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Development of a Zero Extension Line Method for Axially Symmetric Problems in Soil Mechanics

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In this paper, a theory regarding the zero extension line method for axial symmetry has been developed. The method assumes that the soil will yield progressively in accordance with the Mohr-Coulomb failure criterion. A simple approach has been employed for the derivation of equations, which is completely different from the method of characteristics. Equations satisfying equilibrium and yield have been directly written along the zero extension lines to allow the calculation of stresses and displacements at the same points in the soil mass. The governing equations have been shown to be more general in the axi-symmetric case so that those for plane strain cases can be deduced from them. The mobilized strength in soil is related to shear strain. The finite difference form of the equations has also been put forward and the steps towards calculation of the fields have been presented. The most important application of the theory is the prediction of the load-deflection curves for structures in contact with soil. This has been shown using examples of circular footings on clay and sand. It has been concluded that the zero extension line theory provides a relatively simple analytical method for the prediction of the load-deflection curves in both axi-symmetric and plane strain problems in soil mechanics.

INTRODUCTION

Although the concept of Zero Extension Lines (ZEL) has evolved from the characteristic method used in the theory of plasticity, the ZEL theory can be established rather independently. The characteristic method has been used in soil mechanics for many years. It has been used to investigate the limiting equilibrium of soil masses under plane strain [1] and axi-symmetric conditions [2], as well as to obtain the equations for the velocity field when the flow rule is associated [2]. The method has also been extended to the case of variable strength parameters, c and ϕ, in both plane strain [3] and axi-symmetric conditions [4]. Further extension of the method to the case of variable ν was also made [5].

The progressive nature of failure in soil and the need for knowing the load-deflection behavior at loads other than the ultimate, made the researchers pay more attention to the strain field [6]. Attempts were made to predict the pattern of ZEL (velocity characteristics), and to use them in finding the developed field of strains in the soil mass under the effect of different boundary deflections [6,7]. A simple pattern of ZEL was used to find the strain field behind a model retaining wall [7]. The predictions were in agreement with what was observed [7]. Success in the prediction of the strain field by the ZEL persuaded the researchers to implement them in obtaining the mobilized strength at different points of the soil mass [6]. This information would be necessary if a more realistic field of stresses had to be obtained. It was this idea that led to the development of the method of associated field [8-10], which worked well in the prediction of the load-deflection behavior. There were, however, some other difficulties with this method [4] including its use of an iterative process of computations in obtaining convergence and compatibility between ϕ, ν, and ρ fields in each increment and that it required elaborate interpolation routines, since velocities and stresses were not calculated at the same points.

Attempts were then made to find a way of calculating the stress field by the ZEL alone. Two approaches were made to this problem. In the first, the stresses were calculated by considering the force equilibrium of the soil elements between the ZEL [11]. This method was used in the calculation of the static and dynamic bearing capacity and the active and passive pressures [12,13]. It was also used to predict the load-deflection behavior in these problems [14,15]. The second was to transfer the equilibrium-yield equations written along the stress characteristics onto the ZEL

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directions [16]. This approach has more applications although it was shown to yield the same results as the first [16].

Previous work on the ZEL method has all been limited to the plane strain case and it is the aim of this paper to extend the method to the axi-symmetric case. It has also been shown that the ZEL theory can be established independently rather than using the method of characteristics. The equilibrium-yield equations can also be written directly along the ZEL without any reference to the stress characteristics. The most important application of the ZEL theory is the prediction of the load-deflection behavior of structures in contact with soils. This has been shown by some examples of axi-symmetric problems in soil mechanics.

THEORY

The ZEL theory is used for bearing capacity and earth pressure problems in plane strain and axi-symmetric conditions. The shearing of soil in these cases occurs in a principal plane in which the major and minor principal strain increments do not have the same sense. In the axi-symmetric case, these planes are the radial planes on which the intermediate principal stress, σ_θ acts. Figure 1 shows an element of soil in this plane and the Mohr circle for incremental strains. The in-plane dilation angle of soil is defined as:

$$\sin\nu = -\frac{\varepsilon_1+\varepsilon_3}{\varepsilon_1-\varepsilon_3}. \tag{1}$$

As shown in the figure, there would be two directions along which linear incremental strain is zero. These directions, known as the directions of zero extension, make the same angle with the direction of ε_1. The directions of ε_1 and σ_1 are assumed to coincide. If the angle ξ is equal to $45-\nu/2$, there exist two families of ZEL namely, the minus and plus zero extension lines, which make the angles $\psi-\xi$ and $\psi+\xi$ with the direction of r-axis, respectively, so that:

$$\text{Along the minus (-) ZEL}: \frac{dz}{dr} = \tan(\psi-\xi), \tag{2a}$$

$$\text{Along the plus (+) ZEL}: \frac{dz}{dr} = \tan(\psi+\xi). \tag{2b}$$

A network of ZEL is formed by these two families of curves which intersect each other at $90 \pm \nu$.

Velocity Field

The important role of ZEL is that the displacement field can be calculated if they are in hand. If AB is an element of the zero extension line of length, $d\varepsilon$, shown in Figure 2, then it should work as a rigid link between these two points, so that $\overline{AB}$ remains the same after the increment of displacement. This implies that the displacement of B relative to A should be normal to AB so that:

$$\frac{du}{dw} = -\frac{dz}{dr}, \tag{3}$$

where u and w are the components of displacement in r and z directions, respectively. This equation holds for both families of ZEL. The finite difference form of this equation, when written along both directions, can be used to calculate the displacements.

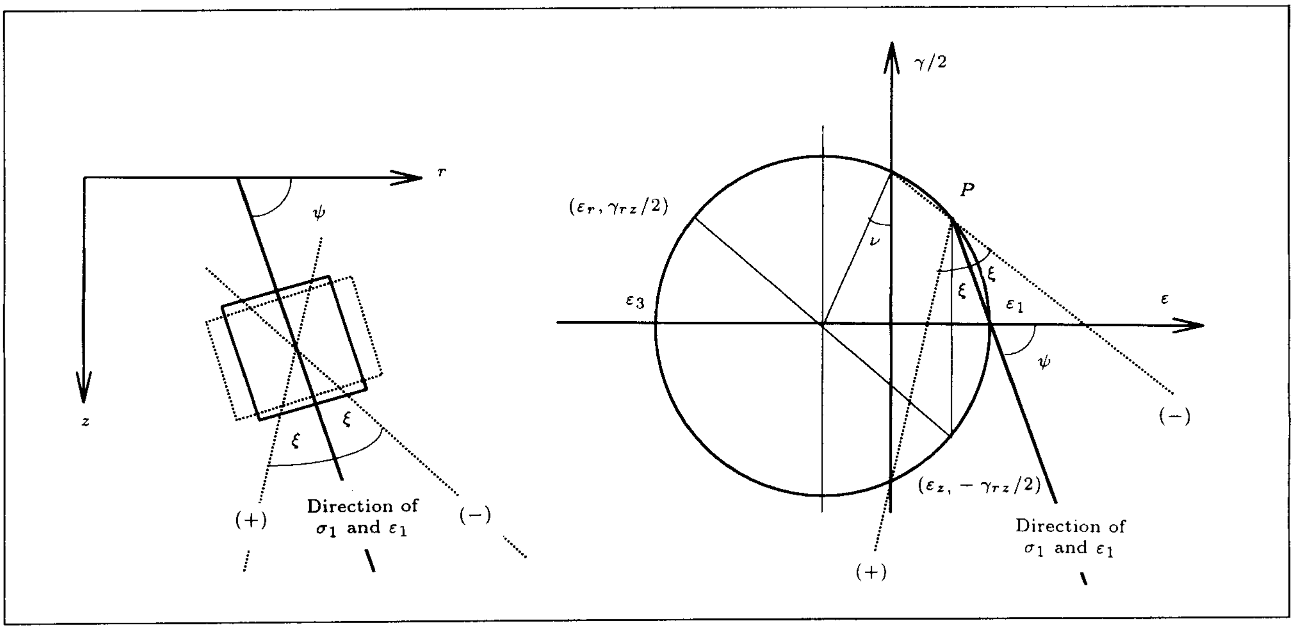

Figure 1. An element of soil in $r-z$ plane and Mohr circle for incremental strains.

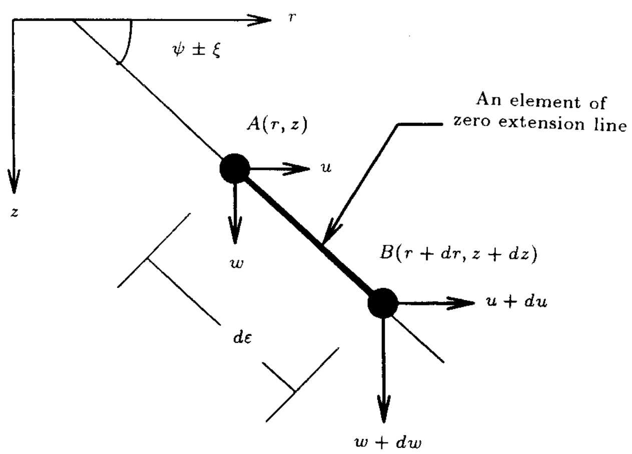

Figure 2. A zero extension line element of soil.

Equilibrium of Progressively Failing Soil

The polar coordinate system (r,θ,z) is used for the axi-symmetric problem. The depth of half-space is measured by the z coordinate and the θ-direction is the direction of intermediate principal strain increments and stress.

The differential equations of equilibrium for the axi-symmetric problem can be written as:

$$\begin{cases} \frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} = f_r \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} = f_z \end{cases}, \tag{4}$$

where:

$$\begin{cases} f_r = R - \frac{n}{r}(\sigma_r - \sigma_\theta) \\ f_z = Z - \frac{n}{r}\tau_{rz} \end{cases}, \tag{5}$$

where n is 1 and R and Z are body and/or inertial forces per unit volume in r and z directions, respectively. The integer n has been used in these equations to show that they reduce to those for the plane strain case if it is set equal to 0. Based on the Harr & Von-Karman hypothesis [17], σ_θ in the axi-symmetric case, is set equal to either of the minor or major principal stresses. If the mobilized friction angle and cohesion are ϕ and c, respectively, progressive failure of soil in the $r-z$ plane can be represented by the Mohr circle of stress drawn in Figure 3. Assuming S to be the average of σ_r and σ_z, T to be the radius of Mohr circle and ψ to be the angle between the direction of σ_1 and r-axis, one can write:

$$\sigma_z = S - T\cos 2\psi,$$

$$\sigma_r = S + T\cos 2\psi,$$

$$\tau_{rz} = T\sin 2\psi. \tag{6}$$

Using these expressions, Equation 4 can be written as:

$$\frac{\partial S}{\partial r} + \cos 2\psi\frac{\partial T}{\partial r} + \sin 2\psi\frac{\partial T}{\partial z} + 2T(\cos 2\psi\frac{\partial \psi}{\partial z} - \sin 2\psi\frac{\partial \psi}{\partial r}) = f_r, \tag{7a}$$

$$\frac{\partial S}{\partial z} + \sin 2\psi\frac{\partial T}{\partial r} - \cos 2\psi\frac{\partial T}{\partial z} + 2T(\cos 2\psi\frac{\partial \psi}{\partial r} + \sin 2\psi\frac{\partial \psi}{\partial z}) = f_z. \tag{7b}$$

These equations consider both equilibrium and yielding of soil in r and z directions, respectively.

Stress Field

As will be shown later, the network of ZEL can be used to find the displacement and strain fields. It would then be advantageous to calculate the stress field using the same network. In this way, stresses, strains and displacements are calculated at the same points and the difficulties involved in methods of associated fields and characteristics would be avoided. An approach was made previously by writing the equilibrium-yield equations along the stress characteristics first and then transferring them onto the ZEL [16]. A rather simpler approach is made here which avoids the elaborate method of characteristics. The equilibrium-yield equations along r and z directions, i.e. Equations 7, can be directly transferred onto the ZEL. This is done by using the covariant law of transformation of partial derivatives of a function F as:

$$\begin{cases} \frac{\partial F}{\partial r} = \frac{1}{\sin 2\xi}[\sin(\psi+\xi)\frac{\partial F}{\partial \varepsilon^-} - \sin(\psi-\xi)\frac{\partial F}{\partial \varepsilon^+}] \\ \frac{\partial F}{\partial z} = \frac{1}{\sin 2\xi}[\cos(\psi-\xi)\frac{\partial F}{\partial \varepsilon^+} - \cos(\psi+\xi)\frac{\partial F}{\partial \varepsilon^-}] \end{cases}, \tag{8}$$

where ε^- and ε^+ are distances along the (-) and (+) ZEL and F stands for S, T, and ψ. Using these expressions, partial derivatives of these functions, with respect to r and z in Equations 7, can be replaced by those with respect to ε^- and ε^+. The new form of these equations would be:

$$\sin(\psi+\xi)\left(\frac{\partial S}{\partial \varepsilon^-} + \frac{\partial T}{\partial \varepsilon^+}\right) - \sin(\psi-\xi)\left(\frac{\partial S}{\partial \varepsilon^+} + \frac{\partial T}{\partial \varepsilon^-}\right) - 2T[\cos(\psi-\xi)\frac{\partial \psi}{\partial \varepsilon^-} - \cos(\psi+\xi)\frac{\partial \psi}{\partial \varepsilon^+}] = f_r\cos\nu, \tag{9a}$$

$$\cos(\psi-\xi)\left(\frac{\partial S}{\partial \varepsilon^+} + \frac{\partial T}{\partial \varepsilon^-}\right) - \cos(\psi+\xi)\left(\frac{\partial S}{\partial \varepsilon^-} + \frac{\partial T}{\partial \varepsilon^+}\right) - 2T[\sin(\psi+\xi)\frac{\partial \psi}{\partial \varepsilon^+} - \sin(\psi-\xi)\frac{\partial \psi}{\partial \varepsilon^-}] = f_z\cos\nu. \tag{9b}$$

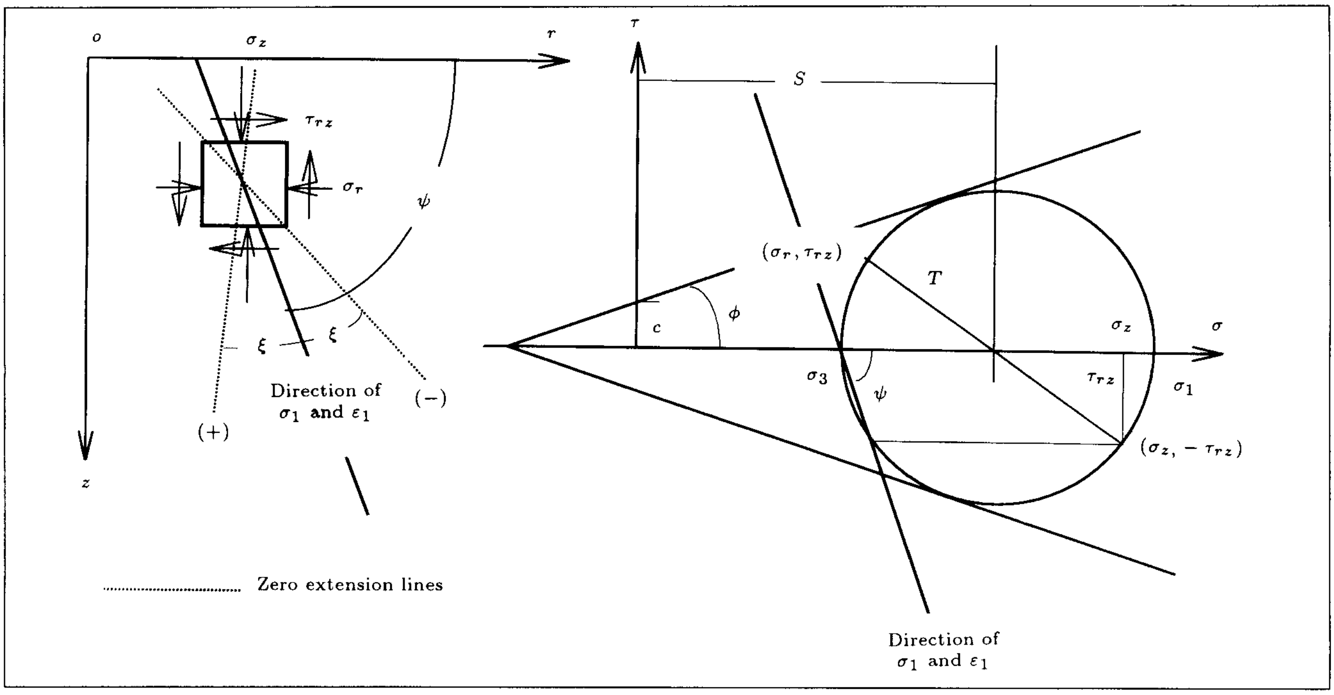

Figure 3. An element of soil in $r - z$ plane and Mohr circle for stresses at failure.

Combining these equations, first, in a way to cancel $\partial S/\partial\varepsilon^+$ and, second, in a way to cancel $\partial S/\partial\varepsilon^-$ will result in the following pair of equations along the zero extension lines:

Along the (-) ZEL:

$$dS + \frac{\partial T}{\partial\varepsilon^+}d\varepsilon^- - \frac{2T}{\cos\nu}\left(d\psi - \sin\nu\frac{\partial\psi}{\partial\varepsilon^+}d\varepsilon^-\right)$$
$$= [f_r\cos(\psi-\xi) + f_z\sin(\psi-\xi)]\,d\varepsilon^-. \qquad (10a)$$

Along the (+) ZEL:

$$dS + \frac{\partial T}{\partial\varepsilon^-}d\varepsilon^+ + \frac{2T}{\cos\nu}\left(d\psi - \sin\nu\frac{\partial\psi}{\partial\varepsilon^-}d\varepsilon^+\right)$$
$$= [f_r\cos(\psi+\xi) + f_z\sin(\psi+\xi)]\,d\varepsilon^+. \qquad (10b)$$

Equations 10 satisfy the equilibrium and yield along the zero extension lines. Note that T, i.e. the radius of Mohr circle, is a function of S, c and ϕ, defined by $S.\sin\phi + c.\cos\phi$. Also, note that c and ϕ are different from point to point due to the differences in strains. Therefore, if the strains due to boundary deflections are determined, the developed strength is known at each point and progressive failure of the mass can be considered by solving Equations 10. The method assumes ν to remain constant during the shearing of soil. This assumption and the coaxiality of ε_1 and σ_1 have been used extensively in the literature [6,7,11-16].

The values of f_r and f_z are obtained from Equations 5. Therefore, Equations 10 hold for both axi-symmetric and plane strain cases provided proper values of f are obtained from Equations 5 and substituted in Equations 10. As mentioned before, n is set equal to zero for the plane strain case. For this case, f's are nothing but the components of body and/or inertial forces in lateral and vertical directions. For the axi-symmetric case, f's are also functions of stress and r-coordinate of the points. It should be mentioned that the above derivation holds only when σ_θ is the intermediate one of the principal stresses, which, in turn, secures the occurrence of yielding in $r - z$ plane. σ_θ is usually assumed to be equal to σ_3 in passive pressure problems [18] and, with this assumption, Equations 5 reduce to:

$$\begin{cases} f_r = R - \frac{nT}{r}(1+\cos 2\psi) \\ f_z = Z - \frac{nT}{r}\sin 2\psi \end{cases}, \qquad (11)$$

in which n is equal to 1. Assumption of σ_θ more than σ_3 (for example, $\sigma_\theta = S$) will cause the calculated pressures in passive pressure problems to reduce. As was shown, the governing equations in axial symmetry are more general so that those for plane strain can be deduced from them by setting $n = 0$.

Strain Field

Once the displacement field is determined, calculation of the strain field can follow simply. The shear strain can be calculated using the strain-displacement relationships or simply by considering the change in the shape of an element of soil between the ZEL [14,15].

Both approaches yield the same result. Using Equations 8 for partial derivatives of u and w, the shear strain-displacement relationship along the ZEL can be written as:

$$\gamma = \Big[\cos(\psi+\xi)\frac{\partial u}{\partial \varepsilon^-} - \cos(\psi-\xi)\frac{\partial u}{\partial \varepsilon^+} - \sin(\psi+\xi)\frac{\partial w}{\partial \varepsilon^-} + \sin(\psi-\xi)\frac{\partial w}{\partial \varepsilon^+}\Big] / \cos\nu. \quad (12)$$

This equation can be used to find the developed shear stain field due to the boundary deflections. The calculated shear strain field can be used to find the values of strength parameters, ϕ and c, at different points of the ZEL net. This is done by using the relation between ϕ or c with maximum shear strain, $\gamma_{\max}$. This relation can be established by using the result of a typical shear test performed in the average stress range.

