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An identification model for a fuzzy time based stationary discrete process

G. Sirbiladze[1]

[1] *Department of Computer Sciences, Ivane Javakhishvili Tbilisi State University, University St. 13, Tbilisi 0186, Georgia*

gia.sirbiladze@tsu.ge

Abstract

A new approach of fuzzy processes, the source of which are expert knowledge reflections on the states on Stationary Discrete Extremal Fuzzy Dynamic System (SDEFDS) in extremal fuzzy time intervals, are considered. A fuzzy-integral representation of a stationary discrete extremal fuzzy process is given. A method and an algorithm for identifying the transition operator of SDEFDS are developed. The SDEFDS transition operator is restored by means of expert knowledge reflections on the states of SDEFDS. The regularization condition for obtaining of the quasi-optimal estimator of the transition operator is represented by the theorem. The corresponding calculating algorithm is provided. The results obtained are illustrated by an example in the case of a finite set of SDEFDS states.

Keywords: Sugeno's type extremal fuzzy measures and integrals, extremal fuzzy time intervals, SDEFDS, identification of the SDEFDS integral model.

1 Introduction

Very often we have no objective, experimental data for parameters and behavioral dynamics of evolutionary phenomena. In these cases, when studying processes' analysis and synthesis problems, the only expert knowledge appears to be applicable. With respect to time expert evaluations constitute expert knowledge streams (processes). The study of these processes is connected with problems of fuzzy-modelling and simulation.

In alternative classical approaches to modeling and when working with complex systems the main accent is placed on the assumption of fuzziness. As the complexity of systems increases, our ability to define exactly their behaviour drops to a certain level, below which such characteristics of information as exactness and uncertainty become mutually excluding. In such situations an exact quantitative analysis of real complex systems is apt to be not quite plausible. Hence, a conclusion comes to mind that problems of this kind should be solved by means of analytic-fundamental methods of fuzzy mathematics, while the system approach to constructing models of complex systems with fuzzy uncertainty guarantees the creation of computer-aided systems forming the instrumental basis of the intelligent technology solutions of expert-analytical problems. It is obvious that the source of fuzzy-statistical samples is the population of fuzzy characteristics of expert knowledge. Fuzziness arises from observations of time moments as well as from other expert measurements.

Applications of the dynamics of fuzzy systems and of the modeling of dynamic systems by fuzzy systems range from physics to biology, to economics, to pattern recognition and to time series prediction. Evidence exists that fuzzy models can explain cooperative processes, occurring in biology, chemistry, material sciences and economics. Relationships between dynamics of fuzzy systems and the performance of decision support systems were found, and chaotic processes in various classes of fuzzy systems were shown as a powerful tool in analyzing complex, weakly structurable systems.

In recent years, investigations of complex dynamic systems with fuzzy uncertainty by means of the theory of fuzzy sets have developed mainly in the following four directions:

I. Lower dynamic systems are described by composition equations in the metric or normed spaces of system states, which can be formally written in terms of fuzzy integral equations or some minmax compositions if a monotone measure is assumed to be a possibilistic one [5, 6, 21, 37, 49].

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II. Quite a number of studies have been devoted to the development of fuzzy differential calculus with an aim of describing dynamic fuzzy systems and their numerical solutions, modeling identification, filtering, control, optimal control, and so on. The main common feature of these approaches is the assumption that the compatibility function is differentiable [8, 10, 15, 22] and others, which to a certain extent facilitates the investigation of a definite class of dynamic fuzzy systems.

III. In some works, the first-order ordinary integral or integral-differential equations (or equation systems) have been constructed, where the constant values are uncertain, and this uncertainty is modeled by substituting fuzzy numbers. The problems of the existence and uniqueness of the solution with various derivatives of fuzzy functions have been studies in the literature [2–4, 33, 46, 51] and others.

IV. In modeling, analyzing, and predicting behaviors of physical and natural phenomena, greater and greater emphasis has been placed on fuzzy-stochastic equations. This is due to combinations of complexity, two kinds of uncertaintyrandomness and fuzziness, and ignorance that are present in the formulation of a great number of these problems. A large class of physically important problems is described by fuzzy-stochastic differential systems [13, 20, 31] and others.

The main difference between items I, II, III, IV (where in dynamic fuzzy systems, the dynamic structure is deterministic and fuzzy uncertainty appears only in the coefficients or initial conditions) and the fuzzy dynamic systems investigated in this work is that time structure and dynamics are fuzzy. However, our consideration is not limited to this class of dynamic systems. So, we construct fuzzy dynamic systems (EFDS), but not dynamic fuzzy systems. This is a new direction in studying and modeling weakly structured dynamic systems.

Fuzzy programming problems have been discussed widely in literature (see [7, 9, 11, 14, 19, 25, 28–30, 37, 41, 44, 47, 48, 50] and others) and applied in such various disciplines as operations research, economic management, business administration, engineering and so on. B. Liu [28] presents a brief review on fuzzy programming models, and classifies them into three broad classes: expected value models, chance-constrained programming and chance-dependent programming. Our further study belongs to the first class, where we use the instrument of fuzzy measures and integrals (see [42] and others) or, speaking more exactly, extremal fuzzy measures and Sugeno's type integrals along with extremal fuzzy expected value[34, 35].

Therefore, in the paper the new approach to the study of weakly structured dynamic systems optimization is presented (here only Stationary Discrete Extremal Fuzzy Dynamic System, where discreteness of SDEFDS is presented with respect to time). Different from other approaches where the source of fuzzy uncertainty in dynamic systems is expert, this approach considers time as long as an expert to be the source of fuzzy uncertainty. This notably widens the area of studied problems. All these is connected to the incomplete, imprecise, anomalous and extremal processes in nature and society, where connections between the system's objects are of subjective (expert) nature, which is caused by lack of objective information about the evolution of studying system, and which often is encountered, for example, in studies of such areas as: economics and business of developing countries, conflictology, sociology, medical diagnosis management of evacuation processes in catastrophe areas, estimation of disease spreading in epidemical regions research of complex systems of applied physics, etc. It is obvious, that in studies of abnormal and complex processes, when objective, statistical data is not sufficient (due to the complexity of process), construction of new technologies for expert knowledge streams engineering becomes highly important. Several researchers have been performed in this direction, which mostly concern construction of mathematical basis.

The dualized aggregation of knowledge streams with respect to fuzzy time intervals in current and future processes is a completely novel approach, since it has, as far as we know, no analogue. The dualized fuzzy models of Extremal Fuzzy Dynamic System (EFDS) were developed by introducing an algebraic structure for fuzzy time intervals. This is a natural way to introduce the expert knowledge of EFDS states into a fuzzy process, which activates an experts intelligence and represents it in pessimistic/optimistic estimates. The proposed new approach is based on a strong mechanism of expert knowledge activation, which makes it possible to evaluate the states of a complex dynamic system and to establish the form of their representation. For this, a minmax aggregation instrument such as Sugeno-type extremal integrals and extremal fuzzy measure theory are developed. For the creation of fuzzy processes with fuzzy time parameter, compositions of a monotone fuzzy time structure and Sugeno-type extremal integrals are constructed. This gives new possibilities for dualized integral representations of weakly structured processes. An important instance is the simplified parametrization of new integrals, which makes the minmax calculation of extremal (low & upper) Sugeno-type integrals much easier in considering process ergodicity, identification, optimization, filtration, and other problems.

This work presents a new approach to the study of EFDSs identification. In contrast to other approaches in which it is assumed that the source of fuzziness in dynamic systems is expert knowledge, in our approach both time and expert knowledge are considered to be factors that account for fuzzy uncertainty. The introduction of such a dualized (fuzzy time + expert knowledge) factors of uncertainty in dynamic systems not only enables experts to use their intellectual ability to the best advantage in the process of knowledge formalization, but also essentially widens the range of problems to be investigated.

Our attention was focused on the rapidly developing theory of fuzzy measures and integrals (see [1, 12, 16–18, 23, 24, 27, 32, 34, 38–40, 42] and others). We employ the part of the theory of fuzzy measures which concerns extremal fuzzy measures [34–36] and which, in our opinion, is rather seldom used. We have constructed a new instrument of a fuzzy measure, the extension of which is based on Sugeno lower and upper integrals. The structure of time is represented by monotone extremal classes of measurable sets. On such structures uncertainty is described by extremal fuzzy measures and problems of fuzzy optimization of extremal fuzzy processes: 1. Fuzzy Identification, 2. Fuzzy Optimal Control, 3. Fuzzy Filtration and so on. We will deal with the fuzzy identification problems SDEFDS, where fuzzy uncertainty arises with discrete time and time structures are monotone classes of measurable sets [34].

As known [34], in fuzzy dynamic processes where fuzziness participates as a time factor, an important role is assigned to the structures of extremal fuzzy time intervals $\{\widetilde{\mathcal{F}I}_*(T), \succeq \underset{*}{\otimes}\}\rangle)$, $(\langle\{\widetilde{\mathcal{F}I}^*(T), \preceq \overset{*}{\otimes}\}$. As the fuzzy time flows, the process of expert knowledge measurement on the system states with respect to time is affected by the incompleteness of the obtained information. The polar characteristics of this information manifest themselves as imprecision and uncertainty. A degree of information imprecision is defined by current fuzzy time moments ($\widetilde{t} \in \widetilde{\mathcal{B}}_1^*$) and future fuzzy time moments ($\widetilde{t} \in \widetilde{\mathcal{B}}_{1*}$), while an uncertainty degree is defined by current fuzzy time intervals ($\widetilde{r} \in \widetilde{\mathcal{B}}_2^*$) and future fuzzy time intervals ($\widetilde{r} \in \widetilde{\mathcal{B}}_{2*}$). We have constructed the corresponding fuzzy monotone structures [34]

$$\{\widetilde{\mathcal{F}I}_*(T), \succeq, \underset{*}{\oplus}\} \quad \text{and} \quad \{\widetilde{\mathcal{F}I}^*(T), \preceq, \overset{*}{\oplus}\},$$

in which sequential extremal fuzzy time intervals are calculated recurrently.

Here only note that when expert describes the dynamics of complex objects and “measures” system states in fuzzy time intervals. It is necessary to carry out “extremal” “dual” measurements, namely, measurements in extended current and compressed future fuzzy time intervals [34].

In the present paper, we represent the extremal fuzzy processes. The subject/matter of our investigation is the existence of an optimal estimation of the transition operator for SDEFDS’s. In Section 2, basic definitions on the extensions of Sugeno’s integral are presented. Section 3 contains some necessary preliminary concepts on general model of an extremal fuzzy process [36]. Based on the fuzzy integral model, Sections 4–5 deals with problems of fuzzy identification problem of SDEFDS. A method and an algorithm, as main results of the work, are developed for identification of the transition operator of SDEFDS. The results obtained are illustrated by the example for the case of a finite set of SDEFDS states.

2 Preliminary concepts: on the space of extended extremal fuzzy measures and extended Sugeno integral

All definitions and results see in [34].

Definition 2.1. *Let X be some nonempty set.*

a) We call some class $\mathcal{B}^* \subset 2^X$ of subsets of X an upper σ^*-monotone class if (i) $\varnothing, X \in \mathcal{B}^*$; (ii) $\forall A, B \in \mathcal{B}^* \Rightarrow A \cup B \in \mathcal{B}^*$; (iii) $\forall \{A_n\} \in \mathcal{B}^*$, $n = 1, 2, \dots,$ $A_n \uparrow A \Rightarrow A \in \mathcal{B}^*$.

b) We call some class $\mathcal{B}_* \subset 2^X$ of subsets of X a lower σ_*-monotone class if (i) $\varnothing, X \in \mathcal{B}_*$; (ii) $\forall A, B \in \mathcal{B}_* \Rightarrow A \cap B \in \mathcal{B}_*$; (iii) $\forall \{A_n\} \in \mathcal{B}_*$, $n = 1, 2, \dots,$ $A_n \downarrow A \Rightarrow A \in \mathcal{B}_*$.

Definition 2.2. *We call the classes $\mathcal{B}^*$ and $\mathcal{B}_*$ extremal if and only if*

$$\forall A \in \mathcal{B}^* \Leftrightarrow \overline{A} \in \mathcal{B}_*.$$

Definition 2.3. *1) $(X, \mathcal{B}^*)$ is called an upper measurable space;*

2) $(X, \mathcal{B}_*)$ is called a lower measurable space;

3) If $\mathcal{B}^*$ and $\mathcal{B}_*$ are extremal σ^*- and σ_*-monotone classes, then $(X, \mathcal{B}_*, \mathcal{B}^*)$ is called an extremal measurable space.

Let we have the following denotation: $\mathbb{R}_0^+ \equiv [0, +\infty]$.

Example 2.4.

$$\mathcal{B}_1^* \triangleq \left\{A \subset \mathbb{R}_0^+ \mid A=(\alpha;+\infty),\ \alpha \in \mathbb{R}_0^+\right\} \cup \{\varnothing\} \cup \{\mathbb{R}_0^+\} \ \textit{ is a } \sigma^*\textit{-monotone class,}$$
$$\mathcal{B}_{1*} \triangleq \left\{A \subset \mathbb{R}_0^+ \mid A=[0;\alpha],\ \ \alpha \in \mathbb{R}_0^+\right\} \cup \{\varnothing\} \cup \{\mathbb{R}_0^+\} \ \textit{ is a } \sigma_*\textit{-monotone class.}$$

$\mathcal{B}_1^*$ and $\mathcal{B}_{1*}$ are called a Borel σ^*-monotone class and a Borel σ_*-monotone class of first kind, respectively. Clearly, $\mathcal{B}_1^*$ and $\mathcal{B}_{1*}$ are extremal.

Example 2.5.

$$\mathcal{B}_2^* \triangleq \left\{A \subset \mathbb{R}_0^+ \mid A=[0;\alpha),\ \ \alpha \in \mathbb{R}_0^+\right\} \cup \{\varnothing\} \cup \{\mathbb{R}_0^+\} \ \textit{ is a } \sigma^*\textit{-monotone class,}$$
$$\mathcal{B}_{2*} \triangleq \left\{A \subset \mathbb{R}_0^+ \mid A=[\alpha;+\infty),\ \alpha \in \mathbb{R}_0^+\right\} \cup \{\varnothing\} \cup \{\mathbb{R}_0^+\} \ \textit{ is a } \sigma_*\textit{-monotone class.}$$

$\mathcal{B}_2^*$ and $\mathcal{B}_{2*}$ are called a Borel σ^*- and a Borel σ_*-monotone class of second kind, respectively. It is obvious that $\mathcal{B}_2^*$ and $\mathcal{B}_{2*}$ are extremal.

Definition 2.6. *Let $(X,\mathcal{B}^*)$ be some upper measurable space. A function $g^* : \mathcal{B}^* \to [0;1]$ is called an upper fuzzy measure if: (i) $g^*(\varnothing)=0$, $g^*(X)=1$; (ii) $\forall A,B \in \mathcal{B}^*$, $A \subset B \Rightarrow g^*(A) \le g^*(B)$; (iii) $\forall \{A_n\} \in \mathcal{B}^*$, $n=1,2,\dots$, $A_n \uparrow A \Rightarrow g^*(A) = \lim\limits_{n\to\infty} g^*(A_n)$.*

Definition 2.7. *Let $(X,\mathcal{B}_*)$ be some lower measurable space. A function $g_* : \mathcal{B}_* \to [0;1]$ is called a lower fuzzy measure if: (i) $g_*(\varnothing)=0$, $g_*(X)=1$; (ii) $\forall A,B \in \mathcal{B}_*$, $A \subset B \Rightarrow g_*(A) \le g_*(B)$; (iii) $\forall \{A_n\} \in \mathcal{B}_*$, $n=1,2,\dots$, $A_n \downarrow A \Rightarrow g_*(A) = \lim\limits_{n\to\infty} g_*(A_n)$.*

Definition 2.8. *Let $(X,\mathcal{B}_*,\mathcal{B}^*)$ be some extremal measurable space, g_* be a lower and g^* an upper fuzzy measure. Then:*

a) $g_* : \mathcal{B}_* \to [0;1]$ and $g^* : \mathcal{B}^* \to [0;1]$ is called extremal if and only if

$$\forall A \in \mathcal{B}_* \ :\ \ g_*(A) = 1 - g^*(\overline{A}).$$

b) $(X,\mathcal{B}_*,\mathcal{B}^*,g_*,g^*)$ is called a space of extremal fuzzy measures.

Definition 2.9. *Let $(X,\mathcal{B}_*,\mathcal{B}^*)$ be some extremal measurable space. Then:*

a) The function $h : X \to \mathbb{R}_0^*$ is called upper measurable if and only if h is measurable with respect to the spaces $(X,\mathcal{B}_*,\mathcal{B}^*)$ and $(\mathbb{R}_0^+,\mathcal{B}_{1*},\mathcal{B}_1^*)$. Then

$$\forall \alpha \ge 0 \quad h^{-1}\left((\alpha;+\infty)\right) \in \mathcal{B}^*, \quad h^{-1}\left([0;\alpha]\right) \in \mathcal{B}_*.$$

b) The function $h : X \to \mathbb{R}_0^+$ is called lower measurable if and only if h is measurable with respect to the spaces $(X,\mathcal{B}_*,\mathcal{B}^*)$ and $(\mathbb{R}_0^+,\mathcal{B}_{2*},\mathcal{B}_2^*)$. Then

$$\forall \alpha \ge 0 \quad h^{-1}\left([0;\alpha)\right) \in \mathcal{B}^*, \quad h^{-1}\left([\alpha;+\infty)\right) \in \mathcal{B}_*.$$

Definition 2.10. *Let $(X,\mathcal{B}_*,\mathcal{B}^*)$ be some extremal measurable space.*

a) The class of fuzzy subsets $\widetilde{A} \subset X$ with lower measurable compatibility functions

$$\widetilde{\mathcal{B}}_* = \left\{\widetilde{A} \subset X \mid \mu_{\widetilde{A}} \ \textit{ is lower measurable}\right\} = \left\{\widetilde{A} \in X \mid \forall\, 0 \le \alpha \le 1,\ \mu_{\widetilde{A}}^{-1}([0;\alpha)) \in \mathcal{B}^*,\ \mu_{\widetilde{A}}^{-1}([\alpha;+\infty)) \in \mathcal{B}^*\right\},$$

is called an extension of the σ_*-monotone class $\mathcal{B}_*$.

b) The class of fuzzy subsets $\widetilde{A} \subset X$ with upper measurable compatibility functions

$$\widetilde{\mathcal{B}}^* = \left\{\widetilde{A} \subset X \mid \mu_{\widetilde{A}} \ \textit{ is upper measurable}\right\} = \left\{\widetilde{A} \in X \mid \forall\, 0 \le \alpha \le 1,\ \mu_{\widetilde{A}}^{-1}([0;\alpha]) \in \mathcal{B}_*,\ \mu_{\widetilde{A}}^{-1}((\alpha;+\infty)) \in \mathcal{B}^*\right\},$$

is called an extension of the σ^*-monotone class $\mathcal{B}^*$.

Definition 2.11. *An extremal measurable space $(X,\widetilde{\mathcal{B}}_*,\widetilde{\mathcal{B}}^*)$ is called an extension of an extremal measurable space $(X,\mathcal{B}_*,\mathcal{B}^*)$.*

Using the Sugeno integral [42], we next introduce the notion of extension of fuzzy extremal measures.

Definition 2.12. *Let $(X, \mathcal{B}_*, \mathcal{B}^*, g_*, g^*)$ be some space of extremal fuzzy measures, and $(X, \widetilde{\mathcal{B}}_*, \widetilde{\mathcal{B}}^*)$ be an extension of the extremal measurable space $(X, \mathcal{B}_*, \mathcal{B}^*)$. Then:*

a) the function

$$\widetilde{g}_*(\widetilde{A}) \equiv \int_{*X} \mu_{\widetilde{A}}(x) \circ g_*(\cdot) \triangleq \bigvee_{0<\alpha\leq 1} \left[\alpha \wedge g_*([\widetilde{A}]_{\bar{\alpha}})\right], \quad \forall \widetilde{A} \in \widetilde{\mathcal{B}}_*; \tag{1}$$

is called an extension of the fuzzy measure g_* on $\widetilde{\mathcal{B}}_*$;

b) the function

$$\widetilde{g}^*(\widetilde{A}) \equiv \int_{X}^{*} \mu_{\widetilde{A}}(x) \circ g^*(\cdot) \triangleq \bigwedge_{0<\alpha\leq 1} \left[\alpha \vee g^*([\widetilde{A}]_{\alpha})\right], \quad \forall \widetilde{A} \in \widetilde{\mathcal{B}}^*, \tag{2}$$

is called an extension of the fuzzy measure g^* on $\widetilde{\mathcal{B}}^*$.

Here $[\widetilde{A}]_\alpha = \{x \in X \mid \mu_{\widetilde{A}}(x) > \alpha\}$, $[\widetilde{A}]_{\bar{\alpha}} = \{x \in X \mid \mu_{\widetilde{A}}(x) \geq \alpha\}$, $0 < \alpha \leq 1$.

