right).$

Since $\mu(X) = \frac{E^T X + f}{\bar{P}^T X + \bar{q}} \in [0,1], \bar{P}^T X + \bar{q} > 0, then \ E^T X + f \geq 0, \forall X \in S$. Now, let us set: $E^T X + f = 0$. Then, $\mu(X^*) = \frac{E^T X^* + f}{\bar{P}^T X^* + \bar{q}} < \frac{E^T \bar{X} + f}{\bar{P}^T \bar{X} + \bar{q}} = 0$ possibly happens due to (13).This is a contradiction since $\mu(X^*) \geq 0$. In a similar way, it can be shown that $G^T \bar{X} + h > 0$. Thus, the (I) is demonstrated. It follows directly from (14) and (15) that:

$$\lambda^* (E^T X^* + f) \leq \lambda_1^* (E^T X^* + f) = \frac{E^T X^* + f}{\bar{P}^T X^* + \bar{q}} \leq \frac{E^T \bar{X} + f}{\bar{P}^T \bar{X} + \bar{q}} = \bar{\lambda}_1 (E^T \bar{X} + f).$$

Thus, $\lambda^* (\frac{E^T X^* + f}{E^T \bar{X} + f}) \leq \bar{\lambda}_1$. As the direct result: $\bar{\theta} - \bar{\lambda}_1 \leq \bar{\theta} - \lambda^* (\frac{E^T X^* + f}{E^T \bar{X} + f})$. Following the same process, it can be also demonstrated that:

$$\bar{\theta} - \bar{\lambda}_2 \leq \bar{\theta} - \lambda^* (\frac{G^T X^* + h}{G^T \bar{X} + h}).$$

Thus, the (II) is proved.

According to (17), we aim to prove the followings are true statements.

$$\bar{\lambda}(\bar{P}^T \bar{X} + \bar{q}) \leq 1, \ \bar{\lambda}(\underline{P}^T \bar{X} + \underline{q}) \leq 1, \quad (18)$$

$$\lambda^* (E^T X^* + f) \leq \bar{\lambda}(E^T \bar{X} + f), \lambda^* (G^T X^* + h) \leq \bar{\lambda}(G^T \bar{X} + h). \quad (19)$$

To ensure the (18), it is resulted from $\bar{\theta} - \bar{\lambda}_1 \leq \epsilon$ that:

$$\bar{\lambda}(\bar{P}^T \bar{X} + \bar{q}) = (\bar{\theta} - \epsilon)(\bar{P}^T \bar{X} + \bar{q}) \leq (\bar{\theta} - (\bar{\theta} - \bar{\lambda}_1))(\bar{P}^T \bar{X} + \bar{q}) = \bar{\lambda}_1 (\bar{P}^T \bar{X} + \bar{q}).$$

It follows directly from (10) that: $\bar{\lambda}_1(\underline{P}^T \bar{X} + \bar{q}) \leq 1$. Thus, $\bar{\lambda}(\underline{P}^T \bar{X} + \bar{q}) \leq 1$. Following a similar way, $\bar{\theta} - \bar{\lambda}_2 \leq \epsilon$ implies $\bar{\lambda}(\underline{P}^T \bar{X} + \underline{q}) \leq 1$. Therefore, (18) is verified.

To ensure (19), It is concluded from $\epsilon \leq \bar{\theta} - \lambda^*(\frac{E^T X^* + f}{E^T \bar{X} + f})$ that: $\lambda^*(\frac{E^T X^* + f}{E^T \bar{X} + f}) \leq \bar{\theta} - \epsilon$. Thus,

$$\lambda^*(E^T X^* + f) \leq (\bar{\theta} - \epsilon)(E^T \bar{X} + f) = \bar{\lambda}(E^T \bar{X} + f).$$

In a similar way, $\epsilon \leq \bar{\theta} - \lambda^*(\frac{G^T X^* + h}{G^T \bar{X} + h})$ results in $\lambda^*(G^T X^* + h) \leq \bar{\lambda}(G^T \bar{X} + h)$. Thus, the (19) is proved.

Let us set: $\bar{Y} = \bar{\lambda}\bar{X}$. We need to show that $(\bar{Y}, \bar{\lambda}) \in \mathcal{U}$. To do this, the followings must hold true.

a) $\bar{\lambda} \geq 0$.

Without loss of generality, let:

$$\bar{\theta} - \lambda^*(\frac{G^T X^* + h}{G^T \bar{X} + h}) = \max \epsilon = \max\{\bar{\theta} - \lambda^*(\frac{E^T X^* + f}{E^T \bar{X} + f}), \ \bar{\theta} - \lambda^*(\frac{G^T X^* + h}{G^T \bar{X} + h})\}.$$

Therefore, $\bar{\lambda} \geq \bar{\theta} - \left(\bar{\theta} - \lambda^*\left(\frac{G^T X^* + h}{G^T \bar{X} + h}\right)\right) = \lambda^*\left(\frac{G^T X^* + h}{G^T \bar{X} + h}\right) \geq 0$.

b) $\bar{Y} \geq 0$.

Since $\bar{X} \in S$, then $\bar{X} \geq 0$. Consequently, $\bar{Y} = \bar{\lambda}\bar{X} \geq 0$.

c) $\left(\bar{P}^T \bar{Y} + \bar{\lambda}\bar{q}\right) \leq 1$.

Considering $\bar{Y} = \bar{\lambda}\bar{X}$ and (18) *prove c.*

d) $\underline{A}\bar{Y} - \bar{\lambda}\underline{b} \leq 0, \ \bar{A}\bar{Y} - \bar{\lambda}\bar{b} \leq 0$.

