

## Construction of 2-uninorms on bounded lattices

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### Abstract

Uninorms and nullnorms are special 2-uninorms. In this work, we construct 2-uninorms on bounded lattices. Let  $L$  be a bounded lattice with a nontrivial element  $d$ . Given two uninorms  $U_1$  and  $U_2$ , defined on sublattices  $[0, d]$  and  $[d, 1]$ , respectively, this paper presents two methods for constructing binary operators on  $L$  which extend both  $U_1$  and  $U_2$ . We show that our first construction is a 2-uninorm on  $L$  if and only if  $U_2$  is conjunctive and our second construction is a 2-uninorm on  $L$  if and only if  $U_1$  is disjunctive. Moreover, we prove that the two 2-uninorms are, respectively, the weakest and the strongest 2-uninorm among all 2-uninorms, the restrictions of which on  $[0, d]^2$  and  $[d, 1]^2$  are respectively  $U_1$  and  $U_2$ .

**Keywords:** Bounded lattices, 2-uninorms, uninorms, nullnorms.

## 1 Introduction

By allowing the *identity element* different from 0 and 1, Yager and Rybalov [42] introduced *uninorms* on the unit interval, which include triangular norms [27] (t-norms henceforth) and triangular conorms [27] (t-conorms henceforth) as special classes. Since then, uninorms have become important aggregation operators, which have applications in expert systems [16, 20, 36], fuzzy logic [28], and fuzzy systems modeling [41]. As an interesting mathematical construction, uninorms have been investigated by many researchers, see [15, 18, 23, 29, 31]. By allowing annihilator to be put anywhere in  $[0, 1]$ , *nullnorms* (or *t-operators*) [5, 30] are another generalization of both t-norms and t-conorms. Later, Akella [1] proposed the important notion of a *2-uninorm*, which has two *local identity elements* in  $[0, 1]$ . 2-uninorms generalize uninorms and nullnorms and they have been applied in related fields such as [13]. Until now, 2-uninorms have attracted some research interests, see [19, 32, 33, 38, 44, 45, 46].

As a bounded lattice, the unit interval is sometimes too special and often cannot be adopted as the underlying value domain of many decision making tasks. Recently, several researchers have considered similar constructions on general bounded lattices. A series of works have been done for uninorms [4, 6, 7, 9, 10, 12, 14, 26, 34, 40], nullnorms [8, 11, 22, 24, 25], uni-nullnorms [37, 39, 43] and null-uninorms [43].

Ertuğrul [21] considered 2-uninorms on a general bounded lattice. Let  $(L, \leq)$  be a bounded lattice. Suppose  $d$  is a nontrivial element of  $L$ ,  $U_1$  ( $U_2$ , resp.) a disjunctive (conjunctive, resp.) uninorm on  $[0, d]$  ( $[d, 1]$ , resp.). Ertuğrul [21] defined a 2-uninorm on  $L$  by extending  $U_1$  and  $U_2$  in a natural way. Furthermore, he showed that the construction could fail to be a 2-uninorm if either the disjunctivity or conjunctivity is not satisfied.

In this work, we consider a similar problem as in [21]. Let  $(L, \leq)$  be a bounded lattice. Suppose  $d$  is a nontrivial element of  $L$ . For any uninorm  $U_1$  on  $[0, d]$  and any uninorm  $U_2$  on  $[d, 1]$ , we construct two operators  $\mathcal{H}_{U_1, U_2}^\wedge$  and  $\mathcal{H}_{U_1, U_2}^\vee$  and show that  $\mathcal{H}_{U_1, U_2}^\wedge$  ( $\mathcal{H}_{U_1, U_2}^\vee$ , resp.) is a 2-uninorm if and only if  $U_2$  is conjunctive ( $U_1$  is disjunctive, resp.). Moreover, we prove that for any 2-uninorm  $\mathcal{H}$  on  $L$ , if  $U_1$  and  $U_2$  are, respectively, the restrictions of  $\mathcal{H}$  on  $[0, d]^2$  and  $[d, 1]^2$ , then  $\mathcal{H}_{U_1, U_2}^\wedge \leq \mathcal{H} \leq \mathcal{H}_{U_1, U_2}^\vee$ . Our 2-uninorms on bounded lattices generalize 2-uninorms on the unit interval and also can be used to obtain uni-nullnorms, null-uninorms and nullnorms on bounded lattices. Moreover, using  $\mathcal{H}_{U_1, U_2}^\vee$ , we can obtain fuzzy implications on bounded lattices, which can be applied in lattice-valued fuzzy set theory.

In the remainder of this work, we first recall some preliminaries in Section 2, then present our constructions and main results in Section 3. A short conclusion as well as an outlook for future research is presented in Section 4.

## 2 Preliminaries

Our reference to basic notions and terminologies of lattice theory is [3]. Suppose  $(L, \leq)$  is a lattice. The binary minimum (meet) and maximum (join) operations on  $L$  are denoted by  $\wedge$  and  $\vee$ , respectively. A lattice  $(L, \leq)$  is called a *bounded lattice* if there exist two elements 0 and 1 in  $L$  such that  $0 \leq x \leq 1$  for any  $x \in L$ . We call 0 and 1, respectively, the bottom and the top of  $L$ . For  $a_1, a_2 \in L$  with  $a_1 < a_2$ , we define  $[a_1, a_2] \equiv \{x \mid a_1 \leq x \leq a_2\}$ . Similarly, we can define  $(a_1, a_2)$ ,  $(a_1, a_2]$  and  $[a_1, a_2)$ . In addition, we define  $I_d \equiv \{x \in L \mid x \text{ is incomparable with } d\}$ .

Let  $(L, \leq)$  be a lattice. Suppose  $F : L^2 \rightarrow L$  is a binary operator on  $L$ . Assume  $L_1$  is a sublattice of  $L$ , i.e., both  $a \vee b$  and  $a \wedge b$  are in  $L_1$  for any  $a, b \in L_1$ . The *restriction* of  $F$  to  $L_1^2$  is denoted as  $F|_{L_1^2}$ . In general,  $F(a, b)$  is not necessarily an element in  $L_1$  despite that  $a, b$  are both in  $L_1$ . In case that  $L_1$  is closed under  $F$ , i.e.,  $F(a, b) \in L_1$  for any  $a, b \in L_1$ , we write  $F|_{L_1}$  to denote the restriction of  $F$  to  $L_1^2$ , which is a binary operator on  $L_1$ . Suppose there is another binary operation  $F'$  on  $L_1$  such that  $F'(a, b) = F(a, b)$  for all  $a, b \in L_1$ . Then  $F$  is called an *extension* of  $F'$  on  $L$ , or  $F$  extends  $F'$  on  $L$ .

In the remainder of this paper, we always denote a bounded lattice  $(L, \leq, 0, 1)$  simply as  $L$ .

All operators considered in this paper are AMC operators in the following sense.

**Definition 2.1.** [1] Assume  $L$  is a bounded lattice. An operator  $F : L^2 \rightarrow L$  is an AMC operator if  $F$  is associative, commutative, and non-decreasing in both variables.

**Definition 2.2.** [17, 27] Assume  $L$  is a bounded lattice. An AMC operator  $F$  on  $L$  is called a *triangular norm* (*t-norm*) if 1 is the identity element of  $F$ , i.e.,  $F(1, a) = a$  for any  $a \in L$ . Analogously, we say  $F$  is a *triangular conorm* (*t-conorm*) if 0 is the identity element of  $F$ , i.e.,  $F(0, a) = a$  for any  $a \in L$ .

The following example gives two special t-norms (t-conorms).

**Example 2.3.** [26, 27] Assume  $L$  is a bounded lattice. Define

$$\begin{aligned} T_M(a, b) &= a \wedge b, \\ T_D(a, b) &= \begin{cases} a, & \text{if } b = 1 \\ b, & \text{if } a = 1 \\ 0, & \text{otherwise,} \end{cases} \\ S_D(a, b) &= \begin{cases} a, & \text{if } b = 0 \\ b, & \text{if } a = 0 \\ 1, & \text{otherwise,} \end{cases} \\ S_M(a, b) &= a \vee b. \end{aligned}$$

$T_M$  ( $T_D$ , resp.) is the strongest (weakest, resp.) t-norm on  $L$  and  $S_D$  ( $S_M$ , resp.) is the strongest (weakest, resp.) t-conorm on  $L$ .

Both t-norms and t-conorms are special nullnorm and uninorm operators.

**Definition 2.4.** [25, 27, 30] A binary operator  $F : L^2 \rightarrow L$  on a bounded lattice  $L$  is a *nullnorm* on  $L$  if it is an AMC operator and has an annihilator  $b \in L$  such that  $F(0, x) = x$  for any  $x \leq b$  and  $F(1, y) = y$  for any  $y \geq b$ .

Every t-conorm  $S$  is a nullnorm with annihilator 1 and every t-norm  $T$  is a nullnorm with annihilator 0.

**Definition 2.5.** [26, 27] A binary operator  $U : L^2 \rightarrow L$  on a bounded lattice  $L$  is called a *uninorm* on  $L$  if  $U$  is an AMC operator and has an identity element  $e$  in  $L$ , i.e.,  $U(e, x) = x$  for all  $x \in L$ . If  $U(0, 1) = 0$ , we say  $U$  is *conjunctive*; if  $U(0, 1) = 1$ , we say  $U$  is *disjunctive*.

Every t-conorm  $S$  (t-norm  $T$ , resp.) on  $L$  is a uninorm with identity element 0 (1, resp.).

Let  $L$  be a bounded lattice and  $e \in L \setminus \{0, 1\}$ . Then  $U_{sc} : L^2 \rightarrow L$  and  $U_{ld} : L^2 \rightarrow L$ , respectively, are the weakest and strongest uninorms on  $L$  with neutral element  $e$  [26], where

$$U_{sc}(x, y) = \begin{cases} x \vee y, & \text{if } (x, y) \in [e, 1]^2 \\ x \wedge y, & \text{if } (x, y) \in [0, e] \times [e, 1] \cup [e, 1] \times [0, e] \\ y, & \text{if } (x, y) \in [e, 1] \times I_e \\ x, & \text{if } (x, y) \in I_e \times [e, 1] \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

$$U_{ld}(x, y) = \begin{cases} x \wedge y, & \text{if } (x, y) \in [0, e]^2 \\ x \vee y, & \text{if } (x, y) \in [0, e] \times [e, 1] \cup [e, 1] \times [0, e] \\ y, & \text{if } (x, y) \in [0, e] \times I_e \\ x, & \text{if } (x, y) \in I_e \times [0, e] \\ 1, & \text{otherwise.} \end{cases} \quad (2)$$

Obviously,  $U_{sc}$  is conjunctive and  $U_{ld}$  is disjunctive.

**Proposition 2.6.** [26] Suppose  $U$  is a uninorm on a bounded lattice  $L$  with identity element  $e \neq 0, 1$ . Then the restriction of  $U$  on  $[0, e]^2$  ( $[e, 1]^2$ , resp.) is a  $t$ -norm ( $t$ -conorm, resp.) on  $[0, e]$  ( $[e, 1]$ , resp.).

**Definition 2.7.** [1, 46] A binary operator  $\mathcal{H} : L^2 \rightarrow L$  on a bounded lattice  $L$  is a 2-uninorm if  $\mathcal{H}$  is an AMC operator and there exist  $e_1, e_2$  and  $d \in (0, 1)$  in  $L$  such that  $0 \leq e_1 \leq d \leq e_2 \leq 1$  and  $\mathcal{H}(x, e_1) = x$  for any  $x \leq d$  and  $\mathcal{H}(y, e_2) = y$  for any  $y \geq d$ . We call  $d$  the cutpoint and call  $e_1, e_2$  the first and, respectively, the second local identity elements of  $\mathcal{H}$ .

**Definition 2.8.** [35] A 2-uninorm with  $e_2 = 1$  is called a uni-nullnorm, and a 2-uninorm with  $e_1 = 0$  is called a null-uninorm.

Assume  $\mathcal{H}$  is a 2-uninorm on  $L$  with local identity elements  $e_1 \leq e_2$  and cutpoint  $d$ . Let  $U_1$  and  $U_2$  be the restrictions of  $\mathcal{H}$  to  $[0, d]^2$  and  $[d, 1]^2$ , respectively. Clearly,  $U_1$  ( $U_2$ , resp.) is a uninorm on  $[0, d]$  ( $[d, 1]$ , resp.) and  $e_1$  ( $e_2$ , resp.) is its identity element.

Obviously, uninorms, nullnorms, uni-nullnorms and null-uninorms are all special 2-uninorms.

**Theorem 2.9.** [21] Let  $L$  be a bounded lattice,  $U_1 : [0, d]^2 \rightarrow [0, d]$  be a disjunctive uninorm with neutral element  $e_1$  and  $U_2 : [d, 1]^2 \rightarrow [d, 1]$  be a conjunctive uninorm with neutral element  $e_2$ . Then the function  $U^2 : L^2 \rightarrow L$  given by

$$U^2(x, y) = \begin{cases} U_1(x, y), & (x, y) \in [0, d]^2 \\ U_2(x, y), & (x, y) \in [d, 1]^2 \\ d, & \text{otherwise} \end{cases}$$

is a 2-uninorm.

### 3 2-uninorms on bounded lattices

Let  $L$  be a bounded lattice. Suppose  $e_1, e_2, d \in L$  with  $0 \leq e_1 \leq d \leq e_2 \leq 1$  and  $0 < d < 1$ . Assume further that  $U_1$  is a uninorm on  $[0, d]$  with identity element  $e_1$  and  $U_2$  is a uninorm on  $[d, 1]$  with identity element  $e_2$ . We give two methods for constructing 2-uninorms by extending  $U_1$  and  $U_2$ . The first extension, denoted  $\mathcal{H}_{U_1, U_2}^\wedge$ , is a 2-uninorm if and only if  $U_2$  is a conjunctive uninorm; the second extension, denoted  $\mathcal{H}_{U_1, U_2}^\vee$ , is a 2-uninorm if and only if  $U_1$  is a disjunctive uninorm.

#### 3.1 The weakest 2-uninorm $\mathcal{H}_{U_1, U_2}^\wedge$

The construction of the first extension is illustrated in Fig. 1.

**Theorem 3.1.** Let  $L$  be a bounded lattice. Suppose  $e_1, e_2, d \in L$  with  $0 < d < 1$  and  $0 \leq e_1 \leq d < e_2 \leq 1$ . Assume further that  $U_1$  is a uninorm on  $[0, d]$  with identity element  $e_1$  and  $U_2$  is a uninorm on  $[d, 1]$  with identity element  $e_2$ . Then the operator  $\mathcal{H}_{U_1, U_2}^\wedge$  given by

$$\mathcal{H}_{U_1, U_2}^\wedge(x, y) = \begin{cases} U_2(x, y), & \text{if } (x, y) \in [d, 1]^2 \\ U_1(x \wedge d, y \wedge d), & \text{otherwise} \end{cases} \quad (3)$$

is a 2-uninorm on  $L$  if and only if  $U_2$  is a conjunctive uninorm.

*Proof.* To simplify the presentation, we use, in (and only in) this proof,  $x \diamond y$  to denote the binary operation  $\mathcal{H}_{U_1, U_2}^\wedge$  of any two elements  $x, y \in L$ , i.e.,  $x \diamond y \equiv \mathcal{H}_{U_1, U_2}^\wedge(x, y)$ .

$I_d$	$U_1(x, y \wedge d)$	$U_1(d, y \wedge d)$	$U_1(x \wedge d, y \wedge d)$
1			
$e_2$	$U_1(x, d)$	$U_2$	$U_1(x \wedge d, d)$
$d$			
$e_1$	$U_1$	$U_1(d, y)$	$U_1(x \wedge d, y)$
0	$e_1$	$d$	$e_2$
		1	$I_d$

Figure 1: The structure of the 2-uniform  $\mathcal{H}_{U_1, U_2}^\wedge$  in Eq. (3)

**Necessity** Suppose  $\mathcal{H}_{U_1, U_2}^\wedge$  is a 2-uniform. We show  $U_2$  is conjunctive, i.e.,  $U_2(d, 1) = d$ . From the definition of  $\diamond$ , we have  $1 \diamond 1 = U_2(1, 1) = 1$  and  $e_1 \diamond 1 = U_1(e_1 \wedge d, d \wedge 1) = d$ . By the associativity of  $\diamond$ , we have

$$U_2(d, 1) = d \diamond 1 = (e_1 \diamond 1) \diamond 1 = e_1 \diamond (1 \diamond 1) = e_1 \diamond 1 = d.$$

**Sufficiency** Suppose  $U_2$  is conjunctive. Then  $d \diamond 1 = U_2(d, 1) = d$ . By  $U_1(e_1, d) = d = U_2(e_2, d)$ , we have  $U_1(d, d) = d = U_2(d, d)$  and  $U_2(x, d) = d$  for any  $x \geq d$ .

We prove that  $\mathcal{H}_{U_1, U_2}^\wedge$  is a 2-uniform with cutpoint  $d$  and local identity elements  $e_1, e_2$ .

Note that we can rewrite  $\mathcal{H}_{U_1, U_2}^\wedge$  as

$$x \diamond y = \begin{cases} U_1(x, y), & \text{if } (x, y) \in [0, d]^2 \\ U_2(x, y), & \text{if } (x, y) \in [d, 1]^2 \\ U_1(d, y \wedge d), & \text{if } (x, y) \in [d, 1] \times I_d \\ U_1(x \wedge d, d), & \text{if } (x, y) \in I_d \times [d, 1] \\ U_1(x, d), & \text{if } (x, y) \in [0, d] \times [d, 1] \\ U_1(d, y), & \text{if } (x, y) \in [d, 1] \times [0, d] \\ U_1(x, y \wedge d), & \text{if } (x, y) \in [0, d] \times I_d \\ U_1(x \wedge d, y), & \text{if } (x, y) \in I_d \times [0, d] \\ U_1(x \wedge d, y \wedge d), & \text{if } (x, y) \in I_d^2. \end{cases} \quad (4)$$

Clearly,  $\diamond$  is commutative and satisfies  $x \diamond e_1 = U_1(x, e_1) = x$  for any  $x \in [0, d]$  and  $x \diamond e_2 = U_2(x, e_2) = x$  for any  $x \in [d, 1]$ . It remains to prove its monotonicity and associativity. Let us first consider the monotonicity. For any  $x, y, z \in L$ , suppose  $x \leq y$ . We show  $x \diamond z \leq y \diamond z$ . The monotonicity clearly holds if  $x, y, z \in [d, 1]$ . Since  $x \leq y$  we see that if  $y \notin [d, 1]$ , then also  $x \notin [d, 1]$  and  $x \diamond z = U_1(x \wedge d, z \wedge d) \leq U_1(y \wedge d, z \wedge d) = y \diamond z$ . The same holds if  $z \notin [d, 1]$ . Finally, if  $x \notin [d, 1]$  and  $y, z \in [d, 1]$ , then  $x \diamond z = U_1(x \wedge d, y \wedge d) \leq d \leq U_2(y, z) = y \diamond z$ .

To show the associativity, we only need to prove

$$x \diamond (y \diamond z) = (x \diamond y) \diamond z,$$

for any  $x, y, z \in L$ .

If  $x, y, z \in [d, 1]$ , the associativity is clear. Otherwise, it is enough to show that in all remaining cases

$$x \diamond (y \diamond z) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d)),$$

and

$$(x \diamond y) \diamond z = U_1(U_1(x \wedge d, y \wedge d), z \wedge d).$$

If fact, if  $y, z \in [d, 1]$ , then  $x \notin [d, 1]$  and  $x \diamond (y \diamond z) = U_1(x \wedge d, U_2(y, z) \wedge d) = U_1(x \wedge d, d) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d))$ . If  $y \notin [d, 1]$  or  $z \notin [d, 1]$ , then  $y \diamond z = U_1(y \wedge d, z \wedge d) \leq d$ . Especially, when  $y \diamond z < d$ , we obtain  $x \diamond (y \diamond z) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d))$ ; when  $y \diamond z = d$  and  $x \notin [d, 1]$ , then  $x \diamond (y \diamond z) = x \diamond d = U_1(x \wedge d, d) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d))$ ; when  $y \diamond z = d$  and  $x \in [d, 1]$ , then  $x \diamond (y \diamond z) = x \diamond d = U_2(x, d) = d = U_1(d, d) = U_1(x \wedge d, d) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d))$ . To sum up,  $x \diamond (y \diamond z) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d))$  if at least one of  $x, y$  and  $z$  is not in  $[d, 1]$ .

Similarly, we can prove that  $(x \diamond y) \diamond z = U_1(U_1(x \wedge d, y \wedge d), z \wedge d)$  if at least one of  $x, y$  and  $z$  is not in  $[d, 1]$ . Hence, the associativity follows from the associativity of  $U_1$ .  $\square$

The next example illustrates the construction of  $\mathcal{H}_{U_1, U_2}^\wedge$  in Theorem 3.1.

**Example 3.2.** Let  $L_1 = \{0, a_1, e_1, a_2, a_3, d, a_4, e_2, a_5, 1, b_1, b_2, c_1, c_2, c_3, c_4\}$  be the bounded lattice shown in Fig. 2. Suppose  $U_1$  ( $U_2$ ) is a conjunctive uninorm on  $[0, d]$  ( $[d, 1]$ ) given by (cf. [4]):

$$U_1(x, y) = \begin{cases} S_M(x, y), & \text{if } (x, y) \in [e_1, d]^2 \\ y, & \text{if } (x, y) \in [e_1, d] \times I_{e_1} \\ x, & \text{if } (x, y) \in I_{e_1} \times [e_1, d] \\ T_D(x \wedge e_1, y \wedge e_1), & \text{otherwise,} \end{cases} \quad (5)$$

and

$$U_2(x, y) = \begin{cases} S_D(x, y), & \text{if } (x, y) \in [e_2, 1]^2 \\ y, & \text{if } (x, y) \in [e_2, 1] \times I_{e_2} \\ x, & \text{if } (x, y) \in I_{e_2} \times [e_2, 1] \\ T_M(x \wedge e_2, y \wedge e_2), & \text{otherwise.} \end{cases} \quad (6)$$

The 2-uninorm  $\mathcal{H}_{U_1, U_2}^\wedge$ , as defined in Theorem 3.1, is given by Table 1.

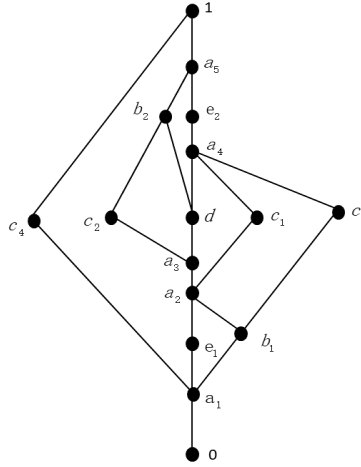


Figure 2: The bounded lattice  $L_1$

**Remark 3.3.** (i) In Theorem 3.1,  $d$  is required to be less than  $e_2$ . Otherwise,  $U_2$  becomes a  $t$ -conorm and it cannot be conjunctive.

(ii) If  $U_2$  is conjunctive and  $L$  is the unit interval  $[0, 1]$ , then  $\mathcal{H}_{U_1, U_2}^\wedge$  corresponds to the 2-uninorms constructed in [46]. Indeed, in this case we have  $\mathcal{H}_{U_1, U_2}^\wedge(0, 1) = \mathcal{H}_{U_1, U_2}^\wedge(0, d) = U_1(0, d) \in \{0, d\}$ . If  $\mathcal{H}_{U_1, U_2}^\wedge(0, 1) = d$ , then it corresponds to the 2-uninorm defined in Theorem 4 of [46]. If  $\mathcal{H}_{U_1, U_2}^\wedge(0, 1) = 0$ , then, by  $\mathcal{H}_{U_1, U_2}^\wedge(1, d) = d$ , this 2-uninorm corresponds to the one introduced in Theorem 5 of [46].

It is interesting to find that  $\mathcal{H}_{U_1, U_2}^\wedge$  is the weakest one among all 2-uninorms which have the same restrictions on  $[0, d]^2$  and  $[d, 1]^2$ .

**Theorem 3.4.** Let  $L$  be a bounded lattice with elements  $e_1, e_2, d$  in  $L$  such that  $0 < d < 1$  and  $0 \leq e_1 \leq d < e_2 \leq 1$ . Suppose  $U_1$  is a uninorm on  $[0, d]$  with identity element  $e_1$  and  $U_2$  is a conjunctive uninorm on  $[d, 1]$  with identity element  $e_2$ . Then  $U_1$  and  $U_2$  are, respectively, the restrictions of  $\mathcal{H}_{U_1, U_2}^\wedge$  on  $[0, d]^2$  and  $[d, 1]^2$ . Moreover, for any 2-uninorm  $\mathcal{H}$  on  $L$  which extends both  $U_1$  and  $U_2$ , we have  $\mathcal{H} \geq \mathcal{H}_{U_1, U_2}^\wedge$ .

*Proof.* By construction,  $U_1$  ( $U_2$ , resp.) is clearly the restriction of  $\mathcal{H}_{U_1, U_2}^\wedge$  on  $[0, d]^2$  ( $[d, 1]^2$ , resp.). Suppose  $\mathcal{H}$  is also a 2-uninorm which extends both  $U_1$  and  $U_2$ . Clearly, for any  $(x, y) \in [0, d]^2 \cup [d, 1]^2$ ,  $\mathcal{H}(x, y) = \mathcal{H}_{U_1, U_2}^\wedge(x, y)$ . For any  $(x, y) \in [0, 1]^2 \setminus ([0, d]^2 \cup [d, 1]^2)$ , we have

$$\mathcal{H}(x, y) \geq \mathcal{H}(x \wedge d, y \wedge d) = U_1(x \wedge d, y \wedge d) = \mathcal{H}_{U_1, U_2}^\wedge(x, y)$$

as  $\mathcal{H}$  is non-decreasing. Thus,  $\mathcal{H} \geq \mathcal{H}_{U_1, U_2}^\wedge$ . □

$\mathcal{H}_{U_1, U_2}^\wedge$	0	$a_1$	$e_1$	$a_2$	$a_3$	$d$	$a_4$	$e_2$	$a_5$	1	$b_1$	$b_2$	$c_1$	$c_2$	$c_3$	$c_4$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$a_1$	0	0	$a_1$	$a_1$	$a_1$	$a_1$	$a_1$	$a_1$	$a_1$	$a_1$	0	$a_1$	$a_1$	$a_1$	0	0
$e_1$	0	$a_1$	$e_1$	$a_2$	$a_3$	$d$	$d$	$d$	$d$	$d$	$b_1$	$d$	$a_2$	$a_3$	$b_1$	$a_1$
$a_2$	0	$a_1$	$a_2$	$a_2$	$a_3$	$d$	$d$	$d$	$d$	$d$	$b_1$	$d$	$a_2$	$a_3$	$b_1$	$a_1$
$a_3$	0	$a_1$	$a_3$	$a_3$	$a_3$	$d$	$d$	$d$	$d$	$d$	$b_1$	$d$	$a_3$	$a_3$	$b_1$	$a_1$
$d$	0	$a_1$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$b_1$	$d$	$d$	$d$	$b_1$	$a_1$
$a_4$	0	$a_1$	$d$	$d$	$d$	$d$	$a_4$	$a_4$	$a_4$	$a_4$	$b_1$	$d$	$d$	$d$	$b_1$	$a_1$
$e_2$	0	$a_1$	$d$	$d$	$d$	$d$	$a_4$	$e_2$	$a_5$	1	$b_1$	$b_2$	$d$	$d$	$b_1$	$a_1$
$a_5$	0	$a_1$	$d$	$d$	$d$	$d$	$a_4$	$a_5$	1	1	$b_1$	$b_2$	$d$	$d$	$b_1$	$a_1$
1	0	$a_1$	$d$	$d$	$d$	$d$	$a_4$	1	1	1	$b_1$	$b_2$	$d$	$d$	$b_1$	$a_1$
$b_1$	0	0	$b_1$	$b_1$	$b_1$	$b_1$	$b_1$	$b_1$	$b_1$	$b_1$	0	$b_1$	$b_1$	$b_1$	0	0
$b_2$	0	$a_1$	$d$	$d$	$d$	$d$	$d$	$b_2$	$b_2$	$b_2$	$b_1$	$d$	$d$	$d$	$b_1$	$a_1$
$c_1$	0	$a_1$	$a_2$	$a_2$	$a_3$	$d$	$d$	$d$	$d$	$d$	$b_1$	$d$	$a_2$	$a_3$	$b_1$	$a_1$
$c_2$	0	$a_1$	$a_3$	$a_3$	$a_3$	$d$	$d$	$d$	$d$	$d$	$b_1$	$d$	$a_3$	$a_3$	$b_1$	$a_1$
$c_3$	0	0	$b_1$	$b_1$	$b_1$	$b_1$	$b_1$	$b_1$	$b_1$	$b_1$	0	$b_1$	$b_1$	$b_1$	0	0
$c_4$	0	0	$a_1$	$a_1$	$a_1$	$a_1$	$a_1$	$a_1$	$a_1$	$a_1$	0	$a_1$	$a_1$	$a_1$	0	0

Table 1: The 2-uninorm  $\mathcal{H}_{U_1, U_2}^\wedge$  in Example 3.2

The above result shows that  $\mathcal{H}_{U_1, U_2}^\wedge$  is the weakest among all 2-uninorms that extend both  $U_1$  and  $U_2$ . A stronger conclusion can be obtained if  $U_2$  is selected to be the weakest uninorm on  $[d, 1]$ .

**Corollary 3.5.** *Let  $L$  be a bounded lattice with elements  $e_1, e_2, d$  such that  $0 < d < 1$  and  $0 \leq e_1 \leq d < e_2 \leq 1$ . Suppose  $U$  is an arbitrary uninorm on  $[0, d]$  and  $U_{sc}$  the weakest uninorm on  $[d, 1]$  with identity element  $e_2$  (cf. Eq.(1)). Then  $\mathcal{H}_{U, U_{sc}}^\wedge$  is the weakest among all 2-uninorms that extend  $U$  and have cutpoint  $d$  and local identity elements  $e_1, e_2$  on  $L$ .*

It is necessary to point out that for any 2-uninorm  $\mathcal{H}$  on a bounded lattice  $L$ ,  $\mathcal{H}(0, 1) = a$  is always its annihilator. Indeed, since  $\mathcal{H}(0, 0) = 0$  and  $\mathcal{H}(1, 1) = 1$ , we get  $\mathcal{H}(a, 0) = \mathcal{H}(0, a) = \mathcal{H}(0, \mathcal{H}(0, 1)) = \mathcal{H}(\mathcal{H}(0, 0), 1) = \mathcal{H}(0, 1) = a$ , and similarly  $\mathcal{H}(1, a) = \mathcal{H}(a, 1) = a$ . Then the monotonicity for every  $x \in L$  gives  $a = \mathcal{H}(a, 0) \leq \mathcal{H}(a, x) \leq \mathcal{H}(a, 1) = a$ , i.e.,  $\mathcal{H}(x, a) = \mathcal{H}(a, x) = a$ . If  $U_1$  is disjunctive in Theorem 3.1, then  $d = U_1(x, d) = \mathcal{H}_{U_1, U_2}^\wedge(x, d)$  for any  $x \in [0, d]$ , i.e.,  $d$  is also the annihilator of  $\mathcal{H}_{U_1, U_2}^\wedge$  (see Corollary 3.6). In this case,  $U_1$  is not a t-norm and therefore  $0 \leq e_1 < d$ .

In the below, for convenience, denote  $X_d = ([0, d] \cup I_d) \times [d, 1] \cup [d, 1] \times ([0, d] \cup I_d)$ .

**Corollary 3.6.** *Let  $L$  be a bounded lattice with elements  $e_1, e_2, d$  such that  $0 < d < 1$  and  $0 \leq e_1 < d < e_2 \leq 1$ . Suppose  $U_1 : [0, d]^2 \rightarrow [0, d]$  is a disjunctive uninorm with identity element  $e_1$  and  $U_2 : [d, 1]^2 \rightarrow [d, 1]$  a conjunctive uninorm with identity element  $e_2$ . Then*

$$\mathcal{H}_{U_1, U_2}^\wedge(x, y) = \begin{cases} U_2(x, y), & \text{if } (x, y) \in [d, 1]^2 \\ d, & \text{if } (x, y) \in X_d \\ U_1(x \wedge d, y \wedge d), & \text{otherwise.} \end{cases} \quad (7)$$

Clearly,  $d$  is the annihilator of the 2-uninorm  $\mathcal{H}_{U_1, U_2}^\wedge$ .

**Remark 3.7.** *Corollary 3.6 has the same conditions as Theorem 1 of [21], i.e., both requiring that  $U_1$  is a disjunctive uninorm and  $U_2$  is a conjunctive uninorm. We find that  $\mathcal{H}_{U_1, U_2}^\wedge$  differs from the 2-uninorm  $U^2$  of [21] only in the region*

$$[0, d] \times I_d \cup I_d \times [0, d] \cup I_d^2 \text{ since } U^2(x, y) = \begin{cases} U_1(x, y), & \text{if } (x, y) \in [0, d]^2 \\ U_2(x, y), & \text{if } (x, y) \in [d, 1]^2 \\ d, & \text{otherwise.} \end{cases}$$

In case  $e_1 = 0$  ( $e_2 = 1$ , resp.) in Theorem 3.1, then  $U_1$  ( $U_2$ , resp.) becomes a t-conorm (t-norm, resp.). This yields a null-uninorm (a uni-nullnorm, resp.) on  $L$ .

**Corollary 3.8.** *Let  $L$  be a bounded lattice with elements  $e, d$  such that  $0 < d < e \leq 1$  and  $0 < d < 1$ . Suppose  $U : [d, 1]^2 \rightarrow [d, 1]$  is a uninorm with identity element  $e$  and  $S : [0, d]^2 \rightarrow [0, d]$  a t-conorm. Then*

$$\mathcal{H}_{S, U}^\wedge(x, y) = \begin{cases} U(x, y), & \text{if } (x, y) \in [d, 1]^2 \\ d, & \text{if } (x, y) \in X_d \\ S(x \wedge d, y \wedge d), & \text{otherwise.} \end{cases} \quad (8)$$

Moreover, the operator  $\mathcal{H}_{S,U}^\wedge$  is a null-uninorm on  $L$  if and only if  $U$  is a conjunctive uninorm. If  $U$  is a conjunctive uninorm, then the null-uninorm  $\mathcal{H}_{S,U}^\wedge$  is also the weakest among all null-uninorms that extend  $U$  and  $S$  with cutpoint  $d$  on  $L$ .

**Remark 3.9.** In Corollary 3.8, if  $S = S_M$  and  $U = U_{sc}$ , then  $H = \mathcal{H}_{S_M, U_{sc}}^\wedge$  is the weakest among all null-uninorms on  $L$  with cutpoint  $d$ . Moreover, we have

$$H(x, y) = \begin{cases} U_{sc}(x, y), & \text{if } (x, y) \in [d, 1]^2 \\ d, & \text{if } (x, y) \in X_d \\ (x \wedge d) \vee (y \wedge d), & \text{otherwise.} \end{cases} \quad (9)$$

**Corollary 3.10.** Let  $L$  be a bounded lattice with elements  $e, d$  such that  $0 \leq e \leq d < 1$  and  $0 < d < 1$ . Suppose  $U$  is a uninorm on  $[0, d]$  with identity element  $e$  and  $T$  a  $t$ -norm on  $[d, 1]$ . Then 2-uninorm  $\mathcal{H}_{U,T}^\wedge$  on  $L$  is a uni-nullnorm. Indeed,  $\mathcal{H}_{U,T}^\wedge$  is the weakest among all uni-nullnorms that extend  $U$  and  $T$  with cutpoint  $d$  on  $L$ .

**Remark 3.11.** (i) The result of Corollary 3.10 is the one of Theorem 3.1 in [39].

(ii) In Corollary 3.10, if  $U = U_{sc}$  and  $T = T_D$ , then  $H = \mathcal{H}_{U_{sc}, T_D}^\wedge$  is the weakest uni-nullnorm among all uni-nullnorms on  $L$  with cutpoint  $d$  and local identity element  $e$ .

Taking  $e = 1$  in Corollary 3.8 or  $e = 0$  in Corollary 3.10, we obtain the nullnorm on  $L$  constructed in [22].

### 3.2 The strongest 2-uninorms $\mathcal{H}_{U_1, U_2}^\vee$

In the previous section, we have defined a 2-uninorm which is the weakest 2-uninorm with given underlying functions. Dually, we can construct the strongest one in a similar way. The construction is illustrated in Fig. 3.

$I_d$	$U_2(d, y \vee d)$	$U_2(x, d \vee y)$	$U_2(x \vee d, y \vee d)$
1			
$e_2$	$U_2(d, y)$	$U_2$	$U_2(x \vee d, y)$
$d$			
$e_1$	$U_1$	$U_2(x, d)$	$U_2(x \vee d, d)$
0	$e_1$	$d$	$e_2$
			1
			$I_d$

Figure 3: The structure of the 2-uninorm  $\mathcal{H}_{U_1, U_2}^\vee$  in Eq. (10)

**Theorem 3.12.** Let  $L$  be a bounded lattice with elements  $e_1, e_2, d$  such that  $0 \leq e_1 < d \leq e_2 \leq 1$  and  $0 < d < 1$ . Suppose  $U_1 : [0, d]^2 \rightarrow [0, d]$  is a uninorm with identity element  $e_1$  and  $U_2 : [d, 1]^2 \rightarrow [d, 1]$  a uninorm with identity element  $e_2$ . The binary operator  $\mathcal{H}_{U_1, U_2}^\vee$  given by

$$\mathcal{H}_{U_1, U_2}^\vee(x, y) = \begin{cases} U_1(x, y), & \text{if } (x, y) \in [0, d]^2 \\ U_2(x \vee d, y \vee d), & \text{otherwise} \end{cases} \quad (10)$$

is a 2-uninorm on  $L$  if and only if  $U_1$  is a disjunctive uninorm.

**Remark 3.13.** (i) In Theorem 3.12,  $d$  is required to be greater than  $e_1$ . Otherwise,  $U_1$  is a  $t$ -norm and it cannot be disjunctive.

(ii) Suppose  $U_1$  is disjunctive. If  $L$  is the unit interval  $[0, 1]$ , then  $\mathcal{H}_{U_1, U_2}^\vee$  corresponds to the 2-uninorms constructed in [46, Theorems 4 & 6].

The 2-uninorm  $\mathcal{H}_{U_1, U_2}^\vee$  is the strongest among all 2-uninorms which extend  $U_1$  and  $U_2$  with cutpoint  $d$ .

**Theorem 3.14.** Let  $L$  be a bounded lattice with elements  $e_1, e_2, d$  such that  $0 \leq e_1 < d \leq e_2 \leq 1$  and  $0 < d < 1$ . Suppose  $U_1$  is a disjunctive uninorm on  $[0, d]$  with identity element  $e_1$  and  $U_2$  a uninorm on  $[d, 1]$  with identity element  $e_2$ . Then  $U_1$  and  $U_2$  are, respectively, the restrictions of  $\mathcal{H}_{U_1, U_2}^\vee$  on  $[0, d]^2$  and  $[d, 1]^2$ . Moreover, for any 2-uninorm  $\mathcal{H}$  on  $L$  which extends both  $U_1$  and  $U_2$ , we have  $\mathcal{H} \leq \mathcal{H}_{U_1, U_2}^\vee$ .

A stronger conclusion can be reached if  $U_1$  is the strongest uninorm on  $[0, d]$ .

**Corollary 3.15.** Let  $L$  be a bounded lattice with elements  $e_1, e_2, d$  such that  $0 \leq e_1 < d < e_2 \leq 1$  and  $0 < d < 1$ . Suppose  $U$  is an arbitrary uninorm on  $[d, 1]$  with identity element  $e_2$  and  $U_{1d}$  the strongest uninorm on  $[0, d]$  with identity element  $e_1$  (cf. Eq.(2)). Then  $\mathcal{H}_{U_{1d}, U}^\vee$  is the strongest among all 2-uninorms on  $L$  that extend  $U$  and have cutpoint  $d$  and local identity elements  $e_1, e_2$ .

In case  $U_2$  is conjunctive, we have a finer representation for  $\mathcal{H}_{U_1, U_2}^\vee$ , which directly implies that  $d$  is the annihilator of the 2-uninorm  $\mathcal{H}_{U_1, U_2}^\vee$ . We denote  $Y_d = [0, d] \times (I_d \cup [d, 1]) \cup (I_d \cup [d, 1]) \times [0, d]$ .

**Corollary 3.16.** Let  $L$  be a bounded lattice with elements  $e_1, e_2, d$  such that  $0 \leq e_1 < d < e_2 \leq 1$  and  $0 < d < 1$ . Suppose  $U_1$  is a disjunctive uninorm on  $[0, d]$  with identity element  $e_1$  and  $U_2$  a conjunctive uninorm on  $[d, 1]$  with identity element  $e_2$ . Then

$$\mathcal{H}_{U_1, U_2}^\vee(x, y) = \begin{cases} U_1(x, y), & \text{if } (x, y) \in [0, d]^2 \\ d, & \text{if } (x, y) \in Y_d \\ U_2(x \vee d, y \vee d), & \text{otherwise.} \end{cases} \quad (11)$$

Clearly,  $d$  is the annihilator of  $\mathcal{H}_{U_1, U_2}^\vee$ .

**Remark 3.17.** Similar to Remark 3.7, when compared with the 2-uninorm  $U^2$  in [21], our 2-uninorm  $\mathcal{H}_{U_1, U_2}^\vee$  differs only in the region  $[d, 1] \times I_d \cup I_d \times [d, 1] \cup I_d^2$ , where  $U^2$  takes the fixed value  $d$ .

**Corollary 3.18.** Let  $L$  be a bounded lattice with elements  $e, d$  such that  $0 \leq e < d < 1$  and  $0 < d < 1$ . Suppose  $T : [d, 1]^2 \rightarrow [d, 1]$  is a  $t$ -norm and  $U : [0, d]^2 \rightarrow [0, d]$  a uninorm with identity element  $e$ . Then

$$\mathcal{H}_{U, T}^\vee(x, y) = \begin{cases} U(x, y), & \text{if } (x, y) \in [0, d]^2 \\ d, & \text{if } (x, y) \in Y_d \\ T(x \vee d, y \vee d), & \text{otherwise.} \end{cases} \quad (12)$$

Moreover,  $\mathcal{H}_{U, T}^\vee$  is a uni-nullnorm on  $L$  if and only if  $U$  is a disjunctive uninorm. If  $U$  is a disjunctive uninorm, then  $\mathcal{H}_{U, T}^\vee$  is indeed the strongest uni-nullnorm among all uni-nullnorms on  $L$  that extend  $U$  and  $T$  with cutpoint  $d$ .

**Remark 3.19.** (i) The result of Corollary 3.18 is the one of Theorem 4.1 in [39].

(ii) In Corollary 3.18, if  $U = U_{1d}$  and  $T = T_M$ , then  $G = \mathcal{H}_{U_{1d}, T_M}^\vee$  is the strongest uni-nullnorm with cutpoint  $d$  and local identity element  $e$ . Moreover, we have

$$G(x, y) = \begin{cases} U_{1d}(x, y), & \text{if } (x, y) \in [0, d]^2 \\ d, & \text{if } (x, y) \in Y_d \\ (x \vee d) \wedge (y \vee d), & \text{otherwise.} \end{cases} \quad (13)$$

**Corollary 3.20.** Let  $L$  be a bounded lattice,  $e, d \in L$ ,  $0 \leq d < e \leq 1$  and  $0 < d < 1$ . Suppose  $U$  is a uninorm on  $[d, 1]$  with identity element  $e$  and  $S$  a  $t$ -conorm on  $[0, d]$ . The 2-uninorm  $\mathcal{H}_{S, U}^\vee$  on  $L$  is a null-uninorm. Indeed, it is the strongest null-uninorm on  $L$  which extends both  $S$  and  $U$  with cutpoint  $d$ .

**Remark 3.21.** In Corollary 3.20, if  $S = S_D$  and  $U = U_{1d}$ , then  $G = \mathcal{H}_{S_D, U_{1d}}^\vee$  is the strongest null-uninorm.

Taking  $e = 1$  in Corollary 3.18 or  $e = 0$  in Corollary 3.20, we obtain the nullnorm on  $L$  constructed in [22].

To illustrate the construction, we also give an example.

**Example 3.22.** Let  $L_1 = \{0, a_1, e_1, a_2, a_3, d, a_4, e_2, a_5, 1, b_1, b_2, c_1, c_2, c_3, c_4\}$  be the bounded lattice as defined in Fig. 2. Suppose  $T_{e_1}$  is a  $t$ -norm on  $[0, e_1]$  and  $S_{e_2}$  a  $t$ -conorm on  $[e_2, 1]$ . Consider the disjunctive uninorm  $U_1$  on  $[0, d]$  with identity element  $e_1$  given by

$$U_1(x, y) = \begin{cases} T_{e_1}(x, y), & \text{if } (x, y) \in [0, e_1]^2 \\ y, & \text{if } (x, y) \in [0, e_1] \times ([0, d] \setminus [0, e_1]) \\ x, & \text{if } (x, y) \in ([0, d] \setminus [0, e_1]) \times [0, e_1] \\ h(x) \vee h(y), & \text{if } (x, y) \in ([0, d] \setminus [0, e_1])^2, \end{cases}$$



and the conjunctive uninorm  $U_2$  on  $[d, 1]$  with identity element  $e_2$  given by

$$U_2(x, y) = \begin{cases} S_{e_2}(x, y), & \text{if } (x, y) \in [e_2, 1]^2 \\ y, & \text{if } (x, y) \in [e_2, 1] \times ([d, 1] \setminus [e_2, 1]) \\ x, & \text{if } (x, y) \in ([d, 1] \setminus [e_2, 1]) \times [e_2, 1] \\ g(x) \wedge g(y), & \text{otherwise,} \end{cases}$$

where  $h$  is a closure operator and  $g$  is an interior operator, respectively (please see the definitions of closure operators and interior operators in [34]). Then  $U_1$  and  $U_2$  are uninorms (see [Theorems 4.1 and 5.1, [34]]).

Now, select  $T_{e_1} = T_M$ ,  $h(x) = x$ ,  $S_{e_2} = S_D$ ,  $g(x) = x$ . Then we obtain 2-uninorm  $\mathcal{H}_{U_1, U_2}^\vee$  on  $L_1$  given in Table 2.

$\mathcal{H}_{U_1, U_2}^\vee$	0	$a_1$	$e_1$	$a_2$	$a_3$	$d$	$a_4$	$e_2$	$a_5$	1	$b_1$	$b_2$	$c_1$	$c_2$	$c_3$	$c_4$
0	0	0	0	$a_2$	$a_3$	$d$	$d$	$d$	$d$	$d$	$b_1$	$d$	$d$	$d$	$d$	$d$
$a_1$	0	$a_1$	$a_1$	$a_2$	$a_3$	$d$	$d$	$d$	$d$	$d$	$b_1$	$d$	$d$	$d$	$d$	$d$
$e_1$	0	$a_1$	$e_1$	$a_2$	$a_3$	$d$	$d$	$d$	$d$	$d$	$b_1$	$d$	$d$	$d$	$d$	$d$
$a_2$	$a_2$	$a_2$	$a_2$	$a_2$	$a_3$	$d$	$d$	$d$	$d$	$d$	$a_2$	$d$	$d$	$d$	$d$	$d$
$a_3$	$a_3$	$a_3$	$a_3$	$a_3$	$a_3$	$d$	$d$	$d$	$d$	$d$	$a_3$	$d$	$d$	$d$	$d$	$d$
$d$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$d$
$a_4$	$d$	$d$	$d$	$d$	$d$	$d$	$a_4$	$a_4$	$a_4$	$a_4$	$d$	$d$	$a_4$	$d$	$a_4$	$a_4$
$e_2$	$d$	$d$	$d$	$d$	$d$	$d$	$a_4$	$e_2$	$a_5$	1	$d$	$b_2$	$a_4$	$b_2$	$a_4$	1
$a_5$	$d$	$d$	$d$	$d$	$d$	$d$	$a_4$	$a_5$	1	1	$d$	$b_2$	$a_4$	$b_2$	$a_4$	1
1	$d$	$d$	$d$	$d$	$d$	$d$	$a_4$	1	1	1	$d$	$b_2$	$a_4$	$b_2$	$a_4$	1
$b_1$	$b_1$	$b_1$	$b_1$	$a_2$	$a_3$	$d$	$d$	$d$	$d$	$d$	$b_1$	$d$	$d$	$d$	$d$	$d$
$b_2$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$b_2$	$b_2$	$b_2$	$d$	$b_2$	$d$	$b_2$	$d$	$b_2$
$c_1$	$d$	$d$	$d$	$d$	$d$	$d$	$a_4$	$a_4$	$a_4$	$a_4$	$d$	$d$	$a_4$	$d$	$a_4$	$a_4$
$c_2$	$d$	$d$	$d$	$d$	$d$	$d$	$d$	$b_2$	$b_2$	$b_2$	$d$	$b_2$	$d$	$b_2$	$d$	$b_2$
$c_3$	$d$	$d$	$d$	$d$	$d$	$d$	$a_4$	$a_4$	$a_4$	$a_4$	$d$	$d$	$a_4$	$d$	$a_4$	$a_4$
$c_4$	$d$	$d$	$d$	$d$	$d$	$d$	$a_4$	1	1	1	$d$	$b_2$	$a_4$	$b_2$	$a_4$	1

Table 2: The 2-uninorm  $\mathcal{H}_{U_1, U_2}^\vee$  in Example 3.22

In [45], the authors discuss the  $(U^2, N)$ -operation  $I_{U^2, N}$  on the unit interval derived from a 2-uninorm  $U^2$  and a fuzzy negation  $N$ , where  $I_{U^2, N}(x, y) = U^2(N(x), y)$ . They prove that  $I_{U^2, N}$  is a fuzzy implication if and only if the related 2-uninorm  $U^2$  is disjunctive, i.e.,  $U^2(0, 1) = 1$ .<sup>1</sup>

Using our 2-uninorms, we can obtain fuzzy implications on bounded lattices as well. In fact, if  $U_1$  and  $U_2$  are disjunctive, then  $\mathcal{H}_{U_1, U_2}^\vee$  is a disjunctive 2-uninorm. Therefore, analogously as in [45], we can use such  $\mathcal{H}_{U_1, U_2}^\vee$  to construct  $(H, N)$ -implications  $I_{H, N}$  on bounded lattices by  $I_{H, N}(x, y) = \mathcal{H}_{U_1, U_2}^\vee(N(x), y)$ .

**Example 3.23.** Let  $L_2 = \{0, a_1, e_1, d, e_2, a_2, b_1, b_2, 1\}$  be the bounded lattice shown in Fig. 4. Define  $N : [0, 1] \rightarrow [0, 1]$  by

$$N(x) = \begin{cases} 1, & \text{if } x = 0 \\ a_2, & \text{if } x = a_1 \\ e_2, & \text{if } x = e_1 \\ d, & \text{if } x \in \{d, b_1, b_2\} \\ e_1, & \text{if } x = e_2 \\ a_1, & \text{if } x = a_2 \\ 0, & \text{if } x = 1. \end{cases}$$

Clearly,  $N$  is a fuzzy negation on  $L_2$ . Let  $U_1$  and  $U_2$  be given, respectively, by

$$U_1(x, y) = \begin{cases} T_M(x, y), & \text{if } (x, y) \in [0, e_1]^2 \\ y, & \text{if } (x, y) \in [0, e_1] \times ([0, d] \setminus [0, e_1]) \\ x, & \text{if } (x, y) \in ([0, d] \setminus [0, e_1]) \times [0, e_1] \\ x \vee y, & \text{if } (x, y) \in ([0, d] \setminus [0, e_1])^2, \end{cases}$$

<sup>1</sup>For definitions of fuzzy implications and fuzzy negations, the reader can refer to [2, 45].

$$U_2(x, y) = \begin{cases} T_D(x, y), & \text{if } (x, y) \in [d, e_2]^2 \\ y, & \text{if } (x, y) \in [d, e_2] \times ([d, 1] \setminus [d, e_2]) \\ x, & \text{if } (x, y) \in ([d, 1] \setminus [d, e_2]) \times [d, e_2] \\ S_D(x \vee e_2, y \vee e_2), & \text{if } (x, y) \in ([d, 1] \setminus [d, e_2])^2. \end{cases}$$

Then  $U_1$  and  $U_2$  are disjunctive, and fuzzy implication  $I_{H,N}$  on  $L_2$  is given in Table 3. The fuzzy implication  $I_{H,N}$  can be useful for lattice-valued fuzzy set theory just as  $(S,N)$ -implications on bounded lattices [2].

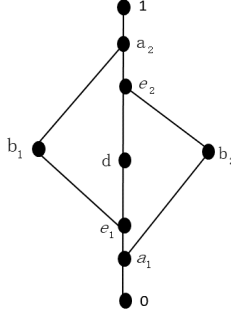


Figure 4: Bounded lattice  $L_2$

$I_{H,N}$	0	$a_1$	$e_1$	$d$	$e_2$	$a_2$	1	$b_1$	$b_2$
0	1	1	1	1	1	1	1	1	1
$a_1$	$a_2$	$a_2$	$a_2$	$a_2$	$a_2$	1	1	1	$a_2$
$e_1$	$d$	$d$	$d$	$d$	$e_2$	$a_2$	1	$a_2$	$e_2$
$d$	$d$	$d$	$d$	$d$	$d$	$a_2$	1	$a_2$	$d$
$e_2$	0	$a_1$	$e_1$	$d$	$d$	$a_2$	1	$a_2$	$d$
$a_2$	0	$a_1$	$a_1$	$d$	$d$	$a_2$	1	$a_2$	$d$
1	0	0	0	$d$	$d$	$a_2$	1	$a_2$	$d$
$b_1$	$d$	$d$	$d$	$d$	$d$	$a_2$	1	$a_2$	$d$
$b_2$	$d$	$d$	$d$	$d$	$d$	$a_2$	1	$a_2$	$d$

Table 3: The  $I_{U^2,N}$  in Example 3.23

## 4 Conclusion

For a bounded lattice  $L$  with elements  $0 \leq e_1 \leq d \leq e_2 \leq 1$  and  $0 < d < 1$ , suppose  $U_1$  is a uninorm on  $[0, d]$  with identity element  $e_1$  and  $U_2$  a uninorm on  $[d, 1]$  with identity element  $e_2$ . We constructed 2-uninorms on  $L$  by extending both  $U_1$  and  $U_2$ . The two 2-uninorms,  $\mathcal{H}_{U_1, U_2}^\wedge$  and  $\mathcal{H}_{U_1, U_2}^\vee$  are the weakest and the strongest 2-uninorms on  $L$  among all 2-uninorms that extend  $U_1$  and  $U_2$ , respectively. Our constructions have also been adapted to construct uni-nullnorms and null-uninorms on  $L$ .

As bounded lattices can be very different from the unit interval  $[0, 1]$ , many nice properties fulfilled by 2-uninorms on  $[0, 1]$  may not hold. It is interesting to extend the classifications obtained in [1] and [46] to general bounded lattices. In addition, it is easy to see that neither  $\mathcal{H}_{U_1, U_2}^\wedge$  nor  $\mathcal{H}_{U_1, U_2}^\vee$  is idempotent, in general. In the future, we will investigate idempotent 2-uninorms on bounded lattices.

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## Construction of 2-uniforms on bounded lattices

A. Xie and Z. Yi

## ساخت ۲-تک نرم‌ها روی شبکه‌های محدود

**چکیده.** تک نرم‌ها و تک نرم‌های پوچ ۲-تک نرم‌های خاص هستند. در این کار ۲-تک نرم‌ها را روی شبکه‌های محدود می‌سازیم. فرض کنید  $L$  یک شبکه محدود با یک عنصر غیربدیهی  $d$  باشد. برای دو تک نرم  $U_1$  و  $U_2$  که به ترتیب روی زیرشبکه‌های  $[0, d]$  و  $[d, 1]$  تعریف شده‌اند، این مقاله دو روش برای ساخت عملگرهای دوتایی روی  $L$  ارائه می‌دهد که هم  $U_1$  و هم  $U_2$  را گسترش می‌دهند. نشان می‌دهیم که ساختار اول ما یک ۲-تک نرم روی  $L$  است اگر و فقط اگر  $U_2$  عطفی باشد و ساختار دوم ما یک ۲-تک نرم روی  $L$  است اگر و فقط اگر  $U_1$  فصلی باشد. علاوه بر این، ثابت می‌کنیم که دو ۲-تک نرم‌ها به ترتیب ضعیف‌ترین و قوی‌ترین ۲-تک نرم در بین تمام ۲-تک نرم‌ها هستند که تحدید آنها به  $[0, d]^2$  و  $[d, 1]^2$  به ترتیب  $U_1$  و  $U_2$  می‌باشد.