Definition 2.13. *A space of extremal fuzzy measures $(X, \widetilde{\mathcal{B}}_*, \widetilde{\mathcal{B}}^*, \widetilde{g}_*, \widetilde{g}^*)$ is called an extension of the space $(X, \mathcal{B}_*, \mathcal{B}^*, g_*, g^*)$.*

Let $(X, \mathcal{B}_*, \mathcal{B}^*, g_*, g^*)$ be some space of extremal fuzzy measures and $(X, \widetilde{\mathcal{B}}_*, \widetilde{\mathcal{B}}^*, \widetilde{g}_*, \widetilde{g}^*)$ be its extension.

Definition 2.14. *a) Let $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{B}}_*$ be any fuzzy sets. Then the lower fuzzy Sugeno integral of the compatibility function $\mu_{\widetilde{B}}$ on the fuzzy set $\widetilde{A}$ is defined with respect to a lower fuzzy measure $\widetilde{g}_*$ by the formula*

$$\int_{*\widetilde{A}} \mu_{\widetilde{B}}(x) \circ \widetilde{g}_*(\cdot) \triangleq \bigvee_{0<\alpha\leq 1} \left[\alpha \wedge \widetilde{g}_*(\widetilde{A} \cap [\widetilde{B}]_{\bar{\alpha}})\right]. \tag{3}$$

b) Let $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{B}}^*$ be any fuzzy sets. Then the upper fuzzy Sugeno integral of the compatibility function $\mu_{\widetilde{B}}$ on the fuzzy set $\widetilde{A}$ is defined with respect to an upper fuzzy measure $\widetilde{g}^*$ by the formula

$$\int_{\widetilde{A}}^{*} \mu_{\widetilde{B}}(x) \circ \widetilde{g}^*(\cdot) \triangleq \bigwedge_{0<\alpha\leq 1} \left[\alpha \vee \widetilde{g}^*(\widetilde{A} \cup [\widetilde{B}]_{\alpha})\right]. \tag{4}$$

3 A general model of an extremal fuzzy dynamic system (EFDS)

As readers may note, the approach represented in this work describes complex evolutionary phenomena which are connected with process development in past, and not only in past (it may concern future time also), when an expert makes evaluations on system state within approximate time moments. These may be extremal and abnormal processes, catastrophes and, generally speaking, emergency situations environment. The expert makes evaluations on system state within approximate time moments which constitute expert knowledge stream. Following [36], in this section, for describing and modelling knowledge streams we construct fuzzy-dynamic system.

We propose the following: the time structure of fuzzy dynamic systems is represented by some space of extended extremal fuzzy measures

$$\langle T, \widetilde{\mathcal{F}I}_*(T), \widetilde{\mathcal{F}I}^*(T), \widetilde{g}_{T*}, \widetilde{g}_T^* \rangle, \quad T = \mathbb{R}_0^*, \tag{5}$$

where $\widetilde{\mathcal{F}I}_*(T)$ and $\widetilde{\mathcal{F}I}^*(T)$ denote monotone classes current and future fuzzy time intervals, respectively; $\widetilde{g}_{T*}$ and $\widetilde{g}_T^*$ are some extremal fuzzy measures on monotone structure $(\widetilde{\mathcal{F}I}_*(T), \widetilde{\mathcal{F}I}^*(T))$, respectively [34].

Let us start describing objects of a fuzzy dynamic system. Let X ($X \neq \varnothing$) be the set of states of some dynamic system to be investigated. Let $(X, \mathcal{B}, g)$ be the space of a fuzzy measure on the measurable space $(X, \mathcal{B})$, as an initial uncertainty measure on the states of EFDS, where $\mathcal{B}$ is a σ-algebra in X.

Let X also be the set of output states of the system under consideration.

Now, let us consider the Cartesian product $X \times T$ and the space of extended composition extremal fuzzy measures [35]

$$\left(X \times T, \widetilde{\mathcal{B} \otimes \mathcal{B}_{T*}}, \widetilde{\mathcal{B} \otimes \mathcal{B}_T^*}, \widetilde{g \otimes g_{T*}}, \widetilde{g \otimes g_T^*}\right),$$

which is induced by the spaces $(X, \mathcal{B}, \mathcal{B}, g, g)$ and $(T, \mathcal{B}_{T*}, \mathcal{B}_T^*, g_{T*}, g_T^*)$ [35], where

$$B_T^* = \{[0,t) \mid t \in T\} \quad \text{and} \quad B_{T*} = \{(t, +\infty) \mid t \in T\},$$

are monotone classes of time intervals [34].

Definition 3.1. [36] *a) A lower measurable binary fuzzy relation $\widetilde{Q}_* \in \widetilde{\mathcal{B} \otimes \mathcal{B}_{T*}}$ is called a future fuzzy process on the measurable states of the system (i.e., $\mu_{\widetilde{Q}_*}(x,t)$ is a lower measurable function).*

b) An upper measurable binary fuzzy relation $\widetilde{Q}^* \in \widetilde{\mathcal{B} \otimes \mathcal{B}^*_T}$ is called a current fuzzy process on the measurable states of the system (i.e., $\mu_{\widetilde{Q}^*}(x,t)$ is an upper measurable function).

c) A pair $(\widetilde{Q}_*, \widetilde{Q}^*)$ of lower and upper measurable binary fuzzy relations is called an extremal fuzzy process on the measurable states of the system (i.e., $Q^* \in \widetilde{\mathcal{B} \otimes \mathcal{B}^*_T}$ and $Q_* \in \widetilde{\mathcal{B} \otimes \mathcal{B}_{T*}}$).

d) An extremal fuzzy process (EFP) is said to be ergodic if there exist the limits $\forall x \in X$, $\lim\limits_{t\to\infty} \mu_{\widetilde{Q}^*}(x,t) \equiv \mu_{\widetilde{A}^*}(x)$, $\lim\limits_{t\to\infty} \mu_{\widetilde{Q}_*}(x,t) \equiv \mu_{\widetilde{A}_*}(x)$, and the limit fuzzy sets $\widetilde{A}^*$ and $\widetilde{A}_*$ are measurable $\widetilde{A}^*, \widetilde{A}_* \in \widetilde{\mathcal{B}}$.

Note that (see [34]) $\forall \tau \in T$, $\forall x \in X$ $E_{\widetilde{Q}_*}(x,\cdot) \in \widetilde{\mathcal{B}}_{T*}$ is a future fuzzy time interval; $E_{\widetilde{Q}^*}(x,\cdot) \in \widetilde{\mathcal{B}}^*_T$ is a current fuzzy time interval; $E_{\widetilde{Q}_*}(\cdot,\tau) \in \widetilde{\mathcal{B}}$ is a fuzzy state of the system, which is "measurable" in the future fuzzy time interval $\widetilde{[\tau,+\infty)}$; $E_{\widetilde{Q}^*}(\cdot,\tau) \in \widetilde{\mathcal{B}}$ is a fuzzy state of the system, which is "measurable" in the current fuzzy time interval $\widetilde{[0,\tau)}$.

It is obvious that model "measurements" of the states of the system at a real time moment $\tau > 0$ are understood as defining pairs of measurable fuzzy sets $E_{\widetilde{Q}_*}(\cdot,\tau)$, $E_{\widetilde{Q}^*}(\cdot,\tau) \in \widetilde{\mathcal{B}}$.

For all $x \in X$, $E_{\widetilde{Q}^*}(x,\cdot)$ and $E_{\widetilde{Q}_*}(x,\cdot)$ are a current fuzzy and a future fuzzy time intervals, respectively, in which the state $x \in X$ of the system is "measured" by the expert.

The family of fuzzy sets $\{E_{\widetilde{Q}_*}(\cdot,\tau)\}_{\tau\geq 0}$ from $\widetilde{\mathcal{B}}$ is called the trajectory of a future fuzzy process, and the family of fuzzy sets $\{E_{\widetilde{Q}^*}(\cdot,\tau)\}_{\tau\geq 0}$ from $\widetilde{\mathcal{B}}$ is called the trajectory of a current fuzzy process. The family of pairs of fuzzy sets $\{E_{\widetilde{Q}_*}(\cdot,\tau), E_{\widetilde{Q}^*}(\cdot,\tau)\}_{\tau\geq 0}$ is called the trajectory of an extremal fuzzy process $(\widetilde{Q}_*, \widetilde{Q}^*)$.

Let $\widetilde{\mathbb{R}}_* \subset X \times T \times X$ be some lower measurable fuzzy relation ($\widetilde{\mathbb{R}}_* \in \widetilde{\mathcal{B} \otimes \mathcal{B}_{T*}} \otimes \widetilde{\mathcal{B}_X}$) describing expert knowledge reflections of fuzzy states of the system on the output values of the system in future fuzzy time intervals, and $\widetilde{\mathbb{R}}^* \subset X \times T \times X$ be some upper measurable fuzzy relation ($\widetilde{\mathbb{R}}^* \in \widetilde{\mathcal{B} \otimes \mathcal{B}^*_T} \otimes \widetilde{\mathcal{B}_X}$) describing expert knowledge reflections of fuzzy states of the system on the output values of the system in current fuzzy time intervals.

Definition 3.2. [36] *a) A lower measurable relation $\widetilde{\mathbb{R}}_* \in \widetilde{\mathcal{B} \otimes \mathcal{B}_{T*}} \otimes \widetilde{\mathcal{B}_X}$ is called a future fuzzy process of expert knowledge reflection of states of the system in future fuzzy time intervals.*

b) An upper measurable relation ($\widetilde{\mathbb{R}}^* \in \widetilde{\mathcal{B} \otimes \mathcal{B}^*_T} \otimes \widetilde{\mathcal{B}_X}$) is called a current fuzzy process of expert knowledge reflection of states of the system in current fuzzy time intervals.

c) A pair $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$ is called an extremal fuzzy process of expert knowledge reflection of states of the system in extremal fuzzy time intervals.

Let $\widetilde{\rho}_* \in (\widetilde{\mathcal{B} \otimes \mathcal{B}_{T*}}) \otimes (\widetilde{\mathcal{B} \otimes \mathcal{B}_{T*}})$ be some lower measurable fuzzy relation in the Cartesian product $(X \times T) \times (X \times T)$. This relation is a future fuzzy transition operator describing the dynamics of the system or, in other words, system state transformations in future fuzzy time intervals.

Let $\widetilde{\rho}^* \in (\widetilde{\mathcal{B} \otimes \mathcal{B}^*_T}) \otimes (\widetilde{\mathcal{B} \otimes \mathcal{B}_{T*}})$ be some upper measurable fuzzy relation in the Cartesian product $(X \times T) \times (X \times T)$. This relation is a current fuzzy transition operator describing the dynamics of the system or, in other words, system state transformations in current fuzzy time intervals.

We call $\widetilde{\rho}_*$ the fuzzy lower transition operator describing the system state dynamics, and $\widetilde{\rho}^*$ the fuzzy upper transition operator describing the system state dynamics. The pair $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ is called the transition operator describing the system state dynamics in extremal fuzzy time intervals.

Definition 3.3. [36] *a) The train*

$$\left\{X, T, \widetilde{\rho}_*, \widetilde{Q}_*, \widetilde{\mathbb{R}}_*\right\}, \tag{6}$$

is called the future fuzzy dynamic system describing the dynamics of the system state in future fuzzy time intervals.

b) The train

$$\left\{X, T, \widetilde{\rho}^*, \widetilde{Q}^*, \widetilde{\mathbb{R}}^*\right\}, \tag{7}$$

is called the current fuzzy dynamic system describing the state dynamics of the system in current fuzzy time intervals.

c) The train

$$\left\{X, T, (\widetilde{\rho}_*, \widetilde{\rho}^*), (\widetilde{Q}_*, \widetilde{Q}^*), (\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)\right\}, \tag{8}$$

is called the extremal fuzzy dynamic system (EFDS) describing the state dynamics of the system in extremal fuzzy time intervals.

It is obvious that the EFDS (8) describes the state dynamics of the system undergoing transformation with fuzzy uncertainty produced by observations at fuzzy time, while the extremality is due to the "measurement" of fuzzy states of the system in current and future fuzzy time intervals.

Definition 3.4. [36] *The system of composition equations*

$$\begin{cases} \widetilde{\mathbb{R}}_* = \widetilde{\rho}_* \underset{*}{\bullet} \widetilde{Q}_*, \\ \widetilde{\mathbb{R}}^* = \widetilde{\rho}^* \overset{*}{\bullet} \widetilde{Q}^* \end{cases} \tag{9}$$

is called the system describing the state dynamics of the extremal dynamic system, where $\underset{*}{\bullet}$ *and* $\overset{*}{\bullet}$ *are some composition operations over fuzzy relations.*

Given $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$, $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ and the initial fuzzy states of the system $\widetilde{A}_{0*}, \widetilde{A}_0^* \in \mathcal{B}$ $(\mu_{\widetilde{A}_{0*}}(x) = \mu_{\widetilde{Q}_*}(x,0),\ \mu_{\widetilde{A}_0^*}(x) = \mu_{\widetilde{Q}^*}(x,0),\ \forall x \in X)$, it is important to find a solution $(\widetilde{Q}_*, \widetilde{Q}^*)$ of (9), which we call an extremal fuzzy process of system state transformation on measurable states of the system.

Below we consider concrete fuzzy systems of form (9) for the discrete case with respect to time. The finding of a system state modeling process $(\widetilde{Q}_*, \widetilde{Q}^*)$ or a transition operator $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ is important when we deal with problems pertaining to optimization problem (here the problems of identifications).

4 Discrete extremal fuzzy process

Let an extremal fuzzy time process of "observation" of system changes be discrete., i.e., the sequence of extremal fuzzy time intervals be given recurrently. This means that from the monotone structures $\{\widetilde{\mathcal{F}I}_*(T), \succeq, \underset{*}{\oplus}\}$ and $\{\widetilde{\mathcal{F}I}^*(T), \preceq, \overset{*}{\oplus}\}$ [35] we choose a monotonically decreasing sequence of future fuzzy time intervals

$$\widetilde{r}_{\tau_{i+1}*} = \widetilde{r}_{\tau_i *} \underset{*}{\oplus} \widetilde{\Delta r}_{\tau_i *}, \quad i = 0, 1, \dots, \tag{10a}$$

and a monotonically increasing sequence of current fuzzy time intervals

$$\widetilde{r}^*_{\tau_{i+1}} = \widetilde{r}^*_{\tau_i} \overset{*}{\oplus} \widetilde{\Delta r}^*_{\tau_i}, \quad i = 0, 1, \dots. \tag{10b}$$

This can be interpreted as follows: while carrying out observation of the system, at the time moment τ_{i+1} the extremal fuzzy time intervals $\widetilde{r}_{\tau_{i+1}*}$ and $\widetilde{r}^*_{\tau_{i+1}}$ are defined recurrently, where $\widetilde{\Delta r}_{\tau_i *}$ is the future fuzzy time interval describing the discreteness of observation of the system at the time moment τ_i, and $\widetilde{\Delta r}_{\tau_i^*}$ is the current fuzzy time interval describing the discreteness of observation of the system at the time moment τ_i. It can be assumed that

$$T_{\mathbb{D}} \triangleq \{\tau_0, \tau_1, \dots\}, \tag{11}$$

is a monotone sequence of time moments $(\tau_0 = 0)$, at which the observation of the system takes place in the course of extremal fuzzy time intervals. It is obvious that $\{\widetilde{r}_{\tau_i *}\} \downarrow$, $\{\widetilde{r}^*_{\tau_i}\} \uparrow$.

Definition 4.1. [36] *a) A sequence* $(\widetilde{r}_{\tau_i *}, \widetilde{r}^*_{\tau_i})_{i \in Z_0}$ *of extremal fuzzy time intervals, which is defined recurrently by formulas* (10)–(11)*, is called a discrete process of extremal fuzzy time intervals.*

b) A sequence

$$(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)_{\mathbb{D}} \triangleq (\widetilde{A}_{\tau_{i*}}, \widetilde{A}^*_{\tau_i})_{i \in Z_0},$$

is called a discrete process of reflection of the process of extremal fuzzy time intervals $(\widetilde{r}_{\tau_i *}, \widetilde{r}^*_{\tau_i})_{i \in Z_0^+}$ *on the measurable space* $(X, \widetilde{\mathcal{B}})$ *of states of the system.*

c) A sequence

$$(\widetilde{Q}_*, \widetilde{Q}^*)_{\mathbb{D}} \triangleq \langle E_{\widetilde{Q}_*}(\cdot, \tau_i), E_{\widetilde{Q}^*}(\cdot, \tau_i) \rangle_{i \in Z_0^+},$$

where $\forall x \in X$

$$\begin{cases} \mu_{E_{\widetilde{Q}_*}(\cdot,\tau_i)}(x) = \displaystyle\fint_{T}{}_{*} \mu_{\widetilde{\rho}'_*}(x,t) \circ \widetilde{g}_{\widetilde{A}_{\tau_i *}}(\cdot) = \fint_{T}{}_{*} \mu_{\widetilde{\rho}'_*}(x,t) \circ \widetilde{g}_{E_{\widetilde{\mathbb{R}}_*}(\cdot,\tau_i)}(\cdot), \\ \mu_{E_{\widetilde{Q}^*}(\cdot,\tau_i)}(x) = \displaystyle\fint_{T}^{*} \mu_{\widetilde{\rho}'^*}(x,t) \circ \widetilde{g}_{\widetilde{A}^*_{\tau_i}}(\cdot) = \fint_{T}^{*} \mu_{\widetilde{\rho}'^*}(x,t) \circ \widetilde{g}_{E_{\widetilde{\mathbb{R}}^*}(\cdot,\tau_i)}(\cdot) \end{cases} \tag{12}$$

is called a discrete fuzzy process describing the system state dynamics in the process of extremal fuzzy time intervals $(\widetilde{r}_{\tau_i *}, \widetilde{r}^*_{\tau_i})_{i\in Z_0^+}$, *where* $\displaystyle\fint_{T}{}_{*}$ *and* $\displaystyle\fint_{T}^{*}$ *denotes Sugeno's extended lower and upper integrals, respectively [34];* $\widetilde{g}_{\widetilde{A}_{\tau_i *}}(\cdot)$ *and* $\widetilde{g}_{\widetilde{A}^*_{\tau_i}}(\cdot)$ *are extended fuzzy measures induced by* $\widetilde{A}_{\tau_i *}$ *and* $\widetilde{A}^*_{\tau_i}$*;* $(\mu_{\widetilde{\rho}'_*}, \mu_{\widetilde{\rho}'^*})$ *is a pair of operators describing the change dynamics of system states at the extremal fuzzy time intervals.*

Here Z_0^+ denotes the set of nonnegative integer numbers.

Theorem 4.2. [36] *The discrete process* $(\widetilde{Q}_*, \widetilde{Q}^*)_{\mathbb{D}}$ *describing the system state change dynamics within* $T_{\mathbb{D}}$ *is defined by the following system of fuzzy integral equations:* $\forall x \in X$

$$\begin{cases} \mu_{E_{\widetilde{Q}_*}(\cdot,\tau_{i+1})}(x) = \displaystyle\fint_{X} \mu_{\widetilde{\rho}_{i*}}(x,x') \circ \widetilde{g}_{E_{\widetilde{Q}_*}(\cdot,\tau_i)}(\cdot), \\ \mu_{E_{\widetilde{Q}^*}(\cdot,\tau_{i+1})}(x) = \displaystyle\fint_{X} \mu_{\widetilde{\rho}^*_i}(x,x') \circ \widetilde{g}_{E_{\widetilde{Q}^*}(\cdot,\tau_i)}(\cdot), \quad i = 0,1,2,\dots, \end{cases} \tag{13}$$

where $E_{\widetilde{Q}_*}(\cdot,\tau_0) \equiv \widetilde{A}_{\tau_0 *} \in \widetilde{\mathcal{B}}$ *and* $E_{\widetilde{Q}^*}(\cdot,\tau_0) \equiv \widetilde{A}^*_{\tau_0} \in \widetilde{\mathcal{B}}$ *are the initial extremal fuzzy states of the system;* $(\widetilde{\rho}_{i*}, \widetilde{\rho}^*_i)$ *is the operator describing the change dynamics of system states at the time moment* τ_i*, while the extended extremal fuzzy measures in* (13) *are induced by extremal fuzzy states* $\langle E_{\widetilde{Q}_*}(\cdot,\tau_i), E_{\widetilde{Q}^*}(\cdot,\tau_i)\rangle$ *of the system.*

Definition 4.3. [36] *A discrete extremal fuzzy process* $\langle E_{\widetilde{Q}_*}(\cdot,\tau_i), E_{\widetilde{Q}^*}(\cdot,\tau_i)\rangle_{i\in Z_0^+}$ *of describing the system state dynamics is called stationary if* $\forall i \in Z_0^+$

$$\widetilde{\rho}_{i*} \equiv \widetilde{\rho}_* \in \widetilde{\mathcal{B}\otimes\mathcal{B}}, \quad \widetilde{\rho}^*_i \equiv \widetilde{\rho}^* \in \widetilde{\mathcal{B}\otimes\mathcal{B}};$$

i.e., an operator describing the system state dynamics does not influence the transformation step.

Now (13) takes the following form: $\forall x \in X$

$$\begin{cases} \mu_{E_{\widetilde{Q}_*}(\cdot,\tau_{i+1})}(x) = \displaystyle\fint_{X} \mu_{\widetilde{\rho}_*}(x,x') \circ \widetilde{g}_{E_{\widetilde{Q}_*}(\cdot,\tau_i)}(\cdot), \\ \mu_{E_{\widetilde{Q}^*}(\cdot,\tau_{i+1})}(x) = \displaystyle\fint_{X} \mu_{\widetilde{\rho}^*}(x,x') \circ \widetilde{g}_{E_{\widetilde{Q}^*}(\cdot,\tau_i)}(\cdot), \end{cases} \tag{14}$$

with initial extremal fuzzy states $\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_0) \in \widetilde{\mathcal{B}}$, $\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_0) \in \widetilde{\mathcal{B}}$.

The problems of ergodicity of SDEFDS are studied and omitted here.

5 Problem of identification of SDEFDS

5.1 Introduction

In this paper we consider some problems of the identification of the model of stationary discrete extremal fuzzy dynamic system (SDEFDS) that we have constructed in Section 4. Such choice is caused by concrete, application purposes, to enable the reader to use constructed model in his theoretical or applied problems for dealing with time series with fuzzy uncertainty.

The basic approaches to the identification of fuzzy process models that have been developed to this day (see [9, 11, 19, 26, 43, 47, 50] and other works) can be divided into two groups – analytical and algorithmic – both of which are oriented to a fuzzy process model written in terms of fuzzy compositional or integro-differential equations or their modifications. Various analytical methods and algorithms were used in order to identify such models, i.e., the corresponding relation of spaces of inputs and outputs of fuzzy dynamic systems. These methods mainly imply the construction of some set-theoretic operation that is inverse to the composition operation and requires a subsequent smoothing of the results [9, 11, 26, 50] and others. In some works [43, 45] and so on, fuzzy models of regression type were identified by means of analytical regularization methods that allowed one to obtain numerical estimators of model coefficients. In this paper, a new approach is proposed to the identification of SDEFDS models.

In Section 4, we have developed a new approach of SDEFDS modeling. The identification of a fuzzy-integral model of stationary discrete extremal fuzzy process is an important task from the standpoint of practical application of the SDEFDS in studying complex dynamic systems. The practice of using fuzzy-integral equations describing complex dynamic systems still lacks methods of the identification of such models. Hence in this paper we develop a new approach to the identification of a model of SDEFDS (formula (14)).

The identification of a fuzzy-integral model of SDEFDS in the form of the system (14) implies the estimation of the transition operator $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ by means of data on the realized transition N pairs so as to obtain "an expert reflection" for which some regularization condition is fulfilled; for example:

$$\bigvee_{i=1}^{N}\left\{d\left(\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_{i+1}),\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})\right)\wedge d\left(\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_{i+1}),\mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_{i+1})\right)\right\}\Rightarrow \min, \tag{15}$$

where d is some distance in the space of fuzzy sets $\widetilde{\mathcal{B}}$; $(\widehat{\widetilde{Q}}_*, \widehat{\widetilde{Q}}^*)$ is the true process of extremal fuzzy output states knowledge (expert reflections on the state) of the SDEFDS in extremal fuzzy time intervals $(\widetilde{r}_{\tau_i *}, \widetilde{r}^*_{\tau_i})$, $i = 1, \ldots, N$. The restoration of the transition operator of the system (14) by means of expert data on transition pairs

$$\langle \mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_i), \mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_i)\rangle, \quad i = 1, 2, \ldots, N,$$

is a nonregular problem for the SDEFDS model (14), since we may obtain a solution which is not unique. Hence we need to introduce solution regularization conditions (like, for example, (15)). Otherwise, one would be unable to obtain a quasi-optimal (in a certain sense) solution of the identification problem.

5.2 An identification method and algorithm for the integral model of stationary discrete extremal fuzzy process

Let a model of a stationary discrete EFP be described by the system of fuzzy-integral equations (14) with unknown pair of fuzzy relations $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ consisting of the transition operator of a SDEFDS at one transformation step with input data $\langle \mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_j), \mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_j)\rangle$ and output data $\langle \mathbb{E}_{\widehat{\widetilde{Q}}_*})\cdot,\tau_{j+1}), \mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_{j+1})\rangle$. At the j-th step of the EFP modeling $(j = 1, 2, \ldots, i)$ any pair is the SDEFDS input, while at the next modeling step the same pair is the SDEFDS output.

Our approach to the identification of the transition operator of a discrete SDEFDS by a priori measurents of fuzzy states

$$\langle \mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_j), \mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_j)\rangle, \quad \tau_1 < \tau_2 < \cdots < \tau_i, \tag{16}$$

in the extremal fuzzy time intervals

$$(\widetilde{r}_{\tau_j *}, \widetilde{r}_{\tau_j^*}), \quad j = 1, 2, \ldots, i,$$

is based on the following arguments. Using the definition of an extended extremal Sugeno fuzzy-integral [34], where $\mathcal{B}_* = \mathcal{B}^* = \mathcal{B}$ and $(g)^* = g^*$ (or dual measure is extremal measure) and assuming that $x \in X$ is fixed, the fuzzy-integral model (14) of a stationary discrete EFP can be written in the form

$$\begin{cases} \mu_{\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_{i+1})}(x) = \bigvee\limits_{0<\alpha\leq 1}\left\{\alpha \wedge g_{\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_i)}\left([\mathbb{E}_{\widetilde{\rho}_*}(x,\cdot)]_{\overline{\alpha}}\right)\right\} \equiv \alpha_{ix}, \\ \mu_{\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_{i+1})}(x) = \bigwedge\limits_{0\leq\alpha< 1}\left\{\alpha \vee g_{\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_i)}\left([\mathbb{E}_{\widetilde{\rho}^*}(x,\cdot)]_{\alpha}\right)\right\} \equiv \alpha^*_{ix}. \end{cases} \tag{17}$$

Then

$$\begin{cases} g_{\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_i)}\left([\mathbb{E}_{\widetilde{\rho}_*}(x,\cdot)]_{\overline{\alpha}_{ix}}\right) \geq \alpha_{ix}, \\ g_{\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_i)}\left([\mathbb{E}_{\widetilde{\rho}^*}(x,\cdot)]_{\alpha^*_{ix}}\right) \leq \alpha^*_{ix} \end{cases}$$

or

$$\begin{cases} g\left([\mathbb{E}_{\widetilde{\rho}_*}(x,\cdot)]_{\overline{\alpha}_{ix}} \cap [\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_i)]_{\overline{\alpha}_{ix}}\right) \geq \alpha_{ix}, \\ g^*\left([\mathbb{E}_{\widetilde{\rho}^*}(x,\cdot)]_{\alpha^*_{ix}} \cup [\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_i)]_{\alpha^*_{ix}}\right) \leq \alpha^*_{ix}. \end{cases} \tag{18}$$

Remark 5.1. *Equalities in (18) are reached when the fuzzy measure functions*

$$g_{\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_i)}([\mathbb{E}_{\widetilde{\rho}_*}(x,\cdot)]_{\overline{\alpha}}) \equiv f_*(\alpha) \;\; \textit{and} \;\; g^*_{\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_i)}([\mathbb{E}_{\widetilde{\rho}^*}(x,\cdot)]_{\alpha}) \equiv f^*(\alpha)$$

are continuous. The main problem is in the fail to reach equalities in (18) *in discrete case.*

This remark shows that the identification of the transition operator $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ in the fuzzy-integral system (14) of SDEFDS with input-output data (16) is a nonregular problem that must be first regularized in order to find some quasi-optimal solution. We propose a variant of the solution of the identification problem that will be obtained by the aggregation of α-level solutions for each input-output pair constructed by the extremalization of the initial fuzzy measure functions from (18), i.e., for fixed $x \in X$ we have

$$\begin{cases} g\big(\big[\mathbb{E}_{\widehat{\widetilde{\rho}}_*}(x,\cdot) \cap \mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_i)\big]_{\overline{\alpha}_{ix}}\big) \xrightarrow[\widehat{\widetilde{\rho}}_*]{\min}, \\ g^*\big(\big[\mathbb{E}_{\widehat{\widetilde{\rho}}^*}(x,\cdot) \cup \mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_i)\big]_{\alpha^*_{ix}}\big) \xrightarrow[\widehat{\widetilde{\rho}}^*]{\max}. \end{cases} \tag{19}$$

Let us introduce some notations. For fixed $x \in X$, we transform the a priori true transition data (16) on fuzzy state of an SDEFDS into the variational series:

$$\begin{pmatrix} \mu_{1*} & > & \mu_{2*} & > & \cdots & > & \mu_{l*} \\ n_{*1} & & n_{*2} & & \cdots & & n_{*l_*} \end{pmatrix}, \tag{20a}$$

where

$$\mu_{*l} \equiv \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{k_l})}(x), \quad l = 1,\dots,l_*, \quad \sum_{l=1}^{l_*} n_{*l} = i,$$

and

$$\begin{pmatrix} \mu^*_1 & < & \mu^*_2 & < & \cdots & < & \mu^*_l \\ n^*_1 & & n^*_2 & & \cdots & & n^*_l \end{pmatrix}, \tag{20b}$$

where

$$\mu^*_l \equiv \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_{m_l})}(x), \quad l = 1,\dots,l^*, \quad \sum_{l=1}^{l^*} n^*_l = i.$$

We construct the following measurable sets of levels μ_{*l} and μ^*_l from $\mathcal{B}$ for input fuzzy states:

$$M^i_{*li_s} \triangleq \big\{x' \in X \mid \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i_s})}(x') \geq \mu_{*l}\big\} \in \mathcal{B}, \tag{21a}$$

if $\mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i_s+1})}(x) = \mu_{*l}$ and $s = 1,\dots,n_{*l_*}$, $l = 1,\dots,l_*$;

$$M^{*i}_{li_q} \triangleq \big\{x' \in X \mid \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_{i_q})}(x') \leq \mu^*_l\big\} \in \mathcal{B}, \tag{21b}$$

if $\mu_{\mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_{i_q+1})}(x) = \mu^*_l$ and $q = 1,\dots,n^*_{l_*}$, $l = 1,\dots,l^*$;

$$\begin{gathered} \Pi^i_{*li_s} \triangleq \big\{M'_{*li_s} \in \mathcal{B} \mid M'_{*li_s} \subset M^i_{*li_s}\big\}, \\ \Pi^{*i}_{li_q} \triangleq \big\{M^{*\prime}_{li_q} \in \mathcal{B} \mid M^{*\prime}_{li_q} \supset M^{*i}_{li_q}.\big\}. \end{gathered} \tag{22}$$

Next, we construct the Cartesian products

$$\Pi^i_{*l} = \prod_{s=1}^{n_{*l}} \times \big(\Pi^i_{*li_s}\big), \quad \Pi^{*i}_{l'} = \prod_{q=1}^{n^*_l} \times \big(\Pi^{*i}_{l'i_q}\big), \tag{23}$$

where $l = 1,\dots,l_*$, $l' = 1,\dots,l^*$.

To find a quasi-optimal solution of the problem of identification of model (14), (16) with the regularization conditions (19), we use the following theorem.

Theorem 5.2. *A quasi-optimal solution of the problem of identification of model* (14), (16) *of SDEFDS with the regularization conditions* (19) *at the* $(i+1)$*-th modeling step is defined by the recurrent equations: for fixed* $x \in X$ *and* $\forall x' \in X$ *we have*

$$\begin{cases} \mu_{\widetilde{\rho}_{*i+1}}(x,x') = \big[\mu_{\widetilde{\rho}_{*i}}(x,x') \vee \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big] \wedge \beta_{*i}(x'), \\ \mu_{\widetilde{\rho}^*_{i+1}}(x,x') = \big[\mu_{\widetilde{\rho}^*_i}(x,x') \wedge \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_{i+1})}(x)\big] \vee \beta^*_i(x'), \end{cases} \tag{24}$$

where expert data $\langle \mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1}), \mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_{i+1})\rangle$ *is some possible extremal fuzzy state of the SDEFDS at the* $(i+1)$*-th modeling step and* $\beta_{*i}(x') \equiv \alpha^i_{*l_*}(x')$, $\beta^*_i(x') \equiv \alpha^{*i}_{l^*}(x')$; α^i_{*l} *and* α^{*i}_l *are* $\mathcal{B}$*-measurable functions which are defined by recursion up to* i: $\forall x' \in X$

$$\begin{cases} \alpha^i_{*l+1}(x') = \alpha^i_{*l}(x') \vee \big[\mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{l+1})}(x') \wedge \mathbb{I}_{A^i_{*l+1}}(x')\big], \;\; l=0,\dots,l_*-1, \\ \alpha^{*i}_{l+1}(x') = \alpha^{*i}_l(x') \wedge \big[\mu_{\mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_{l+1})}(x') \vee \mathbb{I}_{A^{*i}_{l+1}}(x')\big], \;\; l=0,\dots,l^*-1, \end{cases} \tag{25}$$

where $\alpha_{*0}(x') \equiv 0$, $\alpha^*_0(x') \equiv 0$. *The measurable sets* $A^i_{*l} \in \mathcal{B}$ *and* $A^{*i}_l \in \mathcal{B}$ *are defined as follows:*

$$\begin{cases} A^i_{*l+1} = A^i_{*l} \cup \Bigg[\bigcup\limits_{\substack{s=1 \\ \langle \mathring{M}'_{*li_s} \in \Pi^i_{*li_s}\rangle}}^{n_{*l_*}} \mathring{M}'_{*li_s}\Bigg], \\ A^{*i}_{l+1} = A^{*i}_l \cap \Bigg[\bigcap\limits_{\substack{q=1 \\ \langle \mathring{M}^{*\prime}_{l'i_s} \in \Pi^{*i}_{l'i_s}\rangle}}^{n^*_{l_*}} \mathring{M}^{*\prime}_{l'i_s}\Bigg], \end{cases} \tag{26}$$

where for each l $l=0,1,\dots,l_*-1$, $l'=0,1,\dots,l^*-1$; $A^i_{*0} = A^{*i}_0 = \varnothing$ *and* $\langle \mathring{M}'_{*li_1},\dots,\mathring{M}'_{*li_{l_*}}\rangle \in \Pi^i_{*l}$, $\langle \mathring{M}^{*\prime}_{li_1},\dots,\mathring{M}^{*\prime}_{li_{l^*}}\rangle \in \Pi^{*i}_l$ *are solutions of the conditional discrete bicriterial optimization problem:*

$$\begin{cases} \bigwedge\limits_{\langle M'_{*li_1},\dots,M'_{*li_{l_*}}\rangle \in \Pi^i_{*l}} g\Bigg([\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\mu_{*l}} \cap \Bigg[A^i_{*l} \cup \Bigg(\bigcup\limits_{s=1}^{n_{*l_*}} M'_{*li_s}\Bigg)\Bigg]\Bigg) \Rightarrow \min, \\ \bigvee\limits_{\langle M^{*\prime}_{li_1},\dots,M^{*\prime}_{li_{l_*}}\rangle \in \Pi^{*i}_l} g^*\Bigg([\mathbb{E}_{\widetilde{\rho}^*_i}(x,\cdot)]_{\mu^*_l} \cup \Bigg[A^{*i}_l \cap \Bigg(\bigcap\limits_{q=1}^{n^*_{l_*}} M^{*\prime}_{li_q}\Bigg)\Bigg]\Bigg) \Rightarrow \max, \\ g(M'_{*li_s}) \geq \mu_{*l}, \quad s=1,\dots,n_{*l_*}, \\ g^*(M^{*\prime}_{li_q}) \leq \mu^*_l, \quad q=1,\dots,n^*_{l_*}. \end{cases} \tag{27}$$

Proof. The proof consists of two identical parts for the upper and the lower extremal process. We will prove the latter case.

Using the lower measurable "input-output" operator $\mu_{\widetilde{\rho}_{*i}}(x,x') \equiv \mu_{\mathbb{E}_{\widetilde{\rho}_{i*}}(x,\cdot)}(x')$, $x,x' \in X$, at the i-th step we define the fuzzy-integral model of the SDEFDS for the fixed cutting of the compatibility function. If at the $(i+1)$-th step we realize some possible measurable fuzzy state $\mathbb{E}_{\widehat{\widetilde{Q}}_*(\cdot,\tau_{i+1})}(x)$ at the output, then, according to (19), the change of the estimate $\mu_{\widetilde{\rho}_{*i}}(x,x')$ can define only those α-cuttings for which $\alpha \leq \mu_{\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_{i+1})}(x)$. Therefore, the problem consists in finding a $\mathcal{B}$-measurable function $\beta_{*i} : X \to [0,1]$ which changes the estimate $\mu_{\widetilde{\rho}_{*i}}(x,x')$ for such $x' \in X$ that $\mu_{\widetilde{\rho}_{*i}}(x,x') \leq \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)$. The estimate change condition can be written in the following form: $\forall x' \in X$

$$\mu_{\mathbb{E}_{\widetilde{\rho}_{*i+1}}(x,\cdot)}(x') = \big\{\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \wedge \mathbb{I}_{[\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\alpha_i x}}(x')\big\} \vee \beta'_{*i}(x'). \tag{28}$$

Since all $x' \in X$, for which

$$\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \leq \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x),$$

are subject to changing, the function β'_{*i} can be represented as follows: $\forall x' \in X$

$$\beta'_{*i}(x') = \big[\mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x) \vee \mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x')\big] \wedge \big[\gamma'_{*i}(x') \wedge \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big], \tag{29}$$

where

$$\big\{x' \in X \mid \mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \geq \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big\} = \big\{x' \in X \mid \gamma_{*i}(x') \geq \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big\}, \tag{30}$$

and γ_{*i} is a $\mathcal{B}$-measurable function.

Substituting (29) into (28), we obtain

$$\mu_{\mathbb{E}_{\widetilde{\rho}_{*i+1}}(x,\cdot)}(x') = \big[\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \vee \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big] \wedge \big\{\big[\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \wedge \mathbb{I}_{[\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\alpha_i x}}(x')\big] \vee \big[\gamma_{*i}(x') \wedge \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big]\big\}$$

$$
\begin{aligned}
&= \big\{ \big[\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \vee \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big] \wedge \big[\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \vee \mathbb{I}_{[\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\alpha_{ix}}}(x')\big]\big\} \\
&\quad \vee \big\{ \big[\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \vee \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big] \wedge \big[\gamma_{*i}(x') \wedge \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big]\big\} \\
&= \big[\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \vee \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big] \wedge \big\{ \big[\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \vee \mathbb{I}_{[\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\alpha_{ix}}}(x')\big] \vee \gamma_{*i}(x')\big\} \\
&= \big[\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \vee \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big] \wedge \big\{ \big[\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x') \vee \gamma_{*i}(x')\big] \wedge \big[\mathbb{I}_{[\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\alpha_{ix}}}(x') \vee \gamma_{*i}(x')\big]\big\} \\
&= \big[\beta_{*i}(x') \vee \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)\big] \wedge \big[\mathbb{I}_{[\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\alpha_{ix}}}(x') \vee \gamma_{*i}(x')\big].
\end{aligned}
$$

Comparing this result with (24) and assuming that the sought function is

$$\beta_{*i}(x') = \mathbb{I}_{[\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\alpha_{ix}}}(x') \vee \gamma_{*i}(x'), \tag{31}$$

we see that the lower measurable function $\mu_{\mathbb{E}_{\widetilde{\rho}_{*i+1}}(x,\cdot)}(x')$ $\forall x' \in X$ at the $(i+1)$-th modeling step is defined, for fixed $x \in X$, by relation (24) (the first equation). It is obvious that $\forall x' \in [\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\alpha_{ix}}$, $\beta_{*i}(x') = \mathbb{I}_{[\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\alpha_{ix}}}(x')$. In this case it is advisable to assume $\alpha_{*0}^{(i)}(x') = \mathbb{I}_{[\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\alpha_{ix}}}(x')$. The function $\gamma_{*i}(x')$ will be defined by α-levels for $\alpha \le \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_{i+1})}(x)$. Let us consider arbitrary levels μ_{*l} $(l = 1,\dots,l_*)$. At these levels μ_{*l}, the function $\alpha_{*l}^{(i)}(x')$ will have sets of the level

$$A_{*l}^{i} = \big\{x' \in X \mid \alpha_{*l}^{i}(x') \ge \mu_l\big\}. \tag{32}$$

Since $\{\mu_{*l}\}_{l=1}^{l_*}$ is a decreasing sequence, from definitions (21)–(23) we conclude that for validity of conditions (18) (the first inequality) it is sufficient that

$$\begin{cases} [\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\mu_{*l}} \cap [\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_i)]_{\mu_{*l}} \in \Pi_{*l}^{i}, \\ g(M'_{*li_s}) \ge \mu_{*l}, \quad s = 1,\dots,n_{*l}, \end{cases} \tag{33}$$

be fulfilled.

It is obvious that in order that the subset A_{*l}^{i}, $l = 1,\dots,l_*$, be restored, taking into account the embeddedness of sets of levels, it is sufficient for us to use any subset of the form

$$A_{*l}^{i} = A_{*l-1}^{i} \cup \Bigg[\bigcup_{s=1}^{n_{*l_*}} M'_{*li_s}\Bigg], \quad l = 1,\dots.n_{*l_*}. \tag{34}$$

One can easily check that now conditions (24) (the first equation) are fulfilled. To obtain a quasi-optimal solution, in view of the regularization condition (19) it is sufficient, at the i-th modeling step, for each l, $l = 1,\dots,l_*$, to solve the problem of conditional optimization for families of measurable sets $\langle M'_{*li_1},\dots,M_{*li_{n_{*l_*}}}\rangle$

$$\begin{cases} \displaystyle\bigwedge_{\langle M'_{*li_1},\dots,M_{*li_{n_{*l_*}}}\rangle \in \Pi_{*l}^{i}} g\Bigg([\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\mu_{*l}} \cap \Bigg(A_{*l}^{i} \cup \bigcup_{s=1}^{n_{l_*}} M'_{*li_s}\Bigg)\Bigg) \Rightarrow \min, \\ \qquad\qquad g(M'_{*li_s}) \ge \mu_{*l}, \quad s = 1,\dots,n_{l_*}. \end{cases} \tag{35}$$

Let the optimization problem (35) have some solution $\langle \mathring{M}'_{*li_1},\dots,\mathring{M}'_{*li_{n_{*l_*}}}\rangle$ by the condition of the theorem. After substituting this optimal solution into (34), calculating the function $\beta_{*i}(x')$ by (29) and restoring $\mathbb{E}_{\widetilde{\rho}_{*i+1}}(x,\cdot)$, we obtain the estimate of the operator $\widetilde{\rho}_*$ at the $(i+1)$-th modeling step, which satisfies the regularization condition (19). □ □

Remark 5.3. *It is obvious that the solution obtained by Theorem 5.2 is unique in the sense of the embeddedness property.*

Thus, from the standpoint of application, by the proven theorem we can construct, for finite $X = \{x_1,\dots,x_n\}$, an acceptable identification algorithm that makes it possible to obtain a quasi-optimal solution by using the regularization condition (19). This algorithm has the following structure.

5.3 Algorithm 1

1) For fixed $x \in X$ and the lower function $\mu_{\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)}(x')$, at the i-th modeling step from expert knowledge reflections on the SDEFDS states in the fuzzy extremal intervals we define the variational series (20a) as follows:

$$\begin{pmatrix} \mu_{*1} & \mu_{*2} & \dots & \mu_{*l_*} \\ n_{*1} & n_{*2} & \dots & n_{*l_*} \end{pmatrix}, \quad l = 1, \dots, l_*. \tag{36}$$

2) For each SDEFDS input, at each level μ_{*l} $(l = 1, \dots, l_*)$ we define the solution $\langle \mathring{M}'_{*li_1}, \dots, \mathring{M}'_{*li_{n_{*l_*}}} \rangle$ of the conditional discrete optimization problem (such a solution exists because X is finite) as follows:

$$\begin{cases} \bigwedge\limits_{\langle M'_{*li_1}, \dots, M_{*li_{n_{*l_*}}} \rangle \in \Pi^i_{*l}} g\left([\mathbb{E}_{\widetilde{\rho}_{*i}}(x,\cdot)]_{\mu_{*l}} \cap \left(A^i_{*l} \cup \bigcup\limits_{s=1}^{n_{l_*}} M'_{*li_s} \right)\right) \Rightarrow \min, \\ g(M'_{*li_s}) \geq \mu_{*l}, \quad s = 1, \dots, n_{*l}. \end{cases} \tag{37}$$

It may be approximately solved by a genetic algorithm or other heuristic methods.

3) A^i_{*l+1} is defined by formula (26), while the function $\alpha^i_{*l+1}(x')$ by (25).

4) Steps 2)–3) are fulfilled up to $l = l_*$ and, using (25), (26), we calculate the function $\alpha^i_{*l_*}(x') = \beta_{*i}(x')$ for each $x' \in X$.

5) At the $(i+1)$-th modeling step, we construct the estimate of the compatibility function of a fuzzy relation $\widetilde{\rho}_{*i+1}$, taking into account information on $1 \div i$ steps and applying (24).

6) Steps 1)–5) are repeated for each $x \in X$ and thus we restore the function $\mu_{\widetilde{\rho}_{*i+1}}(x, x')$.

7) Steps 1)–6) are repeated, but this time for the upper function $\mu_{\widetilde{\rho}^*_{i+1}}$. We thus complete the restoration of the transition operator of the SDEFDS at the $(i+1)$-th modeling step with information on $1 \div i$ steps taken into account.

8) Upon the arrival of new information on the output of $\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot, \tau_{i+2})$, steps 1)–7) are repeated so as to define more exactly the estimate of the SDEFDS operator $(\widetilde{\rho}_*, \widetilde{\rho}^*)$.

To illustrate the performance of Algorithm 1, let us consider the following numerical example.

5.4 An example of SDEFDS identification

Let $X = \{1, 2, 3, 4\}$ be a finite set of SDEFDS states. Let, on 2^X, a fuzzy measure be given as a possibilistic one with possibilistic distribution

$$\pi \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0.6 & 0.7 & 0.6 & 1.0 \end{pmatrix},$$

so that $\forall A \in 2^X$

$$g^*(A) = \bigvee_{x \in A} \pi(x) \quad g \text{ is a possibility measure.}$$

Let, at the $i = 2$-th modeling step, the estimate $(\widetilde{\rho}_{*2}, \widetilde{\rho}^*_2)$ of the transition operator $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ be represented as follows:

$$\widetilde{\rho}_{*2} = \begin{pmatrix} 1 & 0.8 & 0.5 & 0.2 \\ 0.7 & 0.9 & 0.8 & 0.2 \\ 0.5 & 0.6 & 0.9 & 0.3 \\ 0.1 & 0.4 & 0.8 & 1 \end{pmatrix}, \quad \widetilde{\rho}^*_2 = \begin{pmatrix} 1 & 0.9 & 0.6 & 0.2 \\ 0.6 & 0.9 & 0.7 & 0.1 \\ 0.6 & 0.7 & 0.9 & 0.2 \\ 0.2 & 0.5 & 0.9 & 1 \end{pmatrix}.$$

Let three input fuzzy states (signals) be delivered to the SDEFDS input so that $\widetilde{A}_{0*\tau_i} = \widetilde{A}^*_{0\tau_i}$, $i = 1, 2, 3$, and

$$\begin{aligned} \widetilde{A}_{*0\tau_1} &= \begin{pmatrix} 1/0.8 & 2/0.7 & 3/0.4 & 4/0.2 \end{pmatrix}, \quad i = 1, \\ \widetilde{A}_{*0\tau_2} &= \begin{pmatrix} 1/0.4 & 2/0.6 & 3/0.6 & 4/0.2 \end{pmatrix}, \quad i = 2, \\ \widetilde{A}_{*0\tau_3} &= \begin{pmatrix} 1/0.2 & 2/0.4 & 3/0.5 & 4/0.6 \end{pmatrix}, \quad i = 3, \end{aligned}$$

where $\tau_1 < \tau_2 < \tau_3$ and measurements were taken in some extremal fuzzy time intervals $(\widetilde{r}_{\tau_{i*}}, \widetilde{r}^*_{\tau_i})$, $i = 1, 2, 3$. Let three output fuzzy states (signals) be assumed to have the form:

$$\begin{aligned} \mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot, \tau_1) &= \begin{pmatrix} 1/0.6 & 2/0.5 & 3/0.4 & 4/0.4 \end{pmatrix}, \\ \mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot, \tau_2) &= \begin{pmatrix} 1/0.4 & 2/0.6 & 3/0.5 & 4/0.4 \end{pmatrix}, \\ \mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot, \tau_3) &= \begin{pmatrix} 1/0.4 & 2/0.6 & 3/0.5 & 4/0.5 \end{pmatrix}. \end{aligned}$$

It is required to restore (to define more exactly) the estimate $(\widetilde{\rho}_{*3}, \widetilde{\rho}_3^*)$ of the transition operator $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ at the $i = 3$-th modeling step by the information on the input-output signals presented here.

To solve this problem, we use Algorithm 1.

Step 1: Let $x = 1$ be fixed. Then

$$\mathbb{E}_{\widetilde{\rho}_{*2}}(1,\cdot) = \begin{pmatrix}1/1 & 2/0.8 & 3/0.5 & 4/0.4\end{pmatrix},$$
$$\mathbb{E}_{\widetilde{\rho}_2^*}(1,\cdot) = \begin{pmatrix}1/1 & 2/0.9 & 3/0.6 & 4/0.5\end{pmatrix}.$$

For $x = 1$ the variational series (36) looks like

$$\begin{pmatrix}0.6 & 0.4\\ 1 & 2\end{pmatrix} \qquad \begin{array}{l} l_* = 2,\ \ n_{*1} = 1,\ \ n_{*3} = 2,\ \ i = 2,\\ \mu_{*1} = 0.6,\ \ \mu_{*2} = 0.4.\end{array}$$

Define the sets $M_{*li_s}^i$ and $\Pi_{*li_s}^i$ as follows:
for $l = 1$: $M_{*11}^2 = \{1, 2\}$;
for $l = 2$: $M_{*22}^2 = \{1, 2, 3\}$, $M_{*23}^2 = \{2, 3, 4\}$,

$$\begin{aligned}
\Pi_{*11}^2 &= \big\{\{1\},\ \{2\},\ \{1,2\}\big\},\\
\Pi_{*22}^2 &= \big\{\{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}\big\},\\
\Pi_{*23}^2 &= \big\{\{2\},\ \{3\},\ \{4\},\ \{2,3\},\ \{2,4\},\ \{3,4\},\ \{2,3,4\}\big\}.
\end{aligned}$$

Then

$$\Pi_{*1}^2 = \Pi_{*11}^2, \quad \Pi_{*2}^2 = \Pi_{*22}^2 \times \Pi_{*23}^2.$$

Now define the sets $M_{li_q}^{*i}$ and $\Pi_{li_q}^{*i}$:

$$\mu_1^* = 0.6,\ \ M_{11}^{*2} = \{1, 2\},\ \ \mu_2^* = 0.6,\ \ M_{22}^{*2} = \{1, 2, 3\},\ \ M_{23}^{*2} = \{2, 3, 4\}.$$

Then

$$\Pi_{11}^{*2} = \Pi_{*11}^2, \quad \Pi_{22}^{*2} = \Pi_{*22}^2, \quad \Pi_{23}^{*2} = \Pi_{*23}^2, \quad \Pi_{*1}^2 = \Pi_2^{*2}, \quad \Pi_{*2}^2 = \Pi_2^{*2}.$$

We have

$$\begin{aligned}
&[\mathbb{E}_{\widetilde{\rho}_{*2}}(1,\cdot)]_{\mu_{*1}} = [\mathbb{E}_{\widetilde{\rho}_{*2}}(1,\cdot)]_{0.6} = \{1, 2\}, \quad [\mathbb{E}_{\widetilde{\rho}_2^*}(1,\cdot)]_{\mu_1^*} = \{1, 2, 3\},\\
&[\mathbb{E}_{\widetilde{\rho}_{*2}}(1,\cdot)]_{\mu_{*2}} = \{1, 2, 3\}, \quad [\mathbb{E}_{\widetilde{\rho}_2^*}(1,\cdot)]_{\mu_2^*} = \{1, 2, 3\}.
\end{aligned}$$

Step 2: For $l = 1, 2$, we formulate the conditional optimization problem:
First we consider the case $l = 1$

$$g\big(\{1, 2\} \cap \big[\varnothing \cup \{1\} \cup \{2\} \cup \{1, 2\}\big]\big) = g(\{1, 2\}) = \vee(0.6, 1) = 1 \Rightarrow \min,$$

or

$$\overset{\circ}{M}{}'_{*11} = \{1, 2\}, \ \text{ since } \ g(\overset{\circ}{M}{}'_{*11}) = 0.7 \geq \mu_{*1}.$$

Analogously we obtain $\overset{\circ}{M}{}^{*\prime}_{11} = \{1, 2\}$, $A_{*1}^2 = A_{*0}^2 \cup \overset{\circ}{M}{}'_{*11} = \{1, 2\} = A_1^{*2}$.

Further, by formula (25) we construct the functions α_{*l} and α_l^* for $l = 1$:

$$\alpha_{*1}^2(x') = \alpha_{*0}^2(x') \vee \big[\mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_1)}(x') \wedge \mathbb{I}_{A_{*1}^2}(x')\big] = \begin{cases} 0, & x' \in \{1, 2\},\\ 0.4 & x' \in \{3, 4\}.\end{cases}$$

Analogously, we find that $\alpha_1^{*2}(x') = \alpha_{*1}^2(x')$.

Now let us consider the case $l = 2$:

$$\wedge \big\{g\big(\{1, 2, 3\} \cap [\{1, 2\} \cup (\{1\} \cup \{2\})]\big), \ldots, g\big(\{1, 2, 3\} \cap [\{1, 2\} \cup (\{1, 2\} \cup (\{1, 2, 3\}\{2, 3, 4\}))]\big\} \Rightarrow \min.$$

Clearly, a minimum is attained at $g(\{1, 2\} \cup \{1\} \cup \{2\}) = g(\{1, 2\})$ or $\overset{\circ}{M}{}'_{*22} = \{1\}$, $\overset{\circ}{M}{}'_{*23} = \{2\}$ since $g(\overset{\circ}{M}{}'_{*22}) \geq \mu_{*2}$ and $g(\overset{\circ}{M}{}'_{*23}) \geq \mu_{*2}$.

Thus we obtain $A^2_{*2} = A^2_{*1} \cup \overset{\circ}{M}'_{*22} \cup \overset{\circ}{M}'_{*23} = \{1, 2\}$.

Analogously, we obtain $A^{*2}_2 = \{1, 2\}$. Then

$$\beta_{*2}(x') = \alpha^2_{*2}(x') = \alpha^2_{*1}(x') \vee \big[\mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_2)}(x') \wedge \mathbb{I}_{A^2_{*2}}(x')\big] = (1/0.4, 2/0.6, 3/0.4, 4/0.4).$$

Analogously, $\beta^*_2(x') = \beta_{*2}(x')$.

Step 3: Using (24), we construct $\mathbb{E}_{\widetilde{\rho}_{*3}}(1,\cdot)$ and $\mathbb{E}_{\widetilde{\rho}^*_3}(1,\cdot)$:

$$\mu_{\widetilde{\rho}_{*3}}(1, x') = \big[\mu_{\widetilde{\rho}_{*2}}(1, x') \vee \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_1)}\big] \wedge \beta_{*2}(x').$$

Then

$$\mathbb{E}_{\widetilde{\rho}_{*3}}(1,\cdot) = \big[(1/1, 2/0.8, 3/0.5, 4/0.2) \vee 0.4\big] \wedge \big[1/0.4, 2/0.6, 3/0.4, 4/0.4\big] = \big(1/0.4, 2/0.6, 3/0.5, 4/0.4\big) = \mathbb{E}_{\widetilde{\rho}^*_3}(1,\cdot).$$

The model output estimate is calculated by (13):

$$\begin{aligned}
\mu_{\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_1)}(1) &= \fint_X \big[\mu_{\widetilde{\rho}_{*3}}(1, x') \wedge \mu_{\widetilde{A}_{*0\tau_3}}(x')\big] \circ g(\cdot) \\
&= \fint_X \big[(1/0.4, 2/0.6, 3/0.5, 4/0.4) \wedge (1/0.8, 2/0.7, 3/0.4, 4/0.2)\big] \circ g(\cdot) \\
&= 0.4 = \mu_{\mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_3)}(1),
\end{aligned}$$

i.e., the model value coincides with the true expert valuations (output values). Analogously,

$$\mu_{\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_3)}(i) = \fint_X \big[\mu_{\widetilde{\rho}^*_3}(i, x') \wedge \mu_{\widetilde{A}^*_{0\tau_3}}(i)\big] \circ g(\cdot) = \mu_{\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_3)}(i), \quad i = 2, 3, 4.$$

Repeating steps 1)–3) for $x = 2, 3, 4$, we obtain

$$\begin{aligned}
\mathbb{E}_{\widetilde{\rho}_{*3}}(2,\cdot) &= (1/0.6, 2/0, 3/0, 4/0) = \mathbb{E}_{\widetilde{\rho}^*_3}(2,\cdot), \\
\mathbb{E}_{\widetilde{\rho}_{*3}}(3,\cdot) &= (1/0, 2/0, 3/0.5, 4/0) = \mathbb{E}_{\widetilde{\rho}^*_3}(3,\cdot), \\
\mathbb{E}_{\widetilde{\rho}_{*3}}(4,\cdot) &= (1/0.4, 2/0, 3/0.5, 4/0) = \mathbb{E}_{\widetilde{\rho}^*_3}(4,\cdot).
\end{aligned}$$

This completes the construction of the estimate $(\widetilde{\rho}_{*3}, \widetilde{\rho}^*_3)$ of the transition operator $(\widetilde{\rho}_*, \widetilde{\rho}^*)$:

$$\widetilde{\rho}_{*3} = \widetilde{\rho}^*_3 = \begin{pmatrix} 0.4 & 0.6 & 0.5 & 0.4 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0.4 & 0 & 0.5 & 0 \end{pmatrix}.$$

For the step $i = 3$, the model extremal fuzzy output states of the SDEFDS look like

$$\begin{aligned}
\mu_{\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_j)}(i) &= \fint_X \big[\mu_{\widetilde{\rho}_{*3}}(i, x') \wedge \mu_{\widetilde{A}_{*0\tau_j}}(x')\big] \circ g(\cdot), \\
\mu_{\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_j)}(i) &= \fint_X \big[\mu_{\widetilde{\rho}^*_3}(i, x') \wedge \mu_{\widetilde{A}^*_{0\tau_j}}(x')\big] \circ g^*(\cdot).
\end{aligned}$$

Then, comparing the output results, we obtain

$$\begin{cases} \mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_1) = (1/0.6, 2/0.6, 3/0.5, 4/0.4) = \mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_1), \\ \mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_1) = (1/0.6, 2/0.6, 3/0.4, 4/0.4) = \mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_1), \end{cases}$$

$$\begin{cases} \mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_2) = (1/0.6, 2/0.4, 3/0.5, 4/0.4) = \mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_2), \\ \mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_2) = (1/0.4, 2/0.6, 3/0.4, 4/0.4) = \mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_2), \end{cases}$$

$$\begin{cases} \mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_3) = (1/0.4, 2/0.6, 3/0.5, 4/0.4) = \mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_3), \\ \mathbb{E}_{\widehat{\widetilde{Q}}_*}(\cdot,\tau_3) = (1/0.4, 2/0.6, 3/0.4, 4/0.5) = \mathbb{E}_{\widehat{\widetilde{Q}}^*}(\cdot,\tau_3). \end{cases}$$

It is obvious that there exists a sufficiently good coincidence of model and expert valuations of true extremal fuzzy states of the SDEFDS.

Based on a complete calculation mechanism, the software is designed to solve the constructed discrete optimization model.

6 Conclusion

Based on the results presented in the papers [34–36], we have considered questions of the fuzzy optimization of extremal processes, where:

a) The general EFDS model is described. The dualization of a time structure forms the most important part of the fuzzy instrument of modeling and optimization of SDEFDS;

b) the SDEFDS model with fuzzy uncertainty is introduced, the source of which is "fuzzy measurement" ("expert reflections" on the states of SDEFDS) of the system state in the so-called current and future discrete sequence of fuzzy time intervals.

c) a method has been developed for identifying of the transition operator of an SDEFDS by using the fuzzy-integral model (14) and the information on realized N input-output pairs (16).

d) the restoration of the transition operator by expert data in the model (14) is a nonregular problem. In order to obtain a unique (in a certain sense) quasi-optimal solution of the identification problem, the regularization conditions are introduced: for SDEFDS this is the extremalization of the initial fuzzy measure functions (19) by the corresponding expert-extremal data (16);

e) the respective algorithm of identification of $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ has been developed. The results obtained are illustrated by the example with a finite set of SDEFDS states. A good agreement between the estimates obtained by the proposed method and expert data is observed.

f) currently we work on practical applicability of the results. Namely in modeling and controlling complex industrial processes. We are close to finish the work on analysis and synthesis problems of fuzzy-extremal processes for intuitionistic and q-rung orthopair fuzzy environments.

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An identification model for a fuzzy time based stationary discrete process

G. Sirbiladze

یک مدل‌شناسایی برای یک فرآیند گسسته ثابت مبتنی بر زمان فازی

چکیده. رویکرد جدیدی از فرآیندهای فازی، که منبع آن بازتاب‌های دانش تخصصی در مورد وضعیت های سیستم دینامیکی فازی اکسترمال گسسته ثابت (SDEFDS) در بازه‌های زمانی فازی اکسترمال است، در نظر گرفته می‌شود. یک نمایش انتگرال فازی از یک فرآیند فازی اکسترمال گسسته ثابت داده شده است. یک روش و یک الگوریتم برای شناسایی عملگر انتقال SDEFDS توسعه یافته است. عملگر انتقال SDEFDS با استفاده از بازتاب دانش تخصصی در مورد حالات SDEFDS بازیابی می‌شود. شرط تنظیم برای بدست آوردن برآوردگر شبه بهینه عملگر انتقال با قضیه نشان داده می‌شود. الگوریتم محاسبه مربوطه ارائه شده است. نتایج بدست آمده با یک مثال در مورد یک مجموعه متناهی از حالت‌های SDEFDS نشان داده شده است.