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Original paper

Construction of 2-uninorms on bounded lattices

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Abstract

Uninorms and nullnorms are special 2-uninorms. In this work, we construct 2-uninorms on bounded lattices. Let L be a bounded lattice with a nontrivial element d. Given two uninorms U_1 and U_2, defined on sublattices $[0, d]$ and $[d, 1]$, respectively, this paper presents two methods for constructing binary operators on L which extend both U_1 and U_2. We show that our first construction is a 2-uninorm on L if and only if U_2 is conjunctive and our second construction is a 2-uninorm on L if and only if U_1 is disjunctive. Moreover, we prove that the two 2-uninorms are, respectively, the weakest and the strongest 2-uninorm among all 2-uninorms, the restrictions of which on $[0, d]^2$ and $[d, 1]^2$ are respectively U_1 and U_2.

Keywords: Bounded lattices, 2-uninorms, uninorms, nullnorms.

1 Introduction

By allowing the *identity element* different from 0 and 1, Yager and Rybalov [42] introduced *uninorms* on the unit interval, which include triangular norms [27] (t-norms henceforth) and triangular conorms [27] (t-conorms henceforth) as special classes. Since then, uninorms have become important aggregation operators, which have applications in expert systems [16, 20, 36], fuzzy logic [28], and fuzzy systems modeling [41]. As an interesting mathematical construction, uninorms have been investigated by many researchers, see [15, 18, 23, 29, 31]. By allowing annihilator to be put anywhere in $[0, 1]$, *nullnorms* (or *t-operators*) [5, 30] are another generalization of both t-norms and t-conorms. Later, Akella [1] proposed the important notion of a *2-uninorm*, which has two *local* identity elements in $[0, 1]$. 2-uninorms generalize uninorms and nullnorms and they have been applied in related fields such as [13]. Until now, 2-uninorms have attracted some research interests, see [19, 32, 33, 38, 44, 45, 46].

As a bounded lattice, the unit interval is sometimes too special and often cannot be adopted as the underlying value domain of many decision making tasks. Recently, several researchers have considered similar constructions on general bounded lattices. A series of works have been done for uninorms [4, 6, 7, 9, 10, 12, 14, 26, 34, 40], nullnorms [8, 11, 22, 24, 25], uni-nullnorms [37, 39, 43] and null-uninorms [43].

Ertuğrul [21] considered 2-uninorms on a general bounded lattice. Let $(L, \leq)$ be a bounded lattice. Suppose d is a nontrivial element of L, U_1 (U_2, resp.) a disjunctive (conjunctive, resp.) uninorm on $[0, d]$ ($[d, 1]$, resp.). Ertuğrul [21] defined a 2-uninorm on L by extending U_1 and U_2 in a natural way. Furthermore, he showed that the construction could fail to be a 2-uninorm if either the disjunctivity or conjunctivity is not satisfied.

In this work, we consider a similar problem as in [21]. Let $(L, \leq)$ be a bounded lattice. Suppose d is a nontrivial element of L. For any uninorm U_1 on $[0, d]$ and any uninorm U_2 on $[d, 1]$, we construct two operators $\mathcal{H}^{\wedge}_{U_1,U_2}$ and $\mathcal{H}^{\vee}_{U_1,U_2}$ and show that $\mathcal{H}^{\wedge}_{U_1,U_2}$ ($\mathcal{H}^{\vee}_{U_1,U_2}$, resp.) is a 2-uninorm if and only if U_2 is conjunctive (U_1 is disjunctive, resp.). Moreover, we prove that for any 2-uninorm $\mathcal{H}$ on L, if U_1 and U_2 are, respectively, the restrictions of $\mathcal{H}$ on $[0, d]^2$ and $[d, 1]^2$, then $\mathcal{H}^{\wedge}_{U_1,U_2} \leq \mathcal{H} \leq \mathcal{H}^{\vee}_{U_1,U_2}$. Our 2-uninorms on bounded lattices generalize 2-uninorms on the unit interval and also can be used to obtain uni-nullnorms, null-uninorms and nullnorms on bounded lattices. Moreover, using $\mathcal{H}^{\vee}_{U_1,U_2}$, we can obtain fuzzy implications on bounded lattices, which can be applied in lattice-valued fuzzy set theory.

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In the remainder of this work, we first recall some preliminaries in Section 2, then present our constructions and main results in Section 3. A short conclusion as well as an outlook for future research is presented in Section 4.

2 Preliminaries

Our reference to basic notions and terminologies of lattice theory is [3]. Suppose $(L, \leq)$ is a lattice. The binary minimum (meet) and maximum (join) operations on L are denoted by $\wedge$ and $\vee$, respectively. A lattice $(L, \leq)$ is called a *bounded lattice* if there exist two elements 0 and 1 in L such that $0 \leq x \leq 1$ for any $x \in L$. We call 0 and 1, respectively, the bottom and the top of L. For $a_1, a_2 \in L$ with $a_1 < a_2$, we define $[a_1, a_2] \equiv \{x \mid a_1 \leq x \leq a_2\}$. Similarly, we can define (a_1, a_2), $(a_1, a_2]$ and $[a_1, a_2)$. In addition, we define $I_d \equiv \{x \in L | x \text{ is } \textit{incomparable} \text{ with } d\}$.

Let $(L, \leq)$ be a lattice. Suppose $F : L^2 \to L$ is a binary operator on L. Assume L_1 is a sublattice of L, i.e., both $a \vee b$ and $a \wedge b$ are in L_1 for any $a, b \in L_1$. The *restriction* of F to L_1^2 is denoted as $F|_{L_1^2}$. In general, $F(a, b)$ is not necessarily an element in L_1 despite that a, b are both in L_1. In case that L_1 is closed under F, i.e., $F(a, b) \in L_1$ for any $a, b \in L_1$, we write $F|_{L_1}$ to denote the restriction of F to L_1^2, which is a binary operator on L_1. Suppose there is another binary operation F' on L_1 such that $F'(a, b) = F(a, b)$ for all $a, b \in L_1$. Then F is called an *extension* of F' on L, or F extends F' on L.

In the remainder of this paper, we always denote a bounded lattice $(L, \leq, 0, 1)$ simply as L.

All operators considered in this paper are AMC operators in the following sense.

Definition 2.1. [1] *Assume L is a bounded lattice. An operator $F : L^2 \to L$ is an AMC operator if F is associative, commutative, and non-decreasing in both variables.*

Definition 2.2. [17, 27] *Assume L is a bounded lattice. An AMC operator F on L is called a triangular norm (t-norm) if 1 is the identity element of F, i.e., $F(1, a) = a$ for any $a \in L$. Analogously, we say F is a triangular conorm (t-conorm) if 0 is the identity element of F, i.e., $F(0, a) = a$ for any $a \in L$.*

The following example gives two special t-norms (t-conorms).

Example 2.3. [26, 27] *Assume L is a bounded lattice. Define*

$$T_M(a, b) = a \wedge b,$$

$$T_D(a, b) = \begin{cases} a, & \text{if } b = 1 \\ b, & \text{if } a = 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$S_D(a, b) = \begin{cases} a, & \text{if } b = 0 \\ b, & \text{if } a = 0 \\ 1, & \text{otherwise,} \end{cases}$$

$$S_M(a, b) = a \vee b.$$

T_M (T_D, resp.) is the strongest (weakest, resp.) t-norm on L and S_D (S_M, resp.) is the strongest (weakest, resp.) t-conorm on L.

Both t-norms and t-conorms are special nullnorm and uninorm operators.

Definition 2.4. [25, 27, 30] *A binary operator $F : L^2 \to L$ on a bounded lattice L is a nullnorm on L if it is an AMC operator and has an* annihilator *$b \in L$ such that $F(0, x) = x$ for any $x \leq b$ and $F(1, y) = y$ for any $y \geq b$.*

Every t-conorm S is a nullnorm with annihilator 1 and every t-norm T is a nullnorm with annihilator 0.

Definition 2.5. [26, 27] *A binary operator $U : L^2 \to L$ on a bounded lattice L is called a uninorm on L if U is an AMC operator and has an identity element e in L, i.e., $U(e, x) = x$ for all $x \in L$. If $U(0, 1) = 0$, we say U is conjunctive; if $U(0, 1) = 1$, we say U is disjunctive.*

Every t-conorm S (t-norm T, resp.) on L is a uninorm with identity element 0 (1, resp.).

Let L be a bounded lattice and $e \in L \setminus \{0,1\}$. Then $U_{sc} : L^2 \to L$ and $U_{sc} : L^2 \to L$, respectively, are the weakest and strongest uninorms on L withe neutral element e [26], where

$$U_{sc}(x,y) = \begin{cases} x \vee y, & \text{if } (x,y) \in [e,1]^2 \\ x \wedge y, & \text{if } (x,y) \in [0,e) \times [e,1] \cup [e,1] \times [0,e) \\ y, & \text{if } (x,y) \in [e,1] \times I_e \\ x, & \text{if } (x,y) \in I_e \times [e,1] \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

$$U_{ld}(x,y) = \begin{cases} x \wedge y, & \text{if } (x,y) \in [0,e]^2 \\ x \vee y, & \text{if } (x,y) \in [0,e) \times [e,1] \cup [e,1] \times [0,e) \\ y, & \text{if } (x,y) \in [0,e] \times I_e \\ x, & \text{if } (x,y) \in I_e \times [0,e] \\ 1, & \text{otherwise.} \end{cases} \tag{2}$$

Obviously, U_{sc} is conjunctive and U_{ld} is disjunctive.

Proposition 2.6. [26] *Suppose U is a uninorm on a bounded lattice L with identity element $e \neq 0, 1$. Then the restriction of U on $[0,e]^2$ ($[e,1]^2$, resp.) is a t-norm (t-conorm, resp.) on $[0,e]$ ($[e,1]$, resp.).*

Definition 2.7. [1, 46] *A binary operator $\mathcal{H} : L^2 \to L$ on a bounded lattice L is a 2-uninorm if $\mathcal{H}$ is an AMC operator and there exist e_1, e_2 and $d \in (0,1)$ in L such that $0 \le e_1 \le d \le e_2 \le 1$ and $\mathcal{H}(x, e_1) = x$ for any $x \le d$ and $\mathcal{H}(y, e_2) = y$ for any $y \ge d$. We call d the cutpoint and call e_1, e_2 the first and, respectively, the second local identity elements of $\mathcal{H}$.*

Definition 2.8. [35] *A 2-uninorm with $e_2 = 1$ is called a uni-nullnorm, and a 2-uninorm with $e_1 = 0$ is called a null-uninorm.*

Assume $\mathcal{H}$ is a 2-uninorm on L with local identity elements $e_1 \le e_2$ and cutpoint d. Let U_1 and U_2 be the restrictions of $\mathcal{H}$ to $[0,d]^2$ and $[d,1]^2$, respectively. Clearly, U_1 (U_2, resp.) is a uninorm on $[0,d]$ ($[d,1]$, resp.) and e_1 (e_2, resp.) is its identity element.

Obviously, uninorms, nullnorms, uni-nullnorms and null-uninorms are all special 2-uninorms.

Theorem 2.9. [21] *Let L be a bounded lattice, $U_1 : [0,d]^2 \to [0,d]$ be a disjunctive uninorm with neutral element e_1 and $U_2 : [d,1]^2 \to [d,1]$ be a conjunctive uninorm with neutral element e_2. Then the function $U^2 : L^2 \to L$ given by*

$$U^2(x,y) = \begin{cases} U_1(x,y), & (x,y) \in [0,d]^2 \\ U_2(x,y), & (x,y) \in [d,1]^2 \\ d, & \textit{otherwise} \end{cases}$$

is a 2-uninorm.

3 2-uninorms on bounded lattices

Let L be a bounded lattice. Suppose $e_1, e_2, d \in L$ with $0 \le e_1 \le d \le e_2 \le 1$ and $0 < d < 1$. Assume further that U_1 is a uninorm on $[0,d]$ with identity element e_1 and U_2 is a uninorm on $[d,1]$ with identity element e_2. We give two methods for constructing 2-uninorms by extending U_1 and U_2. The first extension, denoted $\mathcal{H}^{\wedge}_{U_1,U_2}$, is a 2-uninorm if and only if U_2 is a conjunctive uninorm; the second extension, denoted $\mathcal{H}^{\vee}_{U_1,U_2}$, is a 2-uninorm if and only if U_1 is a disjunctive uninorm.

3.1 The weakest 2-uninorm $\mathcal{H}^{\wedge}_{U_1,U_2}$

The construction of the first extension is illustrated in Fig. 1.

Theorem 3.1. *Let L be a bounded lattice. Suppose $e_1, e_2, d \in L$ with $0 < d < 1$ and $0 \le e_1 \le d < e_2 \le 1$. Assume further that U_1 is a uninorm on $[0,d]$ with identity element e_1 and U_2 is a uninorm on $[d,1]$ with identity element e_2. Then the operator $\mathcal{H}^{\wedge}_{U_1,U_2}$ given by*

$$\mathcal{H}^{\wedge}_{U_1,U_2}(x,y) = \begin{cases} U_2(x,y), & \text{if } (x,y) \in [d,1]^2 \\ U_1(x \wedge d, y \wedge d), & \text{otherwise} \end{cases} \tag{3}$$

is a 2-uninorm on L if and only if U_2 is a conjunctive uninorm.

Proof. To simplify the presentation, we use, in (and only in) this proof, $x \diamond y$ to denote the binary operation $\mathcal{H}^{\wedge}_{U_1,U_2}$ of any two elements $x, y \in L$, i.e., $x \diamond y \equiv \mathcal{H}^{\wedge}_{U_1,U_2}(x,y)$.

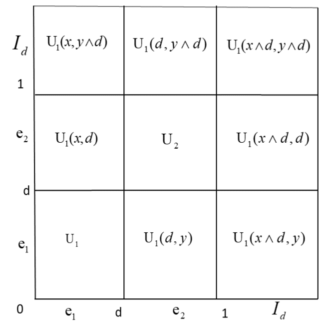

Figure 1: The structure of the 2-uninorm $\mathcal{H}^{\wedge}_{U_1,U_2}$ in Eq. (3)

Necessity Suppose $\mathcal{H}^{\wedge}_{U_1,U_2}$ is a 2-uninorm. We show U_2 is conjunctive, i.e., $U_2(d,1) = d$. From the definition of $\diamond$, we have $1 \diamond 1 = U_2(1,1) = 1$ and $e_1 \diamond 1 = U_1(e_1 \wedge d, d \wedge 1) = d$. By the associativity of $\diamond$, we have

$$U_2(d,1) = d \diamond 1 = (e_1 \diamond 1) \diamond 1 = e_1 \diamond (1 \diamond 1) = e_1 \diamond 1 = d.$$

Sufficiency Suppose U_2 is conjunctive. Then $d \diamond 1 = U_2(d,1) = d$. By $U_1(e_1,d) = d = U_2(e_2,d)$, we have $U_1(d,d) = d = U_2(d,d)$ and $U_2(x,d) = d$ for any $x \geq d$.

We prove that $\mathcal{H}^{\wedge}_{U_1,U_2}$ is a 2-uninorm with cutpoint d and local identity elements e_1, e_2.

Note that we can rewrite $\mathcal{H}^{\wedge}_{U_1,U_2}$ as

$$x \diamond y = \begin{cases} U_1(x,y), & \text{if } (x,y) \in [0,d]^2 \\ U_2(x,y), & \text{if } (x,y) \in [d,1]^2 \\ U_1(d, y \wedge d), & \text{if } (x,y) \in [d,1] \times I_d \\ U_1(x \wedge d, d), & \text{if } (x,y) \in I_d \times [d,1] \\ U_1(x,d), & \text{if } (x,y) \in [0,d] \times [d,1] \\ U_1(d,y), & \text{if } (x,y) \in [d,1] \times [0,d] \\ U_1(x, y \wedge d), & \text{if } (x,y) \in [0,d] \times I_d \\ U_1(x \wedge d, y), & \text{if } (x,y) \in I_d \times [0,d] \\ U_1(x \wedge d, y \wedge d), & \text{if } (x,y) \in I_d^2. \end{cases} \tag{4}$$

Clearly, $\diamond$ is commutative and satisfies $x \diamond e_1 = U_1(x,e_1) = x$ for any $x \in [0,d]$ and $x \diamond e_2 = U_2(x,e_2) = x$ for any $x \in [d,1]$. It remains to prove its monotonicity and associativity. Let us first consider the monotonicity. For any $x,y,z \in L$, suppose $x \leq y$. We show $x \diamond z \leq y \diamond z$. The monotonicity clearly holds if $x,y,z \in [d,1]$. Since $x \leq y$ we see that if $y \notin [d,1]$, then also $x \notin [d,1]$ and $x \diamond z = U_1(x \wedge d, z \wedge d) \leq U_1(y \wedge d, z \wedge d) = y \diamond z$. The same holds if $z \notin [d,1]$. Finally, if $x \notin [d,1]$ and $y,z \in [d,1]$, then $x \diamond z = U_1(x \wedge d, y \wedge d) \leq d \leq U_2(y,z) = y \diamond z$.

To show the associativity, we only need to prove

$$x \diamond (y \diamond z) = (x \diamond y) \diamond z,$$

for any $x,y,z \in L$.

If $x,y,z \in [d,1]$, the associativity is clear. Otherwise, it is enough to show that in all remaining cases

$$x \diamond (y \diamond z) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d)),$$

and

$$(x \diamond y) \diamond z = U_1(U_1(x \wedge d, y \wedge d), z \wedge d).$$

If fact, if $y,z \in [d,1]$, then $x \notin [d,1]$ and $x \diamond (y \diamond z) = U_1(x \wedge d, U_2(y,z) \wedge d) = U_1(x \wedge d, d) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d))$. If $y \notin [d,1]$ or $z \notin [d,1]$, then $y \diamond z = U_1(y \wedge d, z \wedge d) \leq d$. Especially, when $y \diamond z < d$, we obtain $x \diamond (y \diamond z) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d))$; when $y \diamond z = d$ and $x \notin [d,1]$, then $x \diamond (y \diamond z) = x \diamond d = U_1(x \wedge d, d) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d))$; when $y \diamond z = d$ and $x \in [d,1]$, then $x \diamond (y \diamond z) = x \diamond d = U_2(x,d) = d = U_1(d,d) = U_1(x \wedge d, d) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d))$. To sum up, $x \diamond (y \diamond z) = U_1(x \wedge d, U_1(y \wedge d, z \wedge d))$ if at least one of x, y and z is not in $[d,1]$.

Similarly, we can prove that $(x \diamond y) \diamond z = U_1(U_1(x \wedge d, y \wedge d), z \wedge d)$ if at least one of x, y and z is not in $[d,1]$. Hence, the associativity follows from the associativity of U_1. $\square$

The next example illustrates the construction of $\mathcal{H}^{\wedge}_{U_1,U_2}$ in Theorem 3.1.

Example 3.2. *Let $L_1 = \{0, a_1, e_1, a_2, a_3, d, a_4, e_2, a_5, 1, b_1, b_2, c_1, c_2, c_3, c_4\}$ be the bounded lattice shown in Fig. 2. Suppose U_1 (U_2) is a conjunctive uninorm on $[0, d]$ ($[d, 1]$) given by (cf. [4]):*

$$U_1(x,y) = \begin{cases} S_M(x,y), & \text{if } (x,y) \in [e_1,d]^2 \\ y, & \text{if } (x,y) \in [e_1,d] \times I_{e_1} \\ x, & \text{if } (x,y) \in I_{e_1} \times [e_1,d] \\ T_D(x \wedge e_1, y \wedge e_1), & \text{otherwise,} \end{cases} \tag{5}$$

and

$$U_2(x,y) = \begin{cases} S_D(x,y), & \text{if } (x,y) \in [e_2,1]^2 \\ y, & \text{if } (x,y) \in [e_2,1] \times I_{e_2} \\ x, & \text{if } (x,y) \in I_{e_2} \times [e_2,1] \\ T_M(x \wedge e_2, y \wedge e_2), & \text{otherwise.} \end{cases} \tag{6}$$

The 2-uninorm $\mathcal{H}^{\wedge}_{U_1,U_2}$, as defined in Theorem 3.1, is given by Table 1.

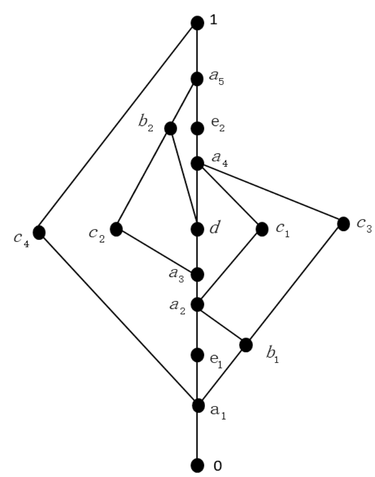

Figure 2: The bounded lattice L_1

Remark 3.3. (i) *In Theorem 3.1, d is required to be less than e_2. Otherwise, U_2 becomes a t-conorm and it cannot be conjunctive.*

(ii) *If U_2 is conjunctive and L is the unit interval $[0, 1]$, then $\mathcal{H}^{\wedge}_{U_1,U_2}$ corresponds to the 2-uninorms constructed in [46]. Indeed, in this case we have $\mathcal{H}^{\wedge}_{U_1,U_2}(0, 1) = \mathcal{H}^{\wedge}_{U_1,U_2}(0, d) = U_1(0, d) \in \{0, d\}$. If $\mathcal{H}^{\wedge}_{U_1,U_2}(0, 1) = d$, then it corresponds to the 2-uninorm defined in Theorem 4 of [46]. If $\mathcal{H}^{\wedge}_{U_1,U_2}(0, 1) = 0$, then, by $\mathcal{H}^{\wedge}_{U_1,U_2}(1, d) = d$, this 2-uninorm corresponds to the one introduced in Theorem 5 of [46].*

It is interesting to find that $\mathcal{H}^{\wedge}_{U_1,U_2}$ is the weakest one among all 2-uninorms which have the same restrictions on $[0, d]^2$ and $[d, 1]^2$.

Theorem 3.4. *Let L be a bounded lattice with elements e_1, e_2, d in L such that $0 < d < 1$ and $0 \le e_1 \le d < e_2 \le 1$. Suppose U_1 is a uninorm on $[0, d]$ with identity element e_1 and U_2 is a conjunctive uninorm on $[d, 1]$ with identity element e_2. Then U_1 and U_2 are, respectively, the restrictions of $\mathcal{H}^{\wedge}_{U_1,U_2}$ on $[0, d]^2$ and $[d, 1]^2$. Moreover, for any 2-uninorm $\mathcal{H}$ on L which extends both U_1 and U_2, we have $\mathcal{H} \ge \mathcal{H}^{\wedge}_{U_1,U_2}$.*

Proof. By construction, U_1 (U_2, resp.) is clearly the restriction of $\mathcal{H}^{\wedge}_{U_1,U_2}$ on $[0, d]^2$ ($[d, 1]^2$, resp.). Suppose $\mathcal{H}$ is also a 2-uninorm which extends both U_1 and U_2. Clearly, for any $(x, y) \in [0, d]^2 \cup [d, 1]^2$, $\mathcal{H}(x, y) = \mathcal{H}^{\wedge}_{U_1,U_2}(x, y)$. For any $(x, y) \in [0, 1]^2 \setminus ([0, d]^2 \cup [d, 1]^2)$, we have

$$\mathcal{H}(x, y) \ge \mathcal{H}(x \wedge d, y \wedge d) = U_1(x \wedge d, y \wedge d) = \mathcal{H}^{\wedge}_{U_1,U_2}(x, y)$$

as $\mathcal{H}$ is non-decreasing. Thus, $\mathcal{H} \ge \mathcal{H}^{\wedge}_{U_1,U_2}$. □

$\mathcal{H}^{\wedge}_{U_1,U_2}$	0	a_1	e_1	a_2	a_3	d	a_4	e_2	a_5	1	b_1	b_2	c_1	c_2	c_3	c_4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_1	0	0	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	0	a_1	a_1	a_1	0	0
e_1	0	a_1	e_1	a_2	a_3	d	d	d	d	d	b_1	d	a_2	a_3	b_1	a_1
a_2	0	a_1	a_2	a_2	a_3	d	d	d	d	d	b_1	d	a_2	a_3	b_1	a_1
a_3	0	a_1	a_3	a_3	a_3	d	d	d	d	d	b_1	d	a_3	a_3	b_1	a_1
d	0	a_1	d	d	d	d	d	d	d	d	b_1	d	d	d	b_1	a_1
a_4	0	a_1	d	d	d	d	a_4	a_4	a_4	a_4	b_1	d	d	d	b_1	a_1
e_2	0	a_1	d	d	d	d	a_4	e_2	a_5	1	b_1	b_2	d	d	b_1	a_1
a_5	0	a_1	d	d	d	d	a_4	a_5	1	1	b_1	b_2	d	d	b_1	a_1
1	0	a_1	d	d	d	d	a_4	1	1	1	b_1	b_2	d	d	b_1	a_1
b_1	0	0	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	0	b_1	b_1	b_1	0	0
b_2	0	a_1	d	d	d	d	d	b_2	b_2	b_2	b_1	d	d	d	b_1	a_1
c_1	0	a_1	a_2	a_2	a_3	d	d	d	d	d	b_1	d	a_2	a_3	b_1	a_1
c_2	0	a_1	a_3	a_3	a_3	d	d	d	d	d	b_1	d	a_3	a_3	b_1	a_1
c_3	0	0	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	0	b_1	b_1	b_1	0	0
c_4	0	0	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	0	a_1	a_1	a_1	0	0

Table 1: The 2-uninorm $\mathcal{H}^{\wedge}_{U_1,U_2}$ in Example 3.2

The above result shows that $\mathcal{H}^{\wedge}_{U_1,U_2}$ is the weakest among all 2-uninorms that extend both U_1 and U_2. A stronger conclusion can be obtained if U_2 is selected to be the weakest uninorm on $[d,1]$.

Corollary 3.5. *Let L be a bounded lattice with elements e_1, e_2, d such that $0 < d < 1$ and $0 \le e_1 \le d < e_2 \le 1$. Suppose U is an arbitrary uninorm on $[0,d]$ and U_{sc} the weakest uninorm on $[d,1]$ with identity element e_2 (cf. Eq.(1)). Then $\mathcal{H}^{\wedge}_{U,U_{sc}}$ is the weakest among all 2-uninorms that extend U and have cutpoint d and local identity elements e_1, e_2 on L .*

It is necessary to point out that for any 2-uninorm $\mathcal{H}$ on a bounded lattice L, $\mathcal{H}(0,1) = a$ is always its annihilator. Indeed, since $\mathcal{H}(0,0) = 0$ and $\mathcal{H}(1,1) = 1$, we get $\mathcal{H}(a,0) = \mathcal{H}(0,a) = \mathcal{H}(0,\mathcal{H}(0,1)) = \mathcal{H}(\mathcal{H}(0,0),1) = \mathcal{H}(0,1) = a$, and similarly $\mathcal{H}(1,a) = \mathcal{H}(a,1) = a$. Then the monotonicity for every $x \in L$ gives $a = \mathcal{H}(a,0) \le \mathcal{H}(a,x) \le \mathcal{H}(a,1) = a$, i.e., $\mathcal{H}(x,a) = \mathcal{H}(a,x) = a$. If U_1 is disjunctive in Theorem 3.1, then $d = U_1(x,d) = \mathcal{H}^{\wedge}_{U_1,U_2}(x,d)$ for any $x \in [0,d]$, i.e., d is also the annihilator of $\mathcal{H}^{\wedge}_{U_1,U_2}$ (see Corollary 3.6). In this case, U_1 is not a t-norm and therefore $0 \le e_1 < d$.

In the below, for convenience, denote $X_d = ([0,d] \cup I_d) \times [d,1] \cup [d,1] \times ([0,d] \cup I_d)$.

Corollary 3.6. *Let L be a bounded lattice with elements e_1, e_2, d such that $0 < d < 1$ and $0 \le e_1 < d < e_2 \le 1$. Suppose $U_1 : [0,d]^2 \to [0,d]$ is a disjunctive uninorm with identity element e_1 and $U_2 : [d,1]^2 \to [d,1]$ a conjunctive uninorm with identity element e_2. Then*

$$\mathcal{H}^{\wedge}_{U_1,U_2}(x,y) = \begin{cases} U_2(x,y), & \text{if } (x,y) \in [d,1]^2 \\ d, & \text{if } (x,y) \in X_d \\ U_1(x \wedge d, y \wedge d), & \text{otherwise.} \end{cases} \tag{7}$$

Clearly, d is the annihilator of the 2-uninorm $\mathcal{H}^{\wedge}_{U_1,U_2}$.

Remark 3.7. *Corollary 3.6 has the same conditions as Theorem 1 of* [21], *i.e., both requiring that U_1 is a disjunctive uninorm and U_2 is a conjunctive uninorm. We find that $\mathcal{H}^{\wedge}_{U_1,U_2}$ differs from the 2-uninorm U^2 of* [21] *only in the region $[0,d] \times I_d \cup I_d \times [0,d] \cup I_d^2$ since* $U^2(x,y) = \begin{cases} U_1(x,y), & \text{if } (x,y) \in [0,d]^2 \\ U_2(x,y), & \text{if } (x,y) \in [d,1]^2 \\ d, & \text{otherwise.} \end{cases}$

In case $e_1 = 0$ ($e_2 = 1$, resp.) in Theorem 3.1, then U_1 (U_2, resp.) becomes a t-conorm (t-norm, resp.). This yields a null-uninorm (a uni-nullnorm, resp.) on L.

Corollary 3.8. *Let L be a bounded lattice with elements e, d such that $0 < d < e \le 1$ and $0 < d < 1$. Suppose $U : [d,1]^2 \to [d,1]$ is a uninorm with identity element e and $S : [0,d]^2 \to [0,d]$ a t-conorm. Then*

$$\mathcal{H}^{\wedge}_{S,U}(x,y) = \begin{cases} U(x,y), & \text{if } (x,y) \in [d,1]^2 \\ d, & \text{if } (x,y) \in X_d \\ S(x \wedge d, y \wedge d), & \text{otherwise.} \end{cases} \tag{8}$$

Moreover, the operator $\mathcal{H}^{\wedge}_{S,U}$ *is a null-uninorm on* L *if and only if* U *is a conjunctive uninorm. If* U *is a conjunctive uninorm, then the null-uninorm* $\mathcal{H}^{\wedge}_{S,U}$ *is also the weakest among all null-uninorms that extend* U *and* S *with cutpoint* d *on* L *.*

Remark 3.9. *In Corollary 3.8, if* $S = S_M$ *and* $U = U_{sc}$*, then* $H = \mathcal{H}^{\wedge}_{S_M,U_{sc}}$ *is the weakest among all null-uninorms on* L *with cutpoint* d *. Moreover, we have*

$$H(x,y) = \begin{cases} U_{sc}(x,y), & \text{if } (x,y) \in [d,1]^2 \\ d, & \text{if } (x,y) \in X_d \\ (x \wedge d) \vee (y \wedge d), & \text{otherwise.} \end{cases} \tag{9}$$

Corollary 3.10. *Let* L *be a bounded lattice with elements* e, d *such that* $0 \le e \le d < 1$ *and* $0 < d < 1$*. Suppose* U *is a uninorm on* $[0,d]$ *with identity element* e *and* T *a t-norm on* $[d,1]$*. Then 2-uninorm* $\mathcal{H}^{\wedge}_{U,T}$ *on* L *is a uni-nullnorm. Indeed,* $\mathcal{H}^{\wedge}_{U,T}$ *is the weakest among all uni-nullnorms that extend* U *and* T *with cutpoint* d *on* L*.*

Remark 3.11. (*i*) *The result of Corollary 3.10 is the one of Theorem 3.1 in [39].*

(*ii*) *In Corollary 3.10, if* $U = U_{sc}$ *and* $T = T_D$*, then* $H = \mathcal{H}^{\wedge}_{U_{sc},T_D}$ *is the weakest uni-nullnorm among all uni-nullnorms on* L *with cutpoint* d *and local identity element* e*.*

Taking $e = 1$ in Corollary 3.8 or $e = 0$ in Corollary 3.10, we obtain the nullnorm on L constructed in *[22]*.

3.2 The strongest 2-uninorms $\mathcal{H}^{\vee}_{U_1,U_2}$

In the previous section, we have defined a 2-uninorm which is the weakest 2-uninorm with given underlying functions. Dually, we can construct the strongest one in a similar way. The construction is illustrated in Fig. 3.

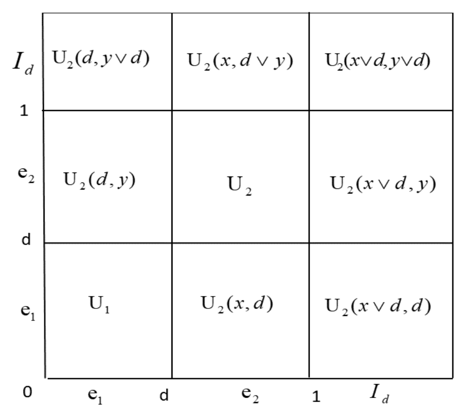

Figure 3: The structure of the 2-uninorm $\mathcal{H}^{\vee}_{U_1,U_2}$ in Eq. (10)

Theorem 3.12. *Let* L *be a bounded lattice with elements* e_1, e_2, d *such that* $0 \le e_1 < d \le e_2 \le 1$ *and* $0 < d < 1$*. Suppose* $U_1 : [0,d]^2 \to [0,d]$ *is a uninorm with identity element* e_1 *and* $U_2 : [d,1]^2 \to [d,1]$ *a uninorm with identity element* e_2*. The binary operator* $\mathcal{H}^{\vee}_{U_1,U_2}$ *given by*

$$\mathcal{H}^{\vee}_{U_1,U_2}(x,y) = \begin{cases} U_1(x,y), & \text{if } (x,y) \in [0,d]^2 \\ U_2(x \vee d, y \vee d), & \text{otherwise} \end{cases} \tag{10}$$

is a 2-uninorm on L *if and only if* U_1 *is a disjunctive uninorm.*

Remark 3.13. (*i*) *In Theorem 3.12,* d *is required to be greater than* e_1*. Otherwise,* U_1 *is a t-norm and it cannot be disjunctive.*

(*ii*) *Suppose* U_1 *is disjunctive. If* L *is the unit interval* $[0,1]$*, then* $\mathcal{H}^{\vee}_{U_1,U_2}$ *corresponds to the 2-uninorms constructed in* [46, Theorems 4 & 6].

The 2-uninorm $\mathcal{H}^{\vee}_{U_1,U_2}$ is the strongest among all 2-uninorms which extend U_1 and U_2 with cutpoint d.

Theorem 3.14. *Let L be a bounded lattice with elements e_1, e_2, d such that $0 \leq e_1 < d \leq e_2 \leq 1$ and $0 < d < 1$. Suppose U_1 is a disjunctive uninorm on $[0, d]$ with identity element e_1 and U_2 a uninorm on $[d, 1]$ with identity element e_2. Then U_1 and U_2 are, respectively, the restrictions of $\mathcal{H}^{\vee}_{U_1,U_2}$ on $[0, d]^2$ and $[d, 1]^2$. Moreover, for any 2-uninorm $\mathcal{H}$ on L which extends both U_1 and U_2, we have $\mathcal{H} \leq \mathcal{H}^{\vee}_{U_1,U_2}$.*

A stronger conclusion can be reached if U_1 is the strongest uninorm on $[0, d]$.

Corollary 3.15. *Let L be a bounded lattice with elements e_1, e_2, d such that $0 \leq e_1 < d < e_2 \leq 1$ and $0 < d < 1$. Suppose U is an arbitrary uninorm on $[d, 1]$ with identity element e_2 and U_{ld} the strongest uninorm on $[0, d]$ with identity element e_1 (cf. Eq.(2)). Then $\mathcal{H}^{\vee}_{U_{ld},U}$ is the strongest among all 2-uninorms on L that extend U and have cutpoint d and local identity elements e_1, e_2.*

In case U_2 is conjunctive, we have a finer representation for $\mathcal{H}^{\vee}_{U_1,U_2}$, which directly implies that d is the annihilator of the 2-uninorm $\mathcal{H}^{\vee}_{U_1,U_2}$. We denote $Y_d = [0, d] \times (I_d \cup [d, 1]) \cup (I_d \cup [d, 1]) \times [0, d]$.

Corollary 3.16. *Let L be a bounded lattice with elements e_1, e_2, d such that $0 \leq e_1 < d < e_2 \leq 1$ and $0 < d < 1$. Suppose U_1 is a disjunctive uninorm on $[0, d]$ with identity element e_1 and U_2 a conjunctive uninorm on $[d, 1]$ with identity element e_2. Then*

$$\mathcal{H}^{\vee}_{U_1,U_2}(x, y) = \begin{cases} U_1(x, y), & \text{if } (x, y) \in [0, d]^2 \\ d, & \text{if } (x, y) \in Y_d \\ U_2(x \vee d, y \vee d), & \textit{otherwise.} \end{cases} \tag{11}$$

Clearly, d is the annihilator of $\mathcal{H}^{\vee}_{U_1,U_2}$.

Remark 3.17. *Similar to Remark 3.7, when compared with the 2-uninorm U^2 in [21], our 2-uninorm $\mathcal{H}^{\vee}_{U_1,U_2}$ differs only in the region $[d, 1] \times I_d \cup I_d \times [d, 1] \cup I_d^2$, where U^2 takes the fixed value d.*

Corollary 3.18. *Let L be a bounded lattice with elements e, d such that $0 \leq e < d < 1$ and $0 < d < 1$. Suppose $T : [d, 1]^2 \to [d, 1]$ is a t-norm and $U : [0, d]^2 \to [0, d]$ a uninorm with identity element e. Then*

$$\mathcal{H}^{\vee}_{U,T}(x, y) = \begin{cases} U(x, y), & \text{if } (x, y) \in [0, d]^2 \\ d, & \text{if } (x, y) \in Y_d \\ T(x \vee d, y \vee d), & \textit{otherwise.} \end{cases} \tag{12}$$

Moreover, $\mathcal{H}^{\vee}_{U,T}$ is a uni-nullnorm on L if and only if U is a disjunctive uninorm. If U is a disjunctive uninorm, then $\mathcal{H}^{\vee}_{U,T}$ is indeed the strongest uni-nullnorm among all uni-nullnorms on L that extend U and T with cutpoint d.

Remark 3.19. (*i*) *The result of Corollary 3.18 is the one of Theorem 4.1 in [39].*

(*ii*) *In Corollary 3.18, if $U = U_{ld}$ and $T = T_M$, then $G = \mathcal{H}^{\vee}_{U_{ld},T_M}$ is the strongest uni-nullnorm with cutpoint d and local identity element e. Moreover, we have*

$$G(x, y) = \begin{cases} U_{ld}(x, y), & \textit{if } (x, y) \in [0, d]^2 \\ d, & \textit{if } (x, y) \in Y_d \\ (x \vee d) \wedge (y \vee d), & \textit{otherwise.} \end{cases} \tag{13}$$

Corollary 3.20. *Let L be a bounded lattice, $e, d \in L$, $0 \leq d < e \leq 1$ and $0 < d < 1$. Suppose U is a uninorm on $[d, 1]$ with identity element e and S a t-conorm on $[0, d]$. The 2-uninorm $\mathcal{H}^{\vee}_{S,U}$ on L is a null-uninorm. Indeed, it is the strongest null-uninorm on L which extends both S and U with cutpoint d.*

Remark 3.21. *In Corollary 3.20, if $S = S_D$ and $U = U_{ld}$, then $G = \mathcal{H}^{\vee}_{S_D,U_{ld}}$ is the strongest null-uninorm.*

Taking $e = 1$ in Corollary 3.18 or $e = 0$ in Corollary 3.20, we obtain the nullnorm on L constructed in [22].

To illustrate the construction, we also give an example.

Example 3.22. *Let $L_1 = \{0, a_1, e_1, a_2, a_3, d, a_4, e_2, a_5, 1, b_1, b_2, c_1, c_2, c_3, c_4\}$ be the bounded lattice as defined in Fig. 2. Suppose T_{e_1} is a t-norm on $[0, e_1]$ and S_{e_2} a t-conorm on $[e_2, 1]$. Consider the disjunctive uninorm U_1 on $[0, d]$ with identity element e_1 given by*

$$U_1(x, y) = \begin{cases} T_{e_1}(x, y), & \textit{if } (x, y) \in [0, e_1]^2 \\ y, & \textit{if } (x, y) \in [0, e_1] \times ([0, d] \setminus [0, e_1]) \\ x, & \textit{if } (x, y) \in ([0, d] \setminus [0, e_1]) \times [0, e_1] \\ h(x) \vee h(y), & \textit{if } (x, y) \in ([0, d] \setminus [0, e_1])^2, \end{cases}$$

and the conjunctive uninorm U_2 *on* $[d,1]$ *with identity element* e_2 *given by*

$$U_2(x,y)=\begin{cases} S_{e_2}(x,y), & \text{if } (x,y)\in[e_2,1]^2\\ y, & \text{if } (x,y)\in[e_2,1]\times([d,1]\setminus[e_2,1])\\ x, & \text{if } (x,y)\in([d,1]\setminus[e_2,1])\times[e_2,1]\\ g(x)\wedge g(y), & \text{otherwise,}\end{cases}$$

where h *is a closure operator and* g *is an interior operator, respectively (please see the definitions of closure operators and interior operators in [34]). Then* U_1 *and* U_2 *are uninorms (see [Theorems 4.1 and 5.1, [34]).*

Now, select $T_{e_1}=T_M$, $h(x)=x$, $S_{e_2}=S_D$, $g(x)=x$. *Then we obtain 2-uninorm* $\mathcal{H}^{\vee}_{U_1,U_2}$ *on* L_1 *given in Table 2.*

$\mathcal{H}^{\vee}_{U_1,U_2}$	0	a_1	e_1	a_2	a_3	d	a_4	e_2	a_5	1	b_1	b_2	c_1	c_2	c_3	c_4
0	0	0	0	a_2	a_3	d	d	d	d	d	b_1	d	d	d	d	d
a_1	0	a_1	a_1	a_2	a_3	d	d	d	d	d	b_1	d	d	d	d	d
e_1	0	a_1	e_1	a_2	a_3	d	d	d	d	d	b_1	d	d	d	d	d
a_2	a_2	a_2	a_2	a_2	a_3	d	d	d	d	d	a_2	d	d	d	d	d
a_3	a_3	a_3	a_3	a_3	a_3	d	d	d	d	d	a_3	d	d	d	d	d
d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d	d
a_4	d	d	d	d	d	d	a_4	a_4	a_4	a_4	d	d	a_4	d	a_4	a_4
e_2	d	d	d	d	d	d	a_4	e_2	a_5	1	d	b_2	a_4	b_2	a_4	1
a_5	d	d	d	d	d	d	a_4	a_5	1	1	d	b_2	a_4	b_2	a_4	1
1	d	d	d	d	d	d	a_4	1	1	1	d	b_2	a_4	b_2	a_4	1
b_1	b_1	b_1	b_1	a_2	a_3	d	d	d	d	d	b_1	d	d	d	d	d
b_2	d	d	d	d	d	d	d	b_2	b_2	b_2	d	b_2	d	b_2	d	b_2
c_1	d	d	d	d	d	d	a_4	a_4	a_4	a_4	d	d	a_4	d	a_4	a_4
c_2	d	d	d	d	d	d	d	b_2	b_2	b_2	d	b_2	d	b_2	d	b_2
c_3	d	d	d	d	d	d	a_4	a_4	a_4	a_4	d	d	a_4	d	a_4	a_4
c_4	d	d	d	d	d	d	a_4	1	1	1	d	b_2	a_4	b_2	a_4	1

Table 2: The 2-uninorm $\mathcal{H}^{\vee}_{U_1,U_2}$ in Example 3.22

In [45], the authors discuss the (U^2,N)- operation $I_{U^2,N}$ on the unit interval derived from a 2-uninorm U^2 and a fuzzy negation N, where $I_{U^2,N}(x,y)=U^2(N(x),y)$. They prove that $I_{U^2,N}$ is a fuzzy implication if and only if the related 2-uninorm U^2 is disjunctive, i.e., $U^2(0,1)=1$. [1]

Using our 2-uninorms, we can obtain fuzzy implications on bounded lattices as well. In fact, if U_1 and U_2 are disjunctive, then $\mathcal{H}^{\vee}_{U_1,U_2}$ is a disjunctive 2-uninorm. Therefore, analogously as in [45], we can use such $\mathcal{H}^{\vee}_{U_1,U_2}$ to construct (H,N)-implications $I_{H,N}$ on bounded lattices by $I_{H,N}(x,y)=\mathcal{H}^{\vee}_{U_1,U_2}(N(x),y)$.

Example 3.23. *Let* $L_2=\{0,a_1,e_1,d,e_2,a_2,b_1,b_2,1\}$ *be the bounded lattice shown in Fig. 4. Define* $N:[0,1]\to[0,1]$ *by*

$$N(x)=\begin{cases} 1, & \text{if } x=0\\ a_2, & \text{if } x=a_1\\ e_2, & \text{if } x=e_1\\ d, & \text{if } x\in\{d,b_1,b_2\}\\ e_1, & \text{if } x=e_2\\ a_1, & \text{if } x=a_2\\ 0, & \text{if } x=1.\end{cases}$$

Clearly, N *is a fuzzy negation on* L_2. *Let* U_1 *and* U_2 *be given, respectively, by*

$$U_1(x,y)=\begin{cases} T_M(x,y), & \text{if } (x,y)\in[0,e_1]^2\\ y, & \text{if } (x,y)\in[0,e_1]\times([0,d]\setminus[0,e_1])\\ x, & \text{if } (x,y)\in([0,d]\setminus[0,e_1])\times[0,e_1]\\ x\vee y, & \text{if } (x,y)\in([0,d]\setminus[0,e_1])^2,\end{cases}$$

[1] For definitions of fuzzy implications and fuzzy negations, the reader can refer to [2, 45].

$$U_2(x,y)=\begin{cases} T_D(x,y), & \text{if } (x,y)\in[d,e_2]^2\\ y, & \text{if } (x,y)\in[d,e_2]\times([d,1]\setminus[d,e_2])\\ x, & \text{if } (x,y)\in([d,1]\setminus[d,e_2])\times[d,e_2]\\ S_D(x\vee e_2,y\vee e_2), & \text{if } (x,y)\in([d,1]\setminus[d,e_2])^2.\end{cases}$$

Then U_1 and U_2 are disjunctive, and fuzzy implication $I_{H,N}$ on L_2 is given in Table 3. The fuzzy implication $I_{H,N}$ can be useful for lattice-valued fuzzy set theory just as (S,N)-implications on bounded lattices [2].

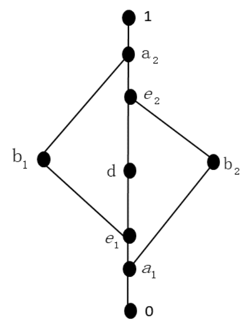

Figure 4: Bounded lattice L_2

$I_{H,N}$	0	a_1	e_1	d	e_2	a_2	1	b_1	b_2
0	1	1	1	1	1	1	1	1	1
a_1	a_2	a_2	a_2	a_2	a_2	1	1	1	a_2
e_1	d	d	d	d	e_2	a_2	1	a_2	e_2
d	d	d	d	d	d	a_2	1	a_2	d
e_2	0	a_1	e_1	d	d	a_2	1	a_2	d
a_2	0	a_1	a_1	d	d	a_2	1	a_2	d
1	0	0	0	d	d	a_2	1	a_2	d
b_1	d	d	d	d	d	a_2	1	a_2	d
b_2	d	d	d	d	d	a_2	1	a_2	d

Table 3: The $I_{U^2,N}$ in Example 3.23

4 Conclusion

For a bounded lattice L with elements $0\le e_1\le d\le e_2\le 1$ and $0<d<1$, suppose U_1 is a uninorm on $[0,d]$ with identity element e_1 and U_2 a uninorm on $[d,1]$ with identity element e_2. We constructed 2-uninorms on L by extending both U_1 and U_2. The two 2-uninorms, $\mathcal{H}^{\wedge}_{U_1,U_2}$ and $\mathcal{H}^{\vee}_{U_1,U_2}$ are the weakest and the strongest 2-uninorms on L among all 2-uninorms that extend U_1 and U_2, respectively. Our constructions have also been adapted to construct uni-nullnorms and null-uninorms on L.

As bounded lattices can be very different from the unit interval $[0,1]$, many nice properties fulfilled by 2-uninorms on $[0,1]$ may not hold. It is interesting to extend the classifications obtained in [1] and [46] to general bounded lattices. In addition, it is easy to see that neither $\mathcal{H}^{\wedge}_{U_1,U_2}$ nor $\mathcal{H}^{\vee}_{U_1,U_2}$ is idempotent, in general. In the future, we will investigate idempotent 2-uninorms on bounded lattices.

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Volume 19, Number 1 (2022), pp. 214

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Construction of 2-uninorms on bounded lattices

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ساخت ۲-تک نرم‌ها روی شبکه‌های محدود

چکیده. تک نرم‌ها و تک نرم‌های پوچ ۲-تک نرم‌های خاص هستند. در این کار ۲-تک نرم‌ها را روی شبکه‌های محدود می‌سازیم. فرض کنید L یک شبکه محدود با یک عنصر غیربدیهی d باشد. برای دو تک نرم U_1 و U_2 که به ترتیب روی زیرشبکه‌های $[0, d]$ و $[d, 1]$ تعریف شده‌اند، این مقاله دو روش برای ساخت عملگرهای دوتایی روی L ارائه می‌دهدکه هم U_1 وهم U_2 را گسترش می‌دهند. نشان می‌دهیم که ساختار اول ما یک ۲-تک نرم روی L است اگروفقط‌اگر U_2 عطفی باشد و ساختار دوم ما یک ۲-تک نرم روی L است اگروفقط‌اگر U_1 فصلی باشد. علاوه براین، ثابت می‌کنیم که دو ۲-تک نرم‌ها به ترتیب ضعیف‌ترین و قوی‌ترین ۲-تک نرم در بین تمام ۲-تک نرم‌ها هستند که تحدید آنها به $[0, d]^2$ و $[d, 1]^2$ به ترتیب U_1 و U_2 می‌باشد.