$\underline{A}\bar{X} - \underline{b} \leq 0, \bar{A}\bar{X} - \bar{b} \leq 0$ since $\bar{X} \in S$. Thus, $\underline{A}\bar{Y} - \bar{\lambda}\underline{b} = \bar{\lambda}(\underline{A}\bar{X} - \underline{b}) \leq 0, \ \bar{A}\bar{Y} - \bar{\lambda}\bar{b} = \bar{\lambda}(\bar{A}\bar{X} - \bar{b}) \leq 0$.

In what follows we aim to create a $\bar{\beta}$ such that $\bar{\beta} \geq \beta^*$ and $(\bar{Y}, \bar{\lambda}, \bar{\beta}) \in \mathcal{U}$.

(19) $\Rightarrow$

$$E^T Y^* + \lambda^* f = \lambda^*(E^T X^* + f) \leq \bar{\lambda}(E^T \bar{X} + f) = E^T + \bar{\lambda}f,$$
$$G^T Y^* + \lambda^* h = \lambda^*(G^T X^* + h) \leq \bar{\lambda}(G^T \bar{X} + h) = G^T \bar{Y} + \bar{\lambda}h. \tag{20}$$

Feasibility of $(Y^*, \lambda^*, \beta^*)$
and (20) $\Rightarrow$

$$\beta^* \leq E^T \bar{Y} + \bar{\lambda}f, \ \beta^* \leq G^T \bar{Y} + \bar{\lambda}h. \tag{21}$$

$$\text{Let } \gamma = \min\{E^T \bar{Y} + \bar{\lambda}f - \beta^*, \ G^T \bar{Y} + \bar{\lambda}h - \beta^*\}, \ \bar{\beta} = \beta^* + \gamma. \tag{22}$$

$$\text{(21) and (22) indicate } \gamma \geq 0, \text{ and as } a \text{ consequence } \beta^* \leq \bar{\beta}. \tag{23}$$

(22) $\Rightarrow$

$$0 \leq \bar{\beta}, \ \bar{\beta} \leq E^T \bar{Y} + \bar{\lambda}f, \ \bar{\beta} \leq G^T \bar{Y} + \bar{\lambda}h. \tag{24}$$

(24) besides $(\bar{Y}, \bar{\lambda}) \in \Omega$ results in $(\bar{Y}, \bar{\lambda}, \bar{\beta}) \in \mathcal{U}$.

In brief, we created $(\bar{Y}, \bar{\lambda}, \bar{\beta}) \in \mathcal{U}$ in such a way that $\bar{\beta} \geq \beta^*$. This contradicts the unique optimality of $(Y^*, \lambda^*, \beta^*)$. The proof is then complete. $\square$

Theorem 3.6. *Let* $(Y^*, \lambda^*, \beta^*)$ *be the optimal solution for (12), then* $\frac{Y^*}{\lambda^*}$ *is an* ϵ*-optimal solution for (7), where*

$$\epsilon = \max\left\{\underline{F}^{\max} - \underline{F}\left(\frac{Y^*}{\lambda^*}\right), \ \bar{F}^{\max} - \bar{F}\left(\frac{Y^*}{\lambda^*}\right)\right\}.$$

Proof. Theorem 3.5 demonstrates $\frac{Y^*}{\lambda^*}$ is efficient for (9). Thus,

Case 1. $\mu(X) < \mu(\frac{Y^*}{\lambda^*}), \forall X \in S$. Thus, $\underline{F}(X) < \underline{F}\left(\frac{Y^*}{\lambda^*}\right), \forall X \in S$. Let us st: $\epsilon_1 = \bar{F}^{\max} - \bar{F}\left(\frac{Y^*}{\lambda^*}\right)$. Therefore,

$$\bar{F}(X) = \left[\underline{F}(X), \ \bar{F}(X)\right] \leq \left[\underline{F}\left(\frac{Y^*}{\lambda^*}\right) + \epsilon_1, \ \bar{F}\left(\frac{Y^*}{\lambda^*}\right) + \epsilon_1\right] = \bar{F}\left(\frac{Y^*}{\lambda^*}\right) + \epsilon_1, \forall X \in S.$$

This indicates $\frac{Y^*}{\lambda^*}$ is an ϵ_1-efficient solution for (7).

Case 2. $\bar{\mu}(X) < \bar{\mu}(\frac{Y^*}{\lambda^*}), \forall X \in S$. As the result, $\bar{F}(X) < \bar{F}\left(\frac{Y^*}{\lambda^*}\right), \forall X \in S$. Let us st: $\epsilon_2 = \underline{F}^{\max} - \underline{F}\left(\frac{Y^*}{\lambda^*}\right)$. Therefore,

$\bar{F}(X) = \left[\underline{F}(X), \bar{F}(X)\right] \leq \left[\underline{F}\left(\frac{Y^*}{\lambda^*}\right) + \epsilon_2, \bar{F}\left(\frac{Y^*}{\lambda^*}\right) + \epsilon_2\right] = \bar{F}\left(\frac{Y^*}{\lambda^*}\right) + \epsilon_2, \forall X \in S$. This indicates $\frac{Y^*}{\lambda^*}$ is an ϵ_2-efficient solution for (7).

If we set $\epsilon = \max\{\epsilon_1, \epsilon_2\}$, then

$$\bar{F}(X) = \left[\underline{F}(X), \bar{F}(X)\right] \leq \left[\underline{F}\left(\frac{Y^*}{\lambda^*}\right) + \epsilon, \bar{F}\left(\frac{Y^*}{\lambda^*}\right) + \epsilon\right] = \bar{F}\left(\frac{Y^*}{\lambda^*}\right) + \epsilon, \forall X \in S.$$

$\square$

Algorithm

This algorithm summarizes the procedure of finding ϵ-optimal solution for the linear fractional programming problem with fuzzy coefficients.

Initial step. Determine $\alpha \in (0, 1]$.

Step 1. Formulate (6), then by using interval arithmetic, formulate (7).

Step 2. Formulate (8), then define the membership functions.

Step 3. Formulate (9), and then (11).

Step 4. Formulate (12), then find $(Y^*, \lambda^*, \beta^*)$ as the optimal solution. Afterwards, set $X^* = \frac{Y^*}{\lambda^*}$.

Final step. Calculate $\epsilon = \max\left\{\underline{F}^{\max} - \underline{F}(X^*), \bar{F}^{\max} - \bar{F}(X^*)\right\}$. Then, introduce the solution X^* as the ϵ-optimal solution for the fuzzy problem 5.

4 Numerical example

4.1 Example 1

$$\text{Maximize} \quad \tilde{F}(X) = \frac{-(0.5, 1, 1.25)X_1 + (2.5, 3, 4)X_2 + (1, 2, 3)}{(0.5, 1, 1.25)X_1 + (1, 2, 3)X_2 + (0.5, 1, 1.25)}, \tag{25}$$

s.t

$$(1, 2, 3)X_1 + (5, 1, 1.25)X_2 \leq (3.5, 4, 4.5), \qquad (2.5, 3, 4)X_1 - (1, 2, 3)X_2 \leq (4, 5, 6),$$
$$(0.5, 1, 1.25)X_1 + (1, 2, 3)X_2 \leq (2.5, 3, 4), \qquad -(0.5, 1, 1.25)X_1 - (2.5, 3, 4)X_2 \leq -(1, 2, 3),$$
$$X_1, X_2 \geq 0.$$

Let us set $\alpha = 0.8$. The (6) is then formulated as follows:

$$\text{Maximize} \quad \bar{F}(X) = \frac{[-1.05, -0.9]X_1 + [2.9, 3.2]X_2 + [1.8, 2.2]}{[0.9, 1.05]X_1 + [1.8, 2.2]X_2 + [0.9, 1.05]}, \tag{26}$$

s.t

$$[1.8, 2.2]X_1 + [0.9, 1.05]X_2 \leq [3.9, 4.1], \qquad [2.9, 3.2]X_1 + [-2.2, -1.8]X_2 \leq [4.8, 5.2],$$
$$[0.9, 1.05]X_1 + [1.8, 2.2]X_2 \leq [2.9, 3.2], \qquad [-1.05, -0.9]X_1 + [-3.2, -2.9]X_2 \leq [-2.2, -1.8],$$
$$X_1, X_2 \geq 0.$$

The (7) is formulated as follows:

$$\text{Maximize} \quad \left[\frac{-1.05X_1 + 2.9X_2 + 1.8}{1.05X_1 + 2.2X_2 + 1.05}, \frac{-0.9X_1 + 3.2X_2 + 2.2}{0.9X_1 + 1.8X_2 + 0.9}\right], \tag{27}$$

s.t $S =$

$$\{1.8X_1 + 0.9x_2 \leq 3.9, \qquad\qquad 2.2X_1 + 1.05X_2 \leq 4.8$$
$$2.9X_1 - 2.2X_2 \leq 4.8 \qquad\qquad 3.2X_1 - 1.8X_2 \leq 5.2$$
$$0.9X_1 + 1.8X_2 \leq 2.9 \qquad\qquad 1.05X_1 + 2.2X_2 \leq 3.2$$
$$-1.05X_1 - 3.2X_2 \leq -2.2 \qquad -0.9X_1 - 2.9X_2 \leq -1.8$$
$$X_1,\ X_2 \geq 0\}.$$

The (8) is then formulated as follows:

$$\underset{X \in S}{\text{Maximize}} \quad \left\{ \underline{F}(X),\ \overline{F}(X) \right\} = \left\{ \frac{-1.05X_1 + 2.9X_2 + 1.8}{1.05X_1 + 2.2X_2 + 1.05},\ \frac{-0.9X_1 + 3.2X_2 + 2.2}{0.9X_1 + 1.8X_2 + 0.9} \right\}. \tag{28}$$

The following individual maxima and minima are obtained using the method of Charnes and Cooper.

$\underline{F}^{\max} = 1.4805,\ \underline{F}^{\min} = 0.1265,\ \overline{F}^{\max} = 2.0585,\ \overline{F}^{\min} = 0.4087.$ Afterwards, the membership functions are defined as follows:

$\underline{\mu}(X) = \frac{-0.8736x_1 + 1.9363X_2 + 1.2313}{1.05X_1 + 2.2X_2 + 1.05},\ \overline{\mu}(X) = \frac{-0.7685x_1 + 1.4937X_2 + 1.1105}{0.9X_1 + 1.8X_2 + 0.9}.$

The (11) is formulated as below by setting:

$\lambda = \min \left\{ \frac{1}{1.05X_1 + 2.2X_2 + 1.05},\ \frac{1}{0.9X_1 + 1.8X_2 + 0.9} \right\},\ Y = \lambda X.$

$$\text{Maximize} \quad \{-0.8736Y_1 + 1.9363Y_2 + 1.2313\,\lambda,\ -0.7685Y_1 + 1.4937Y_2 + 1.1105\,\lambda\}, \tag{29}$$

s.t $\Omega =$

$$1.8Y_1 + 0.9Y_2 - 3.9\lambda \leq 0 \qquad\qquad 2.2Y_1 + 1.05Y_2 - 4.1\lambda \leq 0$$
$$2.9Y_1 - 2.2Y_2 - 4.8\lambda \leq 0 \qquad\qquad 3.2Y_1 - 1.8Y_2 - 5.2\lambda \leq 0$$
$$0.9Y_1 + 1.8Y_2 - 2.9\lambda \leq 0 \qquad\qquad 1.05Y_1 + 2.2Y_2 - 3.2\lambda \leq 0$$
$$-1.05Y_1 - 3.2Y_2 + 2.2\lambda \leq 0 \qquad -0.9Y_1 - 2.9Y_2 + 1.8\lambda \leq 0$$
$$1.05Y_1 + 2.2Y_2 + 1.05\lambda \leq 0 \qquad 0.9Y_1 + 1.8Y_2 + 0.9\lambda \leq 0$$
$$Y_1,\ Y_2,\ \lambda \geq 0.$$

The (12) is then formulated as follows:

$$\text{Maximize} \quad \beta \tag{30}$$

s.t $\mho = \Omega \bigcup \left\{ \beta \leq -0.8736Y_1 + 1.9363Y^2 + 1.2313\lambda,\ \beta \leq -0.7668Y_1 + 1.4937Y^2 + 1.1105\lambda \right\}.$

The problem above is solved and the solution obtained is:

$$(Y^*,\ \lambda^*,\ \beta^*) = (Y_1^*,\ Y_2^*,\ \lambda^*,\ \beta^*) = (0, 0.2683, 0.3902, 0.8341).$$

The optimal solution for the main problem is: $X^* = \frac{Y^*}{\lambda^*} = (0, 0.6875)$, and

$$\tilde{F}(X) = \overline{F}_{\alpha = 0.8}(X^*) = [\underline{F}(X^*),\ \overline{F}(X^*)] = [1.4805,\ 2.0585].$$

Since $\underline{F}^{\max} = \underline{F}(X^*), \overline{F}^{\max} = \overline{F}(X^*)$, then $\epsilon = 0$; this means X^* is the exact optimal solution for the main fuzzy problem.

4.1.1   Numerical analysis

The extreme points of the feasible region $S_{\alpha=0.8}$ are included in Table 1. As we observe numerically, $\underline{F}(\bar{\bar{X}}) < \underline{F}(X^*), \overline{F}(\bar{\bar{X}}) < \overline{F}(X^*)$, where $\bar{\bar{X}}$ is assumed as an extreme point. Therefore, convexity of S along with pseudoconvexity of $\underline{F}(X)$ and $\overline{F}(X)$ implies that: $\underline{F}(X) < \underline{F}(X^*), \overline{F}(X) < \overline{F}(X^*), \forall X \in S$. Thus,

$$\tilde{F}(X) = \left[\underline{F}(X),\ \overline{F}(X) \right] < \left[\underline{F}(X^*),\ \overline{F}(X^*) \right] = \tilde{F}(X^*),\ \forall X \in S;$$

this means X^* is a unique optimal solution for the main fuzzy problem.

Extreme point $\bar{\bar{X}}$	$\tilde{F} = [\underline{F}(\bar{\bar{X}}), \overline{F}(\bar{\bar{X}})]$
$(0, 1.4545)$	$[1.4160, 1.9483]$
$(1.5144, 0.7318)$	$[0.5487, 0.8874]$
$(1.7831, 0.1687)$	$[0.1266, 0.4042]$
$(1.7541, 0.2295)$	$[0.1636, 0.4688]$
$(1.6183, 0.1303)$	$[0.1265, 0.4087]$
$(0, 0.6875) = X^*$	$[1.4805, 2.0585]$

Table 1: Extreme points of $S_{\alpha=0.8}$ and their values of $\tilde{F}(X)$

4.2 Example 2

$$\text{Maximize} \quad \tilde{F}(X) = \frac{(-1.5722, -1.35, -1.1278)X_1 + (9, 10, 15)X_2 + (3, 4, 5)}{(1, 2, 4)X_1 + (4, 5, 7)X_2 + (0, 1, 11)}, \tag{31}$$

$$\text{s.t} \quad S = \{-X_1 \le -1, \ -X_1 + 2X_2 \le 1, \ 2X_1 + X_2 \le 8, \ -2X_2 \le -1, \ X_1, \ X_2 \ge 0\}.$$

If we set $\alpha = 0.55$, then the (8) is formed as follows:

$$\text{Maximize} \quad \left\{\underline{F}(X), \ \overline{F}(X)\right\} = \left\{\frac{-1.45X_1 + 9.55X_2 + 3.55}{2.9X_1 + 5.9X_2 + 5.5}, \ \frac{-1.25X_1 + 12.25X_2 + 4.45}{1.55X_1 + 4.55X_2 + 0.55}\right\}. \tag{32}$$

Taking into account the values:

$$\underline{F}^{\max} = 0.8147, \ \underline{F}^{\min} = 0.1494, \ \overline{F}^{\max} = 2.7029, \ \overline{F}^{\min} = 1.7670,$$

the membership functions are defined as follows:

$$\underline{\mu}(X) = \frac{-2.8307X_1 + 13.0295X_2 + 4.1009}{2.9X_1 + 5.9X_2 + 5.5}, \quad \overline{\mu}(X) = \frac{-1.5908X_1 + 4.4985X_2 + 3.7167}{1.55X_1 + 4.55X_2 + 0.55}.$$

$$\text{Maximize} \quad \beta \tag{33}$$

$$\text{s.t} \quad \mho = \{-Y_1 + \lambda \le 0, \ -Y_1 + 2Y_2 - \lambda \le 0, \ 2Y_1 + Y_2 - 8\lambda \le 0, \ -2Y_2 + \lambda \le 0,$$
$$2.9Y_1 + 5.9Y_2 + 5.5\lambda \le 1, \ 1.55Y_1 + 4.55Y_2 + 0.55\lambda \le 1,$$
$$\beta \le -2.8307Y_1 + 13.0295Y_2 + 4.1009\lambda, \ \beta \le -1.59Y_1 + 4.4985Y_2 + 3.7167\lambda,$$
$$Y_1, \ Y_2, \ \lambda, \ \beta \ge 0\}.$$

The (33) is solved and the obtained solution is $(Y^*, \ \lambda^*, \ \beta^*) = (0.0699, \ 0.0699, \ 0.0699, \ 0.4632)$. Thus, $X^* = \frac{Y^*}{\lambda^*} = (1, \ 1)$ is the proposed solution for (31).

At the solution X^*:

$$\underline{F}(X) = 0.8147, \ \overline{F}(X) = 2.6992, \ \tilde{F}(X) = [0.8147, \ 2.6992],$$

$$\epsilon = \max\left\{\underline{F}^{\max} - \underline{F}(X^*), \ \overline{F}^{\max} - \overline{F}(X^*)\right\} = \max\{0, \ 0.0037\} = 0.0037.$$

Thus,

$$\tilde{F}(X) = \left[\underline{F}(X), \ \overline{F}(X)\right] \le \left[\underline{F}(X^* + \epsilon, \ \overline{F}(X^* + \epsilon\right] = \tilde{F}(X^*) + \epsilon, \ \forall X \in S.$$

4.2.1 Numerical analysis

In Table 2, the extreme point of the feasible region S are listed. Numerically, we see that:

$\underline{F}(\bar{\bar{X}}) + \epsilon < \underline{F}(X^*) + \epsilon, \overline{F}(\bar{\bar{X}}) < \overline{F}(X^*)$, where $\bar{\bar{X}}$ is assumed as an extreme point. Therefore, convexity of S along with

pseudoconvexity of $\underline{F}(X)$ and $\overline{F}(X)$ implies that: $\underline{F}(X) < \underline{F}(X^*) + \epsilon, \ \overline{F}(X) < \overline{F}(X^*) + \epsilon, \ \forall X \in S$. Thus,

$$\tilde{F}(X) = \left[\underline{F}(X), \ \overline{F}(X)\right] < \left[\underline{F}(X^*) + \epsilon, \ \overline{F}(X^*) + \epsilon\right] = \tilde{F}(X^*) + \epsilon, \ \forall X \in S;$$

this means X^* is an ϵ-optimal solution for the main fuzzy problem.

Extreme point $\bar{\bar{X}}$	$\tilde{F} = [\underline{F}(\bar{\bar{X}}), \overline{F}(\bar{\bar{X}})]$
$(1, 0.5)$	$[0.6057, 2.7027]$
$(3, 2)$	$[0.7045, 1.7622]$
$(3.75, 0.5)$	$[0.1496, 0.6816]$
$(1, 1) = X^*$	$[0.8147, 2.6992]$
$\epsilon = 0.0037$	$\tilde{F}(X^*) + \epsilon = [0.8184, 2.7029]$

Table 2: Extreme points of $S_{\alpha=0.55}$ and their values of $\tilde{F}(X)$

4.2.2 Comparison

For example 2, method of Mehra et al. results in solution $X^{Me} = (1, 0.5)$. At the solution X^{Me} :

$$\underline{F}(X) = 0.6057, \ \overline{F}(X) = 2.7029, \ \tilde{F}(X) = [0.6057, 2.7029],$$

$$\epsilon^{Me} = \max\left\{ \underline{F}^{\max} - \underline{F}(X^{Me}), \ \overline{F}^{\max} - \overline{F}(X^{Me}) \right\} = \max\{0.2097, \ 0\} = 0.2097.$$

Thus, our proposed solution X^* is more accurate than the solution X^{Me} due to the fact that: $\epsilon < \epsilon^{Me}$.

4.3 Example 3

In this section, a real life production planning in Taiwan is considered [17]. The original problem modeled as a LFPP with fuzzy coefficients and fuzzy decision variables. In order to be able to solve the problem with the method provided by this study, we set the decision variables to be non-fuzzy. Therefore, we reach the following problem.

$$\text{Maximize} \quad \tilde{F}(X) = \frac{\tilde{f}(X)}{\tilde{g}(X)}, \tag{34}$$

$$\text{s.t} \quad S = \{X_1 + X_2 + X_3 + X_4 \leq (7.2, 8, 8.8), \quad X_5 + X_6 + X_7 + X_8 \leq (12, 14, 13.8),$$
$$X_9 + X_{10} + X_{11} + X_{12} \leq (10.2, 12, 13.8), \quad X_1 + X_5 + X_9 \geq (16.2, 7, 7.8),$$
$$X_2 + X_6 + X_10 \geq (8.9, 10, 11.1), \quad X_3 + X_7 + X_11 \geq (6.5, 8, 9.5),$$
$$X_4 + X_8 + X_{12} \geq (7.8, 9, 10.2), \quad X_i \geq 0 \ i = 1, ..., 12\},$$

where

$$f(X) = (8, 10, 10.8)X_1 + (20.4, 22, 24)X_2 + (8, 10, 10.6)X_3 + (18.8, 20, 22)X_4 + (14, 15, 16)X_5 + (18.2, 20, 22)X_6$$
$$+ (10, 12, 13)X_7 + (6, 8, 8.8)X_8 + (18.4, 20, 21)X_9 + (9.6, 12, 13)X_{10} + (7.8, 10, 10.8)X_{11} + (14, 15, 16)X_{12},$$

$$g(X) = (1.5, 2, 2.5)X_1 + (4, 5, 6)X_2 + (1.3, 2, 2.5)X_3 + (3, 4, 5)X_4 + (2.5, 3, 4)X_5 + (2, 3, 4)X_6 +$$
$$(2.3, 3, 4)X_7 + (1.5, 2, 2.5)X_8 + (3, 4, 5)X_9 + (2, 3, 4)X_{10} + (1.5, 2, 2.7)X_{11} + (2, 3, 4)X_{12}.$$

If we set $\alpha = 0.6$, then (8) is formed as follows:

$$\underset{X \in S}{\text{Maximize}} \quad \left\{ \underline{F}(X), \ \overline{F}(X) \right\} = \left\{ \frac{f_1(X)}{g_1(X)}, \ \frac{f_2(X)}{g_2(X)} \right\}, \tag{35}$$

where

$$f_1(X) = 9.2X_1 + 21.36X_2 + 9.2X_3 + 19.52X_4 + 14.6X_5 + 19.28X_6 + 11.2X_7 + 7.2X_8 + 19.36X_9 + 11.04X_{10} + 9.12X_{11} + 14.8X_{12},$$

$$g_1(X) = 2.2X_1 + 5.4X_2 + 2.2X_3 + 4.4X_4 + 3.4X_5 + 3.4X_6 + 3.4X_7 + 2.2X_8 + 4.4X_9 + 3.4X_{10} + 2.28X_{11} + 3.4X_{12},$$

$$f_2(X) = 10.32X_1 + 22.8X_2 + 10.24X_3 + 20.8X_4 + 15.4X_5 + 20.8X_6 + 12.4X_7 + 8.32X_8 + 20.4X_9 + 12.4X_{10} + 10.32X_{11} + 15.4X_{12},$$

$$g_2(X) = 1.8X_1 + 4.6X_2 + 1.72X_3 + 3.6X_4 + 2.8X_5 + 2.6X_6 + 2.72X_7 + 1.8X_8 + 3.6X_9 + 2.6X_{10} + 1.8X_{11} + 2.6X_{12}.$$

According to $\underline{F}^{\max} = 4.723$, $\underline{F}^{\min} = 3.4741$, $\overline{F}^{\max} = 6.6931$, $\overline{F}^{\min} = 4.9367$, the membership functions are specified as follows:

$$\underline{\mu}(X) = \frac{\underline{\mu}^N(X)}{\underline{\mu}^D(X)}, \quad \overline{\mu}(X) = \frac{\overline{\mu}^N(X)}{\overline{\mu}^D(X)},$$

where

$$\underline{\mu}^N(X) = 1.2467X_1 + 2.0817X_2 + 1.2467X_3 + 4.2340X_4 + 2.2324X_5 + 5.9797X_6 - 0.49X_7 - 0.3547X_8 + 3.262X_9$$
$$- 0.6181X_{10} + 0.9601X_{11} + 2.2324X_{12},$$

$$\underline{\mu}^D(X) = 2.2X_1 + 5.4X_2 + 2.2X_3 + 4.4X_4 + 3.4X_5 + 3.4X_6 + 3.4X_7 + 2.2X_8 + 4.4X_9 + 3.4X_{10} + 2.28X_{11} + 3.4X_{12},$$

$$\overline{\mu}^N(X) = 0.8164X_1 + 0.0519X_2 + 0.9729X_3 + 1.7239X_4 + 0.898X_5 + 4.5346X_6 - 0.5852X_7 - 0.3223X_8 + 1.2912X_9$$
$$- 0.4354X_{10} + 0.8164X_{11} + 1.4601X_{12},$$

$$\overline{\mu}^D(X) = 1.8X_1 + 4.6X_2 + 1.72X_3 + 3.6X_4 + 2.8X_5 + 2.6X_6 + 2.72X_7 + 1.8X_8 + 3.6X_9 + 2.6X_{10} + 1.8X_{11} + 2.6X_{12}.$$

The (12) is formulated as follows:

$$\text{Maximize} \quad \beta \tag{36}$$

$$s.t \quad \mho = \{Y_1 + Y_2 + Y_3 + Y_4 - 7.68\lambda \leq 0, \quad Y_5 + Y_6 + Y_7 + Y_8 - 13.2\lambda \leq 0, \, Y_9 + Y_{10} + Y_{11} + Y_{12} - 11.28\lambda \leq 0,$$
$$Y_1 + Y_5 + Y_9 - 6.68\lambda \geq 0, \quad Y_2 + Y_6 + Y_10 - 9.56\lambda \geq 0,$$
$$Y_3 + Y_7 + Y_{11} - 7.4\lambda \geq 0, \quad Y_4 + Y_8 + Y_{12} - 8.52\lambda \geq 0,$$
$$2.2Y_1 + 5.4Y_2 + 2.2Y_3 + 4.4Y_4 + 3.4Y_5 + 3.4Y_6 + 3.4Y_7 + 2.2Y_8 + 4.4Y_9 + 3.4Y_{10} + 2.28Y_{11} + 3.4Y_{12} \leq 1$$
$$1.8Y_1 + 4.6Y_2 + 1.72Y_3 + 3.6Y_4 + 2.8Y_5 + 2.6Y_6 + 2.72Y_7 + 1.8Y_8 + 3.6Y_9 + 2.6Y_{10} + 1.8Y_{11} + 2.6Y_{12} \leq 1$$
$$\beta \leq 1.2467Y_1 + 2.0817Y_2 + 1.2467Y_3 + 4.2340Y_4 + 2.2324Y_5 + 5.9797Y_6 - 0.49Y_7 - 0.3547Y_8 + 3.262Y_9 - 0.6181Y_{10} +$$
$$0.9601Y_{11} + 2.2324Y_{12}$$
$$\beta \leq 0.8164Y_1 + 0.0519Y_2 + 0.9729Y_3 + 1.7239Y_4 + 0.898Y_5 + 4.5346Y_6 - 0.5852Y_7 - 0.3223Y_8 + 1.2912Y_9 - 0.4354Y_{10} +$$
$$0.8164Y_{11} + 1.4601Y_{12}$$
$$Y_i, \, \lambda, \, \beta \geq 0, i = 1, ..., 12\}.$$

