

Motion Equations Proper for Forward Dynamics of Robotic Manipulator with Flexible Links by Using Recursive Gibbs-Appell Formulation

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Abstract. *In this article, a new systematic method for deriving the dynamic equations of motion for flexible robotic manipulators is developed by using the Gibbs-Appell assumed modes method. The proposed method can be applied to the dynamic simulation and control system design of flexible robotic manipulators. In the proposed method, the link deflection is described by a truncated modal expansion. All the mathematical operations are done by only 3×3 and 3×1 matrices. Also, all dynamic expressions of a link are expressed in the same link local coordinate system. Based on the developed formulation, an algorithm is proposed that recursively and systematically derives the equation of motion, then this method is compared with the recursive Lagrangian method. As shown, this method is computationally simpler and more efficient and it reduces a large amount of computational complexity. Finally, a computational simulation for a manipulator with two elastic links is presented to verify the proposed method.*

Keywords: Manipulator; Flexible link; Recursive; Gibbs-Appell; Complexity.

INTRODUCTION

The derivation of dynamic equations of motion describing the dynamic behavior of robotic manipulators is necessary for dynamic simulation and control system design. Today, many systematic methods can be used for deriving the dynamic equations of robotic manipulators [1-3]. But, these methods are only suitable when the individual links of a robotic manipulator are assumed rigid.

Based on recent advances in robot utilization and also the demand for faster robots with great quality, a light robot usage idea is represented. Robots with elastic links are introduced as a solution for the deformation phenomena in light robots with heavy loads. In this case, deformation causes accuracy reduction and system instability. Therefore, there is an obvious requirement for a complete dynamical model for this kind of robot to control light links at high velocity and in heavy load situations, appropriately. The two main approaches for the dynamic modeling

of flexible robotic manipulators are the finite element method [4] and the assumed modes method [5-9]. The finite element method is a general method and can be applied to manipulators with complex shaped links. But, this method requires sophisticated software for performing assembly and the order reduction of the element equations.

The assumed mode method of modeling flexible manipulators is mainly presented by Book [5]. