Scientia Iranica, Vol. 14, No. 5, pp 414 423
© Sharif University of Technology, October 2007

SCIENTIA
IRANICA

First Order Perturbation Solution for Axial Vibration of Tension Leg Platforms

A.A. Golafshani[1], M.R. Tabeshpour* and M.S. Seif[2]

The dynamic response of the leg (tether) of a Tension Leg Platform (TLP), subjected to axial load at the top of the leg, is presented. The structural model is very simple, but several complicated factors, such as foundation effect, buoyancy and simulated ocean wave load, are considered. As an application, the effect of added mass fluctuation on the dynamic response of the leg subjected to such a load is presented. This effect is important in the fatigue life study of tethers. A first order perturbation method is used, in order to formulate and solve the problem. The differential equation is solved by means of non-harmonic Fourier expansion, in terms of eigenfunctions obtained from a non-regular Sturm-Liouville system.

INTRODUCTION

A Tension Leg Platform (TLP) is a suitable structure for oil exploitation in deep water. Many studies have been carried out to understand the structural behavior of a TLP and to determine the effect of several parameters on the dynamic response and average life time of the structure [1-4]. Angelides et al. (1982) considered the influence of hull geometry, force coefficients, water depth, pre-tension and tendon stiffness on the dynamic responses of the TLP. The floating part of the TLP was modeled as a rigid body with six degrees of freedom and the tendons were represented by linear axial springs [5]. Morgan and Malaeb (1983) investigated the dynamic response of TLPs using a deterministic analysis. The analysis was based on coupled nonlinear stiffness coefficients, closed-form inertia and drag-forcing functions, using the Morison equation [6].

A comprehensive study on the results of tension leg platform responses in random seas, considering all structural and excitation nonlinearities, is presented by Tabeshpour et al. [7]. This kind of interpretation of the results is necessary for the optimum design of a TLP.

There are several issues in design optimization of a TLP. The geometrical optimum design of a TLP hull is presented by using a genetic algorithm method under regular sea waves [8]. Such a method can be used to extend the structural optimization under random wave loads. The optimum pretension of tendons can be determined, based on minimum down time or maximum fatigue life. In minimum down time, the nonlinear time histories of deformations and accelerations are investigated and, in a fatigue study, a first order reliability method is used to estimate the lifetime of the tendons.

Work on closed form solutions of a TLP can be very useful, in order to have a deep view of the structural behavior, due to high nonlinearities in the real structure. A continuous model for the vertical motion of a TLP, considering the effect of a continuous foundation, has been reported [9]. The exact solution of the heave response of the structure can be useful, both in the initial design of the tendons and in the verification of the complete coupled model responses. Added mass fluctuation is an important point, because of its direct effect on the lifetime of the tendons when fatigue analyses are carried out. Fluctuating added mass has a direct relation with the heave response of the hull structure. The effect of added mass fluctuation on the heave response of a

1. *Department of Civil Engineering, Sharif University of Technology, Tehran, I.R. Iran.*
*. *Corresponding Author, Department of Civil Engineering, Sharif University of Technology, Tehran, I.R. Iran.*
2. *Department of Mechanical Engineering, Sharif University of Technology, Tehran, I.R. Iran.*

tension leg platform has been investigated, using a perturbation method both for discrete and continuous models [10,11]. Also, the analytical solution derived can be used to verify the numerical results of the complete model.

Another important problem is the investigation of the effects of radiation and scattering on the hull and tendon responses. An analytical solution for the surge motion of a TLP was proposed and demonstrated [12-14], in which the surge motion of a platform with pre-tensioned tethers was calculated. In that study, however, the elasticity of tethers was only implied and the motion of tethers was also simplified as an on-line rigid-body motion proportional to the top platform. Thus, both the material property and the mechanical behavior for the tether incorporated in the tension leg platform system were ignored. When this simplification was applied, no matter what material was used or what dimension of tethers, the dynamic response of the platform would remain the same, in terms of the vibration mode, periods and the vibration amplitude. An important point in that study was the linearization of the surge motion. But, it is obvious that the structural behavior in the surge motion is highly nonlinear, because of the large deformation of a TLP in the surge motion degree of freedom (geometric nonlinearity) and nonlinear drag forces of the Morison equation. Therefore, the obtained solution is not true for the actual engineering application. For heave degree of freedom, the structural behavior is linear, because there is no geometric nonlinearity in the heave motion degree of freedom and the drag forces on the legs have no vertical component. Similarly, an analytical heave vibration of a TLP with radiation and scattering effects for damped systems has been presented [15]. A similar method is presented for the hydrodynamic pitch response of the structure [16].

The most important point in the design of a TLP is the pretension of the legs. The pretension causes the platform to behave like a stiff structure, with respect to the vertical degrees of freedom (heave, pitch and roll), whereas, with respect to the horizontal degrees of freedom (surge, sway and yaw), it behaves as a floating structure. Among the various degrees of freedom, vertical motion (heave) is very important, because of its direct effect on stress fluctuation, which may lead to the fatigue and fracture of the tethers. Therefore, conceptual studies for understanding the dynamic vertical response of a TLP can be useful for designers. Rossit et al. [17] presented an analytical solution for the dynamic response of the leg of a TLP subjected to a suddenly applied axial load at one end. The applied load was constant and the effect of the buoyancy was not considered.

The aim of this paper is to find a solution to the mentioned problem, using two models. The structural models are very simple, but several complicated factors, such as foundation, buoyancy and simulated ocean wave loading, are considered. In the first model, the foundation is assumed to be rigid, but, in the second model, it is assumed that the foundation is embedded in the ocean bottom, which acts as a Winkler-type elastic foundation. Buoyancy is the force needed to apply a unit displacement to the body in the water. Therefore, it is modeled as a spring at the top of the leg. A concentrated force is applied at the top of the leg, as the simulated load of an ocean wave. The problem is solved by means of a non-harmonic Fourier expansion, in terms of eigenfunctions obtained from a non-regular Sturm-Liouville system [18,19].

ANALYTICAL SOLUTION OF THE MODEL

The structural model of the system is shown in Figure 1. The behavior of the system is described by the following differential equation:

$$\Big([u(y) \quad u(y \quad l_f)]E_f A_f + [u(y \quad l_f)$$

$$u(y \quad l)]E_t A_t\Big)\frac{\partial^2 v}{\partial y^2} + F_h(t)\delta(y \quad l)$$

$$= \Big([u(y) \quad u(y \quad l_f)]\rho_f A_f + [u(y \quad l_f)$$

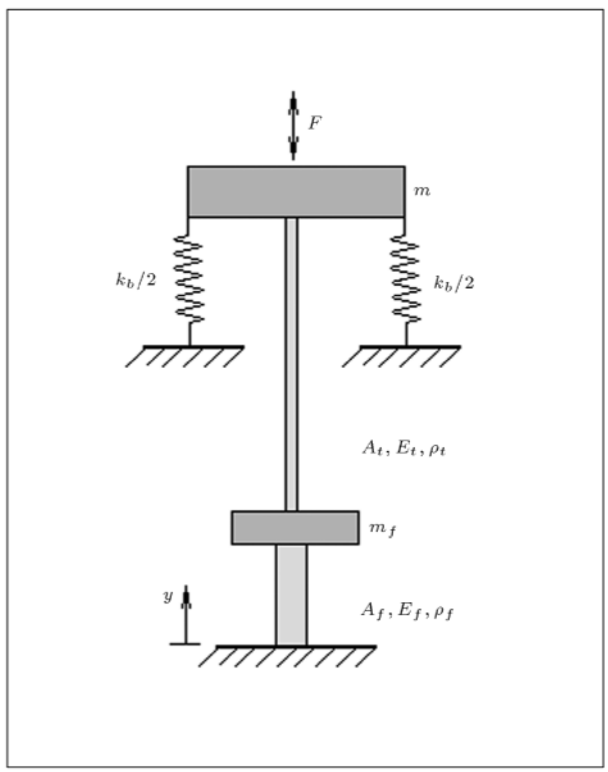

Figure 1. Dynamic structural model.

$$u\,(y \quad l)]\,\rho_t A_t + M_f \delta\,(y \quad l_f)$$

$$+ M\delta\,(y \quad l)\Big)\,\frac{\partial^2 v}{\partial t^2}, \tag{1}$$

where u is the step function, v is the longitudinal deformation, E is the Young modulus of the tether material, A_t and A_f are the cross section areas of the tether and foundation, respectively, ρ_t and ρ_f are the density of the tether and foundation materials, respectively, l_t and l_f are the length of tether and foundation, respectively, and δ denotes the Dirac delta function. The applied vertical load subjected to the mass, m, is the simulated wave load, $F_h(t) = \sum_{j=1}^{N} F_j \sin(\Omega_j t + \phi_j)$, obtained from the wave spectrum.

The system is linear, therefore, one can find the solution of Equation 1 using a single Fourier component, $F_h(t) = F_0 \sin(\Omega t)$, and, then, determine the overall response of the system by the superposition of responses to single Fourier components. The initial conditions are, as follows:

$$v(y,0) = 0,$$

$$\frac{\partial v}{\partial t}(y,0) = 0. \tag{2}$$

And the mass distribution functions are defined, as follows:

$$m(y) = [u(y) \quad u(y \quad l_f)]\rho_f A_f$$

$$+ [u(y \quad l_f) \quad u(y \quad l)]\rho_t A_t + M_f \delta(y \quad l_f)$$

$$+ M\delta(y \quad l). \tag{3}$$

In the case of free vibration, Equation 1 becomes:

$$\Big([u(y) \quad u(y \quad l_f)], E_f A_f + [u(y \quad l_f)$$

$$u(y \quad l)]E_t A_t\Big)\frac{\partial^2 v}{\partial y^2} = m(y)\frac{\partial^2 v}{\partial t^2}, \tag{4}$$

which can be solved by assuming:

$$m(y) = [u(y) \quad u(y \quad l_f)]\rho_f A_f + [u(y \quad l_f)$$

$$u(y \quad l)]\rho_t A_t,$$

subjected to the boundary conditions:

$$v(0,t) = 0, \tag{5}$$

$$E_t A_t\,\frac{\partial v}{\partial y}\,(l_f,t) \quad M_f\,\frac{\partial^2 v}{\partial t^2}\,(l_f,t)$$

$$= E_f A_f\,\frac{\partial v}{\partial y}\,(l_f,t), \tag{6}$$

$$k_b v(l,t) \quad M\frac{\partial^2 v}{\partial t^2}(l,t) = E_t A_t \frac{\partial v}{\partial y}(l,t). \tag{7}$$

One needs to solve Equation 4 for $0 \le y \le l_f$ and $l_f \le y \le l$ independently and match the solutions wherever the two parts are connected.

For $0 \le y = y_1 \le l_f$, one has:

$$E_f A_f \frac{\partial^2 v}{\partial y^2} = m(y)\frac{\partial^2 v}{\partial t^2},$$

or:

$$\frac{\partial^2 v}{\partial y^2} = \frac{1}{c_f^2}\frac{\partial^2 v}{\partial t^2}, \tag{8}$$

where $m(y) = \rho_f A_f$, $c_f^2 = E_f/\rho_f$.

Using the method of separation of variables, the eigenfunctions are determined as:

$$Y_{n1} = B \sin \alpha_{nf} y_1, \tag{9}$$

where α_{nf} is the separation constant and $c_f \alpha_{nf} = \omega_{nf}$ is the angular frequency.

For $l_f \le y = y_2 \le l$, one has:

$$E_t A_t \frac{\partial^2 v}{\partial y^2} = m(y)\frac{\partial^2 v}{\partial t^2},$$

or:

$$\frac{\partial^2 v}{\partial y^2} = \frac{1}{c_t^2}\frac{\partial^2 v}{\partial t^2}, \tag{10}$$

where $m(y) = \rho_t A_t$, $c_t^2 = E_t/\rho_t$.

Similarly, using the method of separation of variables, the eigenfunctions are determined, as follows:

$$Y_{n2} = A[\cos\ \alpha_{nt} y_2 + (\frac{k_b}{k_t}\ \frac{1}{\alpha_{nt} l_t}$$

$$\frac{M}{m_t}\ \alpha_{nt} l_t)\ \sin\ \alpha_{nt} y_2], \tag{11}$$

where α_{nt} is the separation constant and $c_t \alpha_{nt} = \omega_{nt}$ is the angular frequency.

It is clear that the displacement and force fields are continuous at $y_1 = l_f$ and $y_2 = 0$. The continuity of the displacement fields implies:

$$Y_{n1}(l_f) = Y_{n2}(l_t),$$

consequently,

$$A\left[\cos \alpha_{nt} l_t + \left(\frac{k_b}{k_t}\frac{1}{\alpha_{nt} l_t} \quad \frac{M}{m_t}\alpha_{nt} l_t\right)\sin \alpha_{nt} l_t\right]$$

$$B \sin \alpha_{nf} l_f = 0. \tag{12}$$

Also, the continuity of the force field gives:

$$E_tA_t\ \frac{\partial v_1}{\partial y_1}(l_t,t) = \quad E_fA_f\ \frac{\partial v_2}{\partial y_2}(l_f,t)$$

$$M_f\ \frac{\partial^2 v_1}{\partial t^2}(l_t,t). \tag{13}$$

Or, equivalently,

$$A\frac{k_t}{k_f}\alpha_{nt}l_t\Bigg[\sin\alpha_{nt}l_t \quad \Big(\frac{k_b}{k_t}\frac{1}{\alpha_{nt}l_t}$$

$$\frac{M}{m_t}\alpha_{nt}l_t\Big)\cos\alpha_{nt}l_t\Bigg]$$

$$+B\Big(\frac{M_f}{m_f}\alpha_{nf}^2l_f^2\sin\alpha_{nf}l_f$$

$$\alpha_{nf}l_f\cos\alpha_{nf}lf\Big) = 0. \tag{14}$$

The coefficients A and B can be determined by solving Equations 12 and 14. There are non-zero solutions, if:

$$\begin{vmatrix} \cos\alpha_{nt}l_t + \left(\frac{k_b}{k_t}\frac{1}{\alpha_{nt}l_t} \quad \frac{M}{m_t}\alpha_{nt}l_t\right)\sin\alpha_{nt}l_t & \sin\alpha_{nf}l_f \\ \frac{k_t}{k_f}\alpha_{nt}l_t\left[\sin\alpha_{nt}l_t \quad \left(\frac{k_b}{k_t}\frac{1}{\alpha_{nt}l_t} \quad \frac{M}{m_t}\alpha_{nt}l_t\right)\cos\alpha_{nt}l_t\right] & \frac{M_f}{m_f}\alpha_{nf}^2l_f^2\sin\alpha_{nf}l_f \quad \alpha_{nf}l_f\cos\alpha_{nf}l_f \end{vmatrix} = 0, \tag{15}$$

which results in the frequency equation:

$$\left[\left(\frac{k_b}{k_t}\frac{1}{\alpha_{nt}l_t} \quad \frac{M}{m_t}\alpha_{nt}l_t\right)\tan\alpha_{nt}l_t + 1\right]$$

$$\left(\quad\frac{M_f}{m_f}\alpha_{nf}l_f\tan\alpha_{nf}l_f + 1\right)$$

$$\frac{k_t}{k_f}\frac{\alpha_{nt}l_t}{\alpha_{nf}l_f}\tan\alpha_{nf}l_f$$

$$\left[\tan\alpha_{nt}l_t \quad \left(\frac{k_b}{k_t}\frac{1}{\alpha_{nt}l_t} \quad \frac{M}{m_t}\alpha_{nt}l_t\right)\right] = 0, \tag{16}$$

with:

$\rho_tA_tl_t = m_t$: total mass of the tether,
$\rho_fA_fl_f = m_f$: total mass of the foundation,
$E_tA_t/l_t = k_t$: the axial stiffness of the tether,
$E_fA_f/l_f = k_f$: axial stiffness of the foundation.

The response of the tether subjected to axial load can be expressed in terms of the normal modes of the system, as follows:

$$v(y,t) = \sum_{n=1}^{\infty} Y_n(y)T_n(t), \tag{17}$$

$$Y_n(y) = \lfloor u(y) \quad u(y \quad l_f)\rfloor Y_1(y) + [u(y \quad l_f)$$

$$u(y \quad l)]Y_2(y), \tag{18}$$

$$M(y) = [u(y) \quad u(y \quad l_f)]\rho_fA_f + [u(y \quad l_f)$$

$$u(y \quad l)]\rho_tA_t + M_f\delta(y \quad l_f)$$

$$+ M\delta(y \quad l). \tag{19}$$

According to the orthogonality of the normal modes, it can be shown that:

$$\int_0^l M(y)Y_n(y)Y_r(t) = 0\ (n \neq r), \tag{20a}$$

$$\int_0^l M(y)Y_n(y)Y_r(t) = H_r\ (n = r). \tag{20b}$$

Defining:

$$M_1(y_1) = \rho_fAf + M_f\delta(y_1 \quad l_f), \tag{21}$$

and:

$$M_2(y_2) = \rho_tA_t + M\delta(y_2) + M_f\delta(y_2 \quad l_t), \tag{22}$$

Equation 20 is rewritten as:

$$\int_0^{l_f} M_1(y_1)Y_{n1}(y_1)Y_{r1}(y_1)$$

$$+\int_0^{l_t} M_2(y_2)Y_{n2}(y_2)Y_{r2}(y_2) = 0$$

$$(n \neq r), \tag{23a}$$

$$\int_0^{l_f} M_1(y_1)Y_{n1}(y_1)Y_{r1}(y_1)$$

$$+\int_0^{l_t} M_2(y_2)Y_{n2}(y_2)Y_{r2}(y_2) = H_r$$

$$(n = r), \tag{23b}$$

$$H_r = H_{r1} + H_{r2}, \tag{24a}$$

$$H_{r1} = \frac{m_f}{2} + Y_{r1}^2(l_f)\left(\frac{M_f}{2}\right), \tag{24b}$$

$$H_{r2} = \left(\frac{k_b}{k_t}\frac{1}{\alpha_{nt}l_t} \quad \frac{M}{m_t}\alpha_{nt}l_t\right)^2 \frac{m_t}{2} + \frac{m_t}{2}$$

$$+ Y_{r2}^2(l_t)\left(\frac{M_f}{2}\right) + Y_{r2}^2(0)\left(\frac{k_b}{2\alpha_{rt}^2 c_t^2} + \frac{M}{2}\right). \tag{24c}$$

Multiplying Equation 1 by:

$$Y_n(y)dy = \lfloor u(y) \quad u(y \quad l_f)\rfloor Y_1(y)dy$$

$$+ [u(y \quad l_f) \quad u(y \quad l)]Y_2(y)dy,$$

and integrating over $0 \leq y \leq l$, one obtains:

$$\Big([u(y) \quad u(y \quad l_f)]\ E_f A_f$$

$$+ [u(y \quad l_f) \quad u(y \quad l)]\ E_t A_t\Big)$$

$$\times \int_0^l Y_r(\sum Y_n'' T_n)dy + F_h(t)\int_0^l Y_r \delta(y)dy$$

$$= \int_0^l M(y)Y_r(\sum Y_n \ddot{T}_n)dy, \tag{25}$$

or:

$$E_f A_f \int_0^{l_f} Y_{r1}\left(\sum Y_{n1}'' T_n\right)dy_1$$

$$+ E_t A_t \int_{l_f}^{l} Y_{r2}\left(\sum Y_{n2}'' T_n\right)dy_2$$

$$+ F_h(t)Y_{r2}(0) = H_r \ddot{T}_r. \tag{26}$$

Since Y_{n1} satisfies Equation 8 and Y_{n2} satisfies Equation 10, one has:

$$Y_{n1}'' = \quad \frac{M_1(y_1)}{E_f A_f} c_f^2 \alpha_{nf}^2 Y_{n1}, \tag{27}$$

$$Y_{n2}'' = \quad \frac{M_2(y_2)}{E_t A_t} c_t^2 \alpha_{nt}^2 Y_{n2}. \tag{28}$$

Substituting Equation 27 and Equation 28 in Equation 26 and applying Equation 23 results in:

$$\ddot{T}_n + \left(c_f^2\alpha_{nf}^2\frac{H_{n1}}{H_n} + c_t^2\alpha_{nt}^2\frac{H_{n2}}{H_n}\right)T_n$$

$$= \frac{Y_{n2}(0)}{H_n}F_0\sin(\Omega t), \tag{29}$$

which is a non-homogeneous ordinary differential equation, whose solution is given by:

$$T_n(t) = A_n\cos k_n t + B_n \sin k_n t$$

$$+ \frac{F_0}{k_n^2 \quad \Omega^2}\frac{Y_{n2}(0)}{H_n}\sin(\Omega t), \tag{30}$$

where:

$$k_n^2 = c_f^2\alpha_{nf}^2\frac{H_{n1}}{H_n} + c_t^2\alpha_{nt}^2\frac{H_{n2}}{H_n}. \tag{31}$$

Initial conditions result in:

$$A_n = 0,$$

$$B_n = \quad \frac{F_0}{k_n^2 \quad \Omega^2}\frac{\Omega}{k_n}\frac{Y_{n2}(0)}{H_n}, \tag{32}$$

and:

$$T_n(t) = \frac{F_0}{k_n^2 \quad \Omega^2}\frac{\Omega}{k_n}\frac{Y_{n2}(0)}{H_n}$$

$$\times\left(\quad \frac{\Omega}{k_n}\sin k_n t + \sin(\Omega t)\right), \tag{33}$$

or:

$$T_n(t) = \frac{\frac{F_0}{k_n^2 \quad \Omega^2}\frac{\Omega}{k_n}\frac{Y_{n2}(0)}{H_n}\left(\quad \frac{\Omega}{k_n}\sin k_n t + \sin(\Omega t)\right)}{\frac{m_f}{2} + (Y_{r1}^2(l_f) + Y_{r2}^2(l_t))\left(\frac{M_f}{2}\right) + \left(\frac{k_b}{k_t}\frac{1}{\alpha_{nt}l_t} \quad \frac{M}{m_t}\alpha_{nt}l_t\right)^2\frac{m_t}{2} + \frac{m_t}{2} + Y_{r2}^2(0)\left(\frac{k_b}{2\alpha_{rt}^2c_t^2} + \frac{M}{2}\right)}, \tag{34}$$

and:

$$T_{nj}(t) = \frac{\frac{F_j}{k_n^2 \quad \Omega_j^2}\frac{\Omega_j}{k_n}\frac{Y_{n2}(0)}{H_n}\left(\quad \frac{\Omega_j}{k_n}\sin k_n t + \sin(\Omega_j t + \phi_j)\right)}{\frac{m_f}{2} + (Y_{r1}^2(l_f) + Y_{r2}^2(l_t))\left(\frac{M_f}{2}\right) + \left(\frac{k_b}{k_t}\frac{1}{\alpha_{nt}l_t} \quad \frac{M}{m_t}\alpha_{nt}l_t\right)^2\frac{m_t}{2} + \frac{m_t}{2} + Y_{r2}^2(0)\left(\frac{k_b}{2\alpha_{rt}^2c_t^2} + \frac{M}{2}\right)}, \tag{35}$$

and:

$$Y_n(y) = [u(y) \quad u(y \quad l_f)] \sin \alpha_{nf} y_1$$
$$+ [u(y \quad l_f) \quad u(y \quad l)]$$
$$\left[\cos \alpha_{nt} y_2 + \left(\frac{k_b}{k_t} \frac{1}{\alpha_{nt} l_t} \quad \frac{M}{m_t} \alpha_{nt} l_t\right) \sin \alpha_{nt} y_2\right]. \tag{36}$$

Note that the vertical vibration frequency of a typical TLP is between 1.5-2 sec and the excitation frequency content of a sea wave is between 4-15 sec. Therefore, it is not necessary to consider resonance frequency for the vertical vibration of a TLP. Now, the dynamic response of the tether is, as follows:

$$v(y,t) = \sum_{n=1}^{\infty} \sum_{j=1}^{N} Y_n(y) T_{nj}(t). \tag{37}$$

PERTURBATION BASED SOLUTION OF ADDED MASS FLUCTUATION

As a conceptual application of the presented formulation, the problem of added mass fluctuation of the vertical motion of a moored structure is investigated. The structural model of the system has been shown in Figure 2. The behavior of the system is described by the following differential equation:

$$EA_t \frac{\partial^2 v}{\partial y^2} + F_h(t)\delta(y \quad l) = [\rho_t A_t$$
$$+ m(1 + \varepsilon a v)\delta(y \quad l)] \frac{\partial^2 v}{\partial t^2}, \tag{38}$$

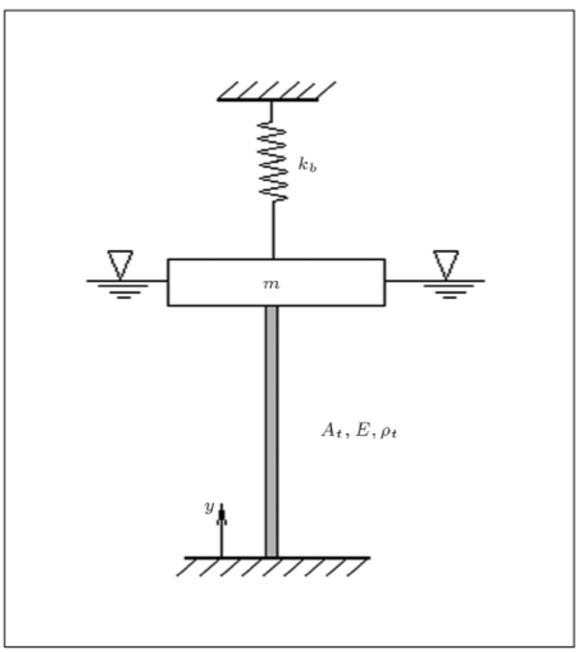

Figure 2. Dynamic structural model.

where ε is the added mass fluctuation parameter and a is the ratio of the added mass to the structural mass. The mass distribution function is defined, as follows:

$$M(y,t) = \rho_t A_t + m[1 + \varepsilon a v(y,t)]\delta(y \quad l). \tag{39}$$

In the case of free vibration, Equation 38 becomes:

$$EA_t \frac{\partial^2 v}{\partial y^2} = M(y,t) \frac{\partial^2 v}{\partial t^2}. \tag{40}$$

Considering the first order perturbation of the response [20], one has:

$$v(y,t) = v_0(y,t) + \varepsilon v_1(y,t). \tag{41}$$

Substituting Equation 41 in Equation 38 results in:

$$EA_t \frac{\partial^2 (v_0 + \varepsilon v_1)}{\partial y^2} + F_h(t)\delta(y \quad l)$$
$$= [\rho_t A_t + m(1 + \varepsilon a(v_0 + \varepsilon v_1))$$
$$\delta(y \quad l)] \frac{\partial^2 (v_0 + \varepsilon v_1)}{\partial t^2}. \tag{42}$$

According to the perturbation theory, one obtains the two following equations:

$$EA_t \frac{\partial^2 v_0}{\partial y^2} + F_h(t)\delta(y \quad l)$$
$$= [\rho_t A_t + m\delta(y \quad l)] \frac{\partial^2 v_0}{\partial t^2}, \tag{43}$$

$$EA_t \frac{\partial^2 v_1}{\partial y^2} + F'_h(t)\delta(y \quad l)$$
$$= [\rho_t A_t + m\delta(y \quad l)] \frac{\partial^2 v_1}{\partial t^2}, \tag{44}$$

where:

$$F'_h(t) = \quad a m v_0 \frac{\partial^2 v_0}{\partial t^2}. \tag{45}$$

Similarly, solving the differential equation gives:

$$T_{n0}(t) = \frac{F_0}{c^2 \alpha_{n0}^2 \quad \Omega^2} \frac{Y_{n0}(l)}{H_{n0}} \Big(\quad \frac{\Omega}{c\alpha_{n0}} \sin c\alpha_{n0} t$$
$$+ \sin(\Omega t)\Big). \tag{46}$$

Now, the dynamic response of the tether is given by:

$$v_0(y,t)=\sum_{n=1}^{\infty}\frac{F_0}{c^2\alpha_{n0}^2-\Omega^2}$$

$$\frac{Y_{n0}(l)\sin\alpha_{n0}y\left(-\frac{\Omega}{c\alpha_{n0}}\sin c\alpha_{n0}t+\sin(\Omega t)\right)}{\frac{m_t}{2}+Y_{r0}^2(l)\left(\frac{k_b}{2\alpha_{r0}^2c^2}+\frac{m}{2}\right)}. \quad (47)$$

Using the separation of variables, the eigenfunctions are determined, as follows:

$$Y_{n1}=\sin\alpha_{n1}y, \quad (48)$$

and the frequency equation becomes:

$$\tan\alpha_{n1}l=\frac{\alpha_{n1}l}{\frac{m}{m_t}\alpha_{n1}^2l^2-\frac{k_b}{k_t}}, \quad (49)$$

where α_{n1} is the separation constant.

The response of a tether subjected to axial load can be expressed in terms of the normal modes of the system, as follows:

$$v_1(y,t)=\sum_{n=1}^{\infty}Y_{n1}(y)T_{n1}(t), \quad (50)$$

where:

$$H_{r1}=\frac{m_t}{2}+Y_{r1}^2(l)\left(\frac{k_b}{2\alpha_{r1}^2c^2}+\frac{m}{2}\right). \quad (51)$$

As previously, one has:

$$Y_{n1}''=-\frac{M(y)}{EA_t}c^2\alpha_{n1}^2Y_{n1}, \quad (52)$$

and:

$$\ddot{T}_{n1}+c^2\alpha_{n1}^2T_{n1}=\frac{Y_{n1}(l)}{H_{n1}}F_h'(t). \quad (53)$$

Substituting Equation 45 in Equation 53 results in:

$$\ddot{T}_{n1}+c^2\alpha_{n1}^2T_{n1}=am\frac{Y_{n1}(l)}{H_{n1}}Y_{n0}^2(l)T_{n0}\ddot{T}_{n0},$$

and:

$$T_{n0}\ddot{T}_{n0}=\Omega c\alpha_{n0}\left(\frac{F_0}{c^2\alpha_{n0}^2-\Omega^2}\frac{Y_{n0}(l)}{H_{n0}}\right)^2$$

$$\left[\frac{\Omega}{c\alpha_{n0}}(-\sin^2 c\alpha_{n0}t-\sin^2\Omega t)\right.$$

$$\left.+\left(1+\frac{\Omega^2}{c^2\alpha_{n0}^2}\right)\sin c\alpha_{n0}t\sin\Omega t\right]. \quad (54)$$

After some mathematical calculation, Equation 31 is reformed as:

$$T_{n0}\ddot{T}_{n0}=\frac{\Omega c\alpha_{n0}}{2}\left(\frac{F_0}{c^2\alpha_{n0}^2-\Omega^2}\frac{Y_{n0}(l)}{H_{n0}}\right)^2$$

$$\left[\frac{\Omega}{c\alpha_{n0}}(\cos 2c\alpha_{n0}t+\cos 2\Omega t-2)\right.$$

$$+\left(1+\frac{\Omega^2}{c^2\alpha_{n0}^2}\right)\Big(\cos(c\alpha_{n0}-\Omega)t$$

$$\left.-\cos(c\alpha_{n0}+\Omega)t\Big)\right]. \quad (55)$$

As previously, one has:

$$\ddot{T}_{n1}+c^2\alpha_{n1}^2T_{n1}=am\Omega c\alpha_{n0}\frac{Y_{n1}(l)Y_{n0}^2(l)}{2H_{n1}}$$

$$\left(\frac{F_0}{c^2\alpha_{n0}^2-\Omega^2}\frac{Y_{n0}(l)}{H_{n0}}\right)^2$$

$$\times\left[\frac{\Omega}{c\alpha_{n0}}(\cos 2c\alpha_{n0}t+\cos 2\Omega t-2)\right.$$

$$+\left(1+\frac{\Omega^2}{c^2\alpha_{n0}^2}\right)(\cos(c\alpha_{n0}-\Omega)t$$

$$\left.-\cos(c\alpha_{n0}+\Omega)t)\right]. \quad (56)$$

Solving the differential Equation 53, one obtains:

$$T_{n1}(t)=A_{n1}\cos c\alpha_{n1}t+B_{n1}\sin c\alpha_{n1}t$$

$$+am\Omega c\alpha_{n0}\frac{Y_{n1}(l)Y_{n0}^2(l)}{2H_{n1}}$$

$$\left(\frac{F_0}{c^2\alpha_{n0}^2-\Omega^2}\frac{Y_{n0}(l)}{H_{n0}}\right)^2$$

$$\times\left[\frac{\Omega}{c\alpha_{n0}}\left(-\frac{2}{c^2\alpha_{n0}^2}\right.\right.$$

$$\left.-\frac{\cos 2c\alpha_{n0}t}{3c^2\alpha_{n0}^2}+\frac{\cos 2\Omega t}{c^2\alpha_{n0}^2-4\Omega^2}\right)$$

$$+(1+\frac{\Omega^2}{c^2\alpha_{n0}^2})\left(\frac{\cos(c\alpha_{n0}-\Omega)t}{\Omega(2c\alpha_{n0}-\Omega)}\right.$$

$$\left.\left.+\frac{\cos(c\alpha_{n0}+\Omega)t}{\Omega(2c\alpha_{n0}+\Omega)}\right)\right]. \quad (57)$$

From initial conditions, one has:

$$B_{n1} = 0,$$

and:

$$A_{n1} = \quad am\Omega c\alpha_{n0}\frac{Y_{n1}(l)Y_{n0}^2(l)}{2H_{n1}}$$
$$\left(\frac{F_0}{c^2\alpha_{n0}^2 \quad \Omega^2}\frac{Y_{n0}(l)}{H_{n0}}\right)^2$$
$$\left[\frac{\Omega}{c\alpha_{n0}}\left(\quad \frac{7}{3c^2\alpha_{n0}^2} + \frac{1}{c^2\alpha_{n0}^2 \quad 4\Omega^2}\right)\right.$$
$$\left. + (1 + \frac{\Omega^2}{c^2\alpha_{n0}^2})\left(\frac{4c\alpha_{n0}}{\Omega(4c^2\alpha_{n0}^2 \quad \Omega^2)}\right)\right],$$

and, therefore:

$$T_{n1}(t) = \quad am\Omega c\alpha_{n0}\frac{Y_{n1}(l)Y_{n0}^2(l)}{2H_{n1}}$$
$$\left(\frac{F_0}{c^2\alpha_{n0}^2 \quad \Omega^2}\frac{Y_{n0}(l)}{H_{n0}}\right)^2$$
$$\times\left\{\left[\frac{\Omega}{c\alpha_{n0}}(\quad \frac{7}{3c^2\alpha_{n0}^2} + \frac{1}{c^2\alpha_{n0}^2 \quad 4\Omega^2})\right.\right.$$
$$\left. + (1 + \frac{\Omega^2}{c^2\alpha_{n0}^2})\left(\frac{4c\alpha_{n0}}{\Omega(4c^2\alpha_{n0}^2 \quad \Omega^2)}\right)\right]$$
$$\cos c\alpha_{n1}t \quad \left[\frac{\Omega}{c\alpha_{n0}}\left(\quad \frac{2}{c^2\alpha_{n0}^2}\right.\right.$$
$$\left.\frac{\cos 2c\alpha_{n0}t}{3c^2\alpha_{n0}^2} + \frac{\cos 2\Omega t}{c^2\alpha_{n0}^2 \quad 4\Omega^2}\right)$$
$$+ \left(1 + \frac{\Omega^2}{c^2\alpha_{n0}^2}\right)\left(\frac{\cos(c\alpha_{n0} \quad \Omega)t}{\Omega(2c\alpha_{n0} \quad \Omega)}\right.$$
$$\left.\left.\left. + \frac{\cos(c\alpha_{n0} + \Omega)t}{\Omega(2c\alpha_{n0} + \Omega)}\right)\right]\right\}. \tag{58}$$

By substituting Equations 48 and 58 into 50, v_1 is determined as:

$$v_1(y,t) = \sum_{n=1}^{\infty} \quad am\Omega c\alpha_{n0}\frac{Y_{n1}(l)Y_{n0}^2(l)}{2H_{n1}}$$
$$\left(\frac{F_0}{c^2\alpha_{n0}^2 \quad \Omega^2}\frac{Y_{n0}(l)}{H_{n0}}\right)^2 \sin\alpha_{n1}y$$
$$\times\left\{\left[\frac{\Omega}{c\alpha_{n0}}\left(\quad \frac{7}{3c^2\alpha_{n0}^2} + \frac{1}{c^2\alpha_{n0}^2 \quad 4\Omega^2}\right)\right.\right.$$
$$\left. + \left(1 + \frac{\Omega^2}{c^2\alpha_{n0}^2}\right)\left(\frac{4c\alpha_{n0}}{\Omega(4c^2\alpha_{n0}^2 \quad \Omega^2)}\right)\right]$$
$$\cos c\alpha_{n1}t \quad \left[\frac{\Omega}{c\alpha_{n0}}\left(\quad \frac{2}{c^2\alpha_{n0}^2}\right.\right.$$
$$\left.\frac{\cos 2c\alpha_{n0}t}{3c^2\alpha_{n0}^2} + \frac{\cos 2\Omega t}{c^2\alpha_{n0}^2 \quad 4\Omega^2}\right)$$
$$+ \left(1 + \frac{\Omega^2}{c^2\alpha_{n0}^2}\right)\left(\frac{\cos(c\alpha_{n0} \quad \Omega)t}{\Omega(2c\alpha_{n0} \quad \Omega)}\right.$$
$$\left.\left.\left. + \frac{\cos(c\alpha_{n0} + \Omega)t}{\Omega(2c\alpha_{n0} + \Omega)}\right)\right]\right\}. \tag{59}$$

From Equation 49, one has:

$$\alpha_{n0} = \alpha_{n1} = \alpha_n.$$

Now, the dynamic response of the tether becomes:

$$v(y,t) = \sum_{n=1}^{\infty}\sum_{j=1}^{N}\frac{F_j}{c^2\alpha_n^2 \quad \Omega_j^2}$$
$$\frac{\sin\alpha_n l \sin\alpha_n y\left(\quad \frac{\Omega_j}{c\alpha_n}\sin c\alpha_n t + \sin\Omega_j t\right)}{\frac{m_t}{2} + \sin^2\alpha_n l\left(\frac{k_b}{2\alpha_n^2 c^2} + \frac{m}{2}\right)}$$
$$\varepsilon\sum_{n=1}^{\infty}\sum_{j=1}^{N}\frac{1}{2}am\Omega_j c\alpha_n\left(\frac{F_j}{c^2\alpha_n^2 \quad \Omega_j^2}\right)^2$$
$$\frac{\sin^5\alpha_n l \sin\alpha_n y}{\left[\frac{m_t}{2} + \sin^2\alpha_n l\left(\frac{k_b}{2\alpha_n^2 c^2} + \frac{m}{2}\right)\right]^3}$$
$$\times\left\{\left[\frac{\Omega_j}{c\alpha_n}\left(\quad \frac{7}{3c^2\alpha_n^2} + \frac{1}{c^2\alpha_n^2 \quad 4\Omega_j^2}\right)\right.\right.$$
$$\left. + \left(1 + \frac{\Omega_j^2}{c^2\alpha_n^2}\right)\left(\frac{4c\alpha_n}{\Omega_j(4c^2\alpha_n^2 \quad \Omega_j^2)}\right)\right]\cos c\alpha_n t$$

$$\left[\frac{\Omega_j}{c\alpha_n}\left(\quad\frac{2}{c^2\alpha_n^2}\right.\right.$$

$$\left.\frac{\cos 2c\alpha_n t}{3c^2\alpha_n^2}+\frac{\cos 2\Omega_j t}{c^2\alpha_n^2 \quad 4\Omega_j^2}\right)$$

$$+\left(1+\frac{\Omega_j^2}{c^2\alpha_n^2}\right)\left(\frac{\cos(c\alpha_n \quad \Omega_j)t}{\Omega_j(2c\alpha_n \quad \Omega_j)}\right.$$

$$\left.\left.\left.+\frac{\cos(c\alpha_n+\Omega_j)t}{\Omega_j(2c\alpha_n+\Omega_j)}\right)\right]\right\}. \tag{60}$$

CONCLUSION

The analytical solution of the tether response of a TLP was presented for a simple continuous model. The applied load is simulation of an ocean wave. Some complicated factors, such as foundation effect and buoyancy, were considered. The presented solution gives a conceptual view of the heave response of a TLP under sea wave loads. The formulation presented herein can be used in analytical studies on the fatigue life of tethers.

The analytical solutions of the tether response of a TLP were presented for a continuous model, considering the buoyancy and the effect of added mass fluctuation under the load simulated as an ocean wave. A first order perturbation method was used to solve the differential equation, approximately. The presented solution gives a conceptual view of the heave response of a TLP under sea wave loads. The formulation presented here can also be used in analytical studies on the fatigue life of tethers.

The importance of the solved example is an investigation into the added mass fluctuation in fatigue analysis on the tendons (tethers) of the moored structure. Such closed form studies can be used for various studies, such as: Verifying the numerical results of the 6 degrees of freedom TLP system, the conceptual investigation of added mass fluctuation, simple fatigue analysis for uncoupled TLP systems, comparing the structural dynamics response with wave radiation scattering interaction models (see references) and the effect of higher modes in the axial vibration of tendons, etc.

REFERENCES

1. Ahmad, S. "Stochastic TLP response under long crested random sea", *Journal of Computers and Structures*, **61**(6), pp 975-993 (1996).
2. Jain, A.K. "Nonlinear coupled response of offshore tension leg platforms to regular wave forces", *Ocean Engineering*, **24**(7), pp 577-592 (1997).
3. Faltinsen, O.M., Van Hooff, R.W., Fylling, I.J. and Teigen, P.S. "Theoretical and experimental investigations tension leg platform behavior", *Proceedings of BOSS 1*, pp 411-423 (1982).
4. Chandrasekaran, S. and Jain, A.K. "Triangular configuration tension leg platform behavior under random sea wave loads", *Ocean Engineering*, **29**(15), pp 1895-1928 (2002).
5. Angelides, D.C., Chen, C. and Will, S.A. "Dynamic response of tension leg platform", *Proceedings of BOSS 2*, pp 100-120 (1982).
6. Morgan, J.R. and Malaeb, D. "Dynamic analysis of tension leg platforms", in *Proceedings of the Second International Conference of Offshore Mechanics and Arctic Engineering*, Houston, pp 31-37 (1983).
7. Tabeshpour, M.R., Golafshani, A.A. and Seif, M.S. "A comprehensive study on the results of tension leg platform responses in random sea", *Journal of Zhejiang University*, **7**(8), pp 1305-1317 (2006).
8. Seif, M.S., Tabeshpour, M.R., Davoodi, M. and Golafshani, A.A. "Geometrical optimum design of TLP using genetic algorithm method", *The 1st International Conference on Design Engineering and Science*, Vienna, Australia (2005).
9. Tabeshpour, M.R., Golafshani, A.A. and Seif, M.S. "Simple model for exact heave motion of tension leg platform", *WSEAS Transactions on Applied Mathematics*, **5**, pp 500-506 (2006).
10. Tabeshpour, M.R., Golafshani, A.A. and Seif, M.S. "The effect of added mass fluctuation on vertical vibration of a TLP", *International Journal of Engineering*, **18**(3), pp 219-228 (2005).
11. Tabeshpour, M.R., Seif, M.S. and Golafshani, A.A. "Vertical response of TLP with the effect of added mass fluctuation", *16th Symposium on Theory and Practice of Shipbuilding*, Croatia (2004).
12. Lee, C.P. and Lee, J.F. "Wave induced surge motion of a tension leg structure", *Ocean Engineering*, **20**(2), pp 171-186 (1993).
13. Lee, C.P. "Dragged surge motion of a tension leg structure", *International Journal of Ocean Engineering*, **21**(3), pp 311-328 (1994).
14. Lee, H.H., Wang, P.-W. and Lee, C.-P. "Dragged surge motion of tension leg platforms and strained elastic tethers", *Ocean Engineering*, **26**(2), pp 579-594 (1999).
15. Tabeshpour, M.R., Golafshani, A.A., Ataie Ashtiani, B. and Seif, M.S. "Analytical solution of heave vibration of tension leg platform", *Int. J. Hydrology and Hydromechanics*, **54**(3), pp 280-289 (2006).
16. Tabeshpour, M.R., Ataie Ashtiani, B., Seif, M.S. and Golafshani, A.A. "Wave interaction pitch response of tension leg structures", *13th International Congress on Sound and Vibration*, Austria, Vienna (2006).
17. Rossit, C.A., Laura, P.A.A. and Bambill, D.V. "Dynamic response of the leg of a TLP subjected to an axial suddenly applied load at one end", *Ocean Engineering*, **23**(3), pp 219-224 (1996).

18. Benedek, A., Guichal, E. and Panzone, R. "On certain non-harmonic Fourier expansions as eigenfunctions of non-regular Sturm-Liouville systems", *Instituto de Matematica*, Publication No. 4, Universidad Nacional del Sur, Bahia Blanca, Argentina (1974).

19. Laura, P.A.A., Reyes, J.A. and Rossi, R.E. "Analysis of a cable-like system suddenly stopped at one end", *J. Sound Vibr.*, **37**, pp 195-204 (1974).

20. Nayfeh, A.H., *Perturbation Methods*, John Wiley & Sons, New York, USA (1973).