The above problem is solved and the obtained solution is:
$(Y^*, \, \lambda^*, \, \beta^*) = (0.0309, \, 0, \, 0.0471, \, 0, \, 0.037, \, 0.0971, \, 0, \, 0, \, 0, \, 0, \, 0.0280, \, 0.0865, \, 0.0102, \, 0.6933)$. Thus, the proposed solution is:
$X^* = \frac{Y^*}{\lambda^*} = (3.04, \, 0, \, 4.64, \, 0, \, 3.64, \, 9.56, \, 0, \, 0, \, 0, \, 0, \, 2.76, \, 8.52)$.
At the solution X^*:

$$\underline{F}(X) = 4.7165, \; \overline{F}(X) = 6.6931, \; \tilde{F}(X) = [4.7165, \, 6.6931],$$

$$\epsilon = \max\left\{ \underline{F}^{\max} - \underline{F}(X^*), \; \overline{F}^{\max} - \overline{F}(X^*) \right\} = \max\{0, 0.0065\} = 0.0065,$$

$$\tilde{F}(X) = \left[\underline{F}(X), \; \overline{F}(X) \right] \leq \left[\underline{F}(X^*) + \epsilon, \; \overline{F}(X^*) + \epsilon \right] = \tilde{F}(X^*) + \epsilon, \; \forall X \in S.$$

4.3.1   Comparison

Method of [17] results in a solution for which $\tilde{F}(X) = [3.688, 6.576]$. Therefore,

$$\epsilon^{Da} = \max\{4.7230 - 3.688, 6.6931 - 6.5760\} = 1.035.$$

Since $\epsilon \leq \epsilon^{Da}$, it is then concluded that this study provide a better result.

4.4   Example 4

Our proposed approach can be used to solve bi-objective linear fractional programming problem. In this section, we consider a real life example taken from [29].

$$\text{Maximize} \{Z_1(X), \ Z_2(X)\} \tag{37}$$

$$\text{s.t} \quad S = \{0.3X_1 + 0.4X_2 + 0.4X_3 + 0.98X_4 + 0.97X_5 + 0.98X_6 \leq 600,$$
$$2280000X_1 + 9200X_2 + 16000X_3 + 22500X_4 + 20000X_5 + 20000X_6 \leq 20000000,$$
$$650X_1 + 630X_2 + 320X_3 + 660X_4 + 360X_5 + 640X_6 \leq 500000,$$
$$20X_1 + 22X_2 + 20X_3 + 18X_4 + 20X_5 + 17X_6 \leq 15000,$$
$$11400X_1 + 3220X_2 + 1800X_3 + 12750X_4 + 3250X_5 + 3000X_6 \leq 6000000,$$
$$148X_1 + 238X_4 + 135X_6 \leq 50000,$$
$$180X_1 + 220X_2 + 200X_3 + 150X_4 + 100X_5 + 160X_6 \leq 120000,$$
$$60X_1 + 40X_2 + 35X_3 + 50X_4 + 30X_5 + 45X_6 \leq 30000,$$
$$30X_1 + 32X_2 + 28X_3 + 35X_4 + 26X_5 + 20X_6 \leq 200000,$$
$$15X_1 + 18X_2 + 16X_3 + 14X_4 + 17X_5 + 18X_6 \leq 10000,$$
$$42X_1 + 38X_2 + 36X_3 + 40X_4 + 37X_5 + 35X_6 \leq 25000,$$
$$X_i \geq 0 \ \ i = 1, ..., 6\},$$

where

$$Z_1(X) = \frac{59890X_1 + 23390X_2 + 30750X_3 + 59750X_4 + 40700X_5 + 59435X_6}{35345X_1 + 13420X_2 + 18455X_3 + 39455X_4 + 23840X_5 + 24070X_6 + 500000},$$

$$Z_2(X) = \frac{59890X_1 + 23390X_2 + 30750X_3 + 59750X_4 + 40700X_5 + 59435X_6}{96X_1 + 120X_2 + 144X_3 + 144X_4 + 84X_5 + 120X_6 + 480}.$$

Consider:

$$Z_1^{\min} = 0, \ Z_1^{\max} = 2.3381, \ Z_2^{\min} = 0, \ Z_2^{\max} = 491.51.51,$$

$$\mu_{Z_1}(X) = \frac{25615X_1 + 10004X_2 + 13152X_3 + 25555X_4 + 17407X_5 + 25420X_6}{35345X_1 + 13420X_2 + 18455X_3 + 39455X_4 + 23840X_5 + 24070X_6 + 500000},$$

$$\mu_{Z_2}(X) = \frac{121.8477X_1 + 47.5876X_2 + 62.5617X_3 + 121.5629X_4 + 82.8052X_5 + 120.992X_6}{96X_1 + 120X_2 + 144X_3 + 144X_4 + 84X_5 + 120X_6 + 480},$$

the (12) is formed for this example as follows:

$$\text{Maximize} \quad \beta \tag{38}$$

$$\text{s.t} \quad \mho = \{0.3Y_1 + 0.4Y_2 + 0.4Y_3 + 0.98Y_4 + 0.97Y_5 + 0.98Y_6 - 600\lambda \leq 0,$$
$$2280000Y_1 + 9200Y_2 + 16000Y_3 + 22500Y_4 + 20000Y_5 + 20000Y_6 - 20000000\lambda \leq 0,$$
$$650Y_1 + 630Y_2 + 320Y_3 + 660Y_4 + 360Y_5 + 640Y_6 - 500000\lambda \leq 0,$$
$$20Y_1 + 22Y_2 + 20Y_3 + 18Y_4 + 20Y_5 + 17Y_6 - 15000\lambda \leq 0,$$
$$11400Y_1 + 3220Y_2 + 1800Y_3 + 12750Y_4 + 3250Y_5 + 3000Y_6 - 6000000\lambda \leq 0,$$
$$148Y_1 + 238Y_4 + 135Y_6 - 50000\lambda \leq 0,$$
$$180Y_1 + 220Y_2 + 200Y_3 + 150Y_4 + 100Y_5 + 160Y_6 - 120000\lambda \leq 0,$$
$$60Y_1 + 40Y_2 + 35Y_3 + 50Y_4 + 30Y_5 + 45Y_6 - 30000\lambda \leq 0,$$
$$30Y_1 + 32Y_2 + 28Y_3 + 35Y_4 + 26Y_5 + 20Y_6 - 200000\lambda \leq 0,$$
$$15Y_1 + 18Y_2 + 16Y_3 + 14Y_4 + 17Y_5 + 18Y_6 - 10000\lambda \leq 0,$$
$$42Y_1 + 38Y_2 + 36Y_3 + 40Y_4 + 37Y_5 + 35Y_6 - 25000\lambda \leq 0,$$
$$35345Y_1 + 13420Y_2 + 18455Y_3 + 39455Y_4 + 23840Y_5 + 24070Y_6 + 500000\lambda \leq 1,$$
$$96Y_1 + 120Y_2 + 144Y_3 + 144Y_4 + 84Y_5 + 120Y_6 + 480\lambda \leq 1,$$
$$\beta \leq 25615Y_1 + 10004Y_2 + 13152Y_3 + 25555Y_4 + 17407Y_5 + 25420Y_6,$$
$$\beta \leq 121.8477Y_1 + 47.5876Y_2 + 62.5617Y_3 + 121.5629Y_4 + 82.8052Y_5 + 120.0492_6,$$
$$Y_i \geq 0, \ i = 1, ..., 6, \ \lambda, \ \beta \geq 0\}.$$

The (38) is solved and the solution $X^* = \frac{Y^*}{\lambda^*} = (0, 0, 0, 0, 0, 370)$ is obtained as an efficient solution for (37).
At the solution X^*:

$$Z_1(X) = 2.3380, \quad Z_2(X) = 489.9944, \quad \mu_{Z_1}(X) = 0.9999, \quad \mu_{Z_2}(X) = 0.9948.$$

The average of $\mu_{Z_1}(X)$ and $\mu_{Z_2}(X)$ is: 0.9948.

4.4.1 Comparison

The solution proposed by Pramy and Islam is: $\hat{X} = (0, 0, 0, 0, 196.078, 370.37)$.
At the solution $\hat{X}$:

$$Z_1(X) = 2.1288, \quad Z_2(X) = 488.531, \quad \mu_{Z_1}(X) = 0.9105, \quad \mu_{Z_2}(X) = 0.9887.$$

The average of $\mu_{Z_1}(X)$, $\mu_{Z_2}(X)$ is: 0.9496.
The results show that solution X^* proposed by this study dominates the solution $\hat{X}$ provided by Pramy and Islam due to the fact that:

$$Z_1(\hat{X}) < Z_1(X^*), \; Z_2(\hat{X}) < Z_2(X^*).$$

5   Conclusions

In this paper, an approach was proposed to address the linear fractional programming with fuzzy coefficients (FLFPP). In the method, the fuzzy problem was finally changed into a LPP. It was proven that the solution resulted by the LPP is an ϵ-optimal solution for the main problem. To construct our methodology, the concept of α-cuts, the membership function, max-min technique, and variable transformations were used. Although we only used triangular fuzzy numbers for convenience, this article covers the LFPPs with any kind of fuzzy numbers. Four numerical examples were solved in order to illustrate the method and comparisons were made to show the efficiency. For the first example, our outcome is an exact optimal solution. The second example was solved for $\alpha = 0.55$ and found that the solution proposed by this article dominates the outcome of Mehra et al. For example 3, the solution provided by this study also dominated the solution proposed by [17]. Since our proposed method can be used to address the bi-objective linear fractional programming problem, we considered example 4 taken from [29]. This bi objective linear fractional programming problem was solved and the results demonstrated that our solution dominated the solution of Pramy and Islam. In brief, we conclude that our proposed approach is reliable to address the LFPP with fuzzy coefficients, and bi-objective LFPP. It should be mentioned that, in this paper, the Linprog documentation of Optimization Toolbox of MATLAB R2016 was employed to solve the linear programming problems.

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An approach based on α-cuts and max-min technique to linear fractional programming with fuzzy coefficients

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رویکردی مبتنی بر تکنیک α-برش و Max-Min برای برنامه ریزی کسری خطی با ضرائب فازی

چکیده. این مقاله یک روش کارآمد و سرراست با پیچیدگی‌های محاسباتی کمتر برای پرداختن به برنامه‌ریزی کسری خطی با ضرائب فازی (FLFPP) ارائه می‌دهد. برای ساخت رویکرد، از مفهوم α-برش برای مقابله با اعداد فازی، علاوه بر رتبه‌بندی آنها استفاده می‌شود. بر این اساس، مسئله فازی با استفاده از حساب بازه‌ای به یک مسئله برنامه‌ریزی کسری خطی دو منظوره (BOLFPP) تبدیل می شود. پس از آن یک BOLFPP معادل برحسب توابع عضویت اهداف، تعریف می‌شود که با اعمال تبدیل‌های متغیر غیرخطی مناسب به یک مسئله برنامه‌ریزی خطی دو منظوره (BOLPP) تبدیل می شود. نظریه Max-Min برای تغییر BOLPP به یک مسئله برنامه‌ریزی خطی (LPP) مورد استفاده قرار می گیرد. ثابت شده است که راه حل بهینه LPP یک راه حل ϵ — بهینه برای مسئله فازی است. چهار مثال عددی برای نشان دادن روش، ارائه شده و برای نشان دادن کارایی مقایسه‌هایی صورت گرفته است.