CALCULATION PROCEDURE AND RECURRENCE FORMULAE

Calculations consist of two major steps. In the first step, the ZEL net is constructed. This is done by writing Equations 2 and 10 in finite difference form and using them in a calculation of r, z, S, T and ψ of any point like C from the information at points A and B (Figure 4). Note that T is not independent of S. Starting from the boundary at which the stress state is defined, the ZEL net is constructed and the initial stress field is obtained at the nodes of the ZEL net in a procedure similar to that used by the method of characteristics (for more information see [19]). In the second step, the constructed ZEL net is used to find the displacement and strain fields. Displacements are calculated by writing Equation 3 in finite difference form for both zero extension directions as:

$$(u_B - u_C)(r_B - r_C) + (w_B - w_C)(z_B - z_C) = 0, \quad (13a)$$

$$(u_A - u_C)(r_A - r_C) + (w_A - w_C)(z_A - z_C) = 0. \quad (13b)$$

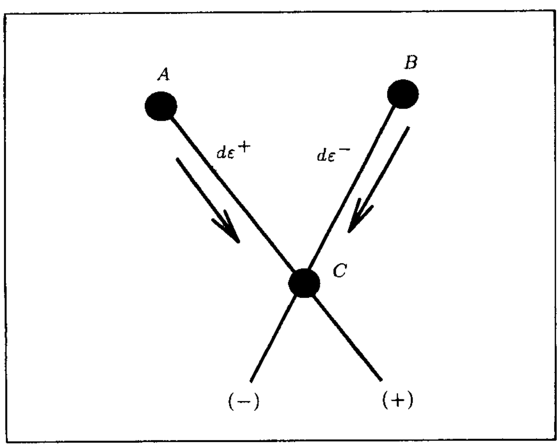

Figure 4. Obtaining the information at C from those of points A and B using ZEL net.

Starting from the boundary at which the displacements are known, u and w of any point like C can be calculated from those of A and B using Equations 13.

The shear strain field is calculated from the displacement field. This can be done simply by writing Equation 12 in finite difference form or considering the distortion of soil elements between the ZEL. The mobilized c or ϕ at each point can also be obtained from the stress-strain curve of the material using the strain of the point. The obtained field of c or ϕ is then used in calculating the stress field under this deformation field using the finite difference form of Equation 10a. In solving this equation, the following approximations are used in calculating the rate of change of functions ψ and T along the ZEL:

Along BC:

$$\left(\frac{\partial F}{\partial \varepsilon^+}\right)_B d\varepsilon^- \cong \left(\frac{\partial F}{\partial \varepsilon^+}\right)_A d\varepsilon^- = (F_C - F_A)\frac{d\varepsilon^-}{d\varepsilon^+}. \quad (14a)$$

Along AC:

$$\left(\frac{\partial F}{\partial \varepsilon^-}\right)_A d\varepsilon^+ \cong \left(\frac{\partial F}{\partial \varepsilon^-}\right)_B d\varepsilon^+ = (F_C - F_B)\frac{d\varepsilon^+}{d\varepsilon^-}. \quad (14b)$$

where F stands for ψ and T, and $d\varepsilon^-$ and $d\varepsilon^+$ are lengths $\overline{BC}$ and $\overline{AC}$, respectively. For obtaining better results from the solution of finite difference equations, the quantities multiplying the differences are averaged along the linear zero extension line elements in iterations after the first. The average stress S for the points along any (-) ZEL can be calculated from Equation 10a when written in finite difference form. Calculations of stress starts from the boundary at which the stresses are known and proceeds toward the boundary at which the stresses are to be determined. Iterations are required to obtain convergence since S_C is involved in calculation of the right side. A computer program has been written to solve the problems by this procedure.

SOME APPLICATIONS

Examples are provided here to illustrate the capability of the model in solving the axi-symmetric problems. The main purpose here is to show the required data for the model in each case and the capability of the model in providing the load-deflection curves.

Smooth and Rough Circular Footings on Clay

The behavior of circular footings on a saturated soft clay is considered as the first example. A smooth footing of 1 meter diameter is compared with a rough one of the same size. It has generally been accepted that Hill's mechanism will develop in the smooth case and Prandtl's in the rough case. In the rough case a conical wedge is assumed to form under the footing which plays the role of a relatively rigid cone [20]. Soil volume is assumed to remain constant in undrained shearing. The stress-strain curve of the soil in a typical unconfined compression test has been used to construct the relationship between the undrained shear strength and maximum shear strain of the soil. This relation has been shown in Figure 5a. The unit weight of the soil is 19 kN/m^3. Load-settlement curves for both footings in the form of average footing pressure versus settlement have been drawn in Figure 5b. The rate of increase in average pressure at higher settlements is much slower. The values of pressure at final settlement for rough and smooth footings are about 2% less than the ultimate pressure obtained by the method of characteristics [2,4]. The pattern of the ZEL field for the rough footing is shown in Figure 5c, together with the deformed net indicating the developed velocity field and heave at the ground surface. The pressure distribution under the rough footing for 0%, 50%, and 100% of the final settlement has also been shown on the same figure.

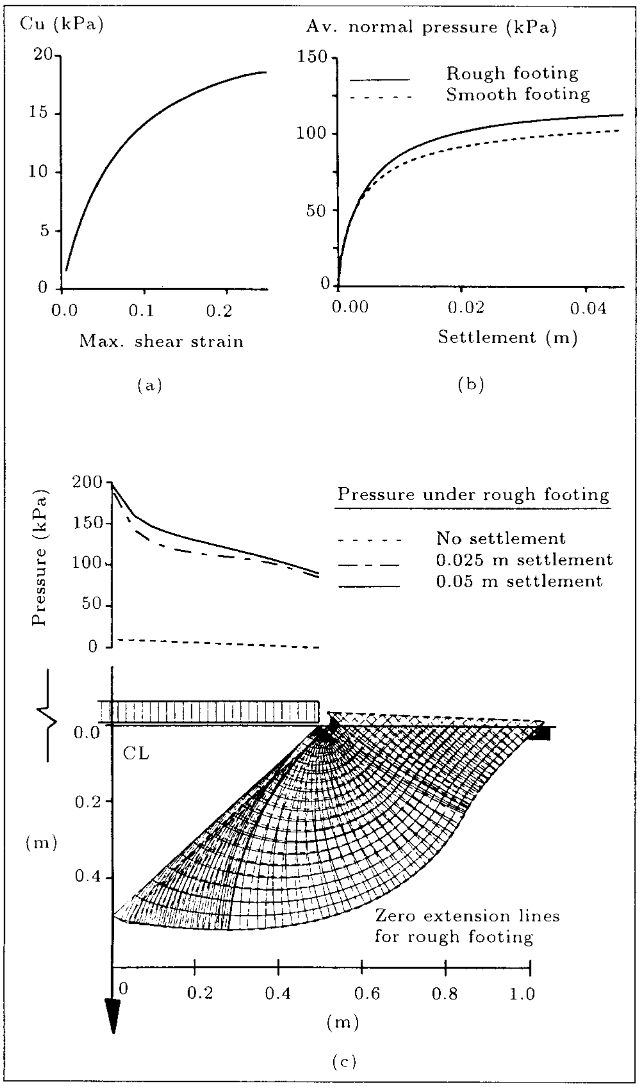

Figure 5. Load-deflection behavior of circular footings on saturated clay.

Circular Footing on Sand

A rough 1 meter diameter circular footing on a medium dense sand would be considered as another example. The stress-strain curve of the sand in a triaxial test performed at an average stress range has been used to construct the $\phi - \gamma_{\max}$ relation of the sand shown in Figure 6a. The unit weight is 19.6 kN/m^3 and the dilation angle is 10°. The pattern of the ZEL net for this case is shown in Figure 6c. The average pressure-settlement curve obtained from the analysis is shown in Figure 6b. The developed average pressure at 0.15 m settlement is 1553 kPa and is still increasing with more settlement. The ultimate bearing capacity for this condition using Bolton's suggested bearing capacity factors is 3418 kPa [20]. The pressure distribution under the footing for different percentages of the final settlement has also been shown in the figure. The velocity field induced in this case is also shown by the deformed ZEL net in the figure.

CONCLUSION

In this paper, a simple derivation has been put forward for the zero extension line theory for both axi-symmetric and plane strain cases, which is different from the method of characteristics. The zero extension line theory provides a rather simple method by which the solution of earth pressure problems can be obtained in the form of the load-deflection curves. The method is capable of predicting the pattern of development of the velocities and strains in the soil mass. It provides a simple way for predicting how the strength is mobilized at different locations of the soil mass due to the boundary deflections. It can also predict the pressure distribution on the structures in contact with soil. The zero extension line method assumes coaxiality of principal stress and principal strain increments. It relates the mobilized strength to the strain but assumes the dilation angle to remain constant during the shearing of soil. The method satisfies the equilibrium and progressive yielding conditions by writing the relevant equations directly along the zero extension lines. In this way, the stresses and displacements are

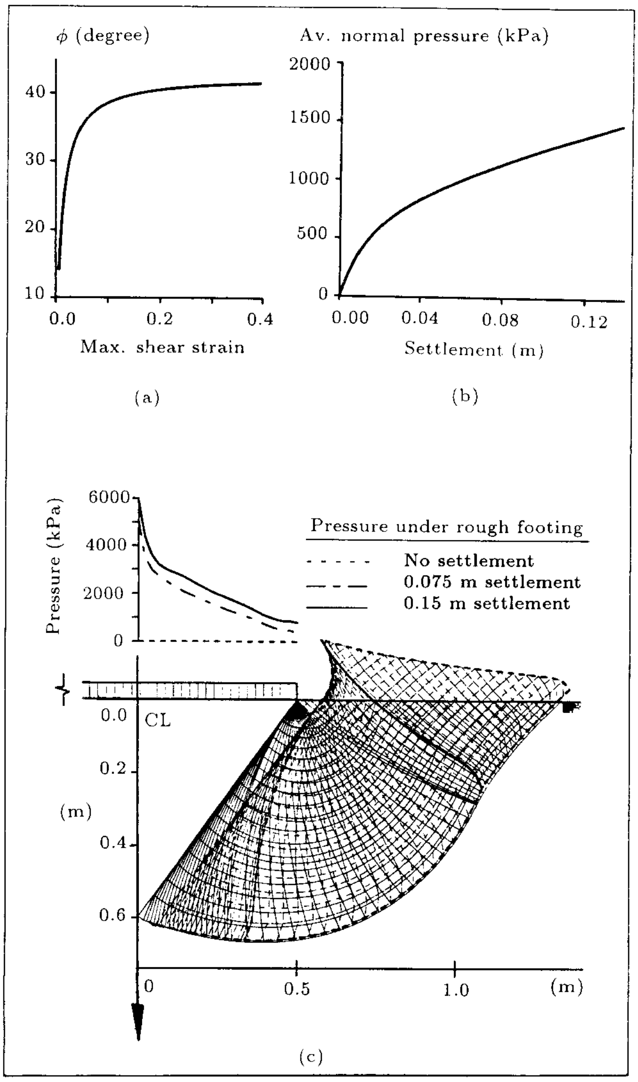

Figure 6. Load-deflection behavior of a rough circular footing on sand.

calculated at the same points in the soil mass and the difficulties involved in the associated fields method are avoided. It can then be concluded that the method provides a simple analytical tool for predicting the soil-structure interaction behaviors in plane strain and axially symmetric conditions. This has been shown by examples of smooth and rough circular footings on clay and sand. Attempt has not been made to verify the model quantitatively. This requires extensive full scale tests and proper laboratory tests on the same soil under the same conditions. Despite this, the calculated pressures at final settlements have been compared with the ultimate pressures obtained by the method of characteristics and were found to be generally lower than them. This is because the ultimate bearing capacity or passive loads obtained by the characteristic method are relevant to the conditions at which the full strength of the soil is mobilized everywhere with which the flow of soil is associated. These conditions rarely happen in applying the ZEL method to different problems. The ZEL method provides a more realistic analysis of the ultimate load, as well as the load-deformation behavior, since it considers the pattern of development of strains and strength in soil.

ACKNOWLEDGMENT

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NOMENCLATURE

c	cohesion
f_r, f_z	as defined by Equations 5 and 11
F	function of variables r and z or ε^- and ε^+
n	an integer equal to 1 for axi-symmetric and 0 for plane strain problems
S	average stress $=(\sigma_r + \sigma_z)/2$
u, w	components of displacement in r and z directions, respectively
r, z	coordinates along r and z axes
R, Z	body and/or inertial forces in r and z directions, respectively
$\varepsilon_1, \varepsilon_3$	major and minor principal incremental strains
$d\varepsilon^-, d\varepsilon^+$	small distances along the - and + zero extension lines
ϕ	angle of internal friction
γ	shear strain
$\xi = \pi/4 - \nu/2$	angle between zero extension lines and the direction of σ_1 or ε_1
ν	in plane dilation angle of soil
ρ	density of soil
ψ	angle between σ_1 or ε_1 and r direction
$\sigma_r, \sigma_z, \tau_{rz}$	components of stress tensor with respect to $r - z$ coordinates
σ_1	major principal stress
σ_θ	circumferential stress in the axi-symmetric problem
τ	shear stress
θ	annular coordinate in the axi-symmetric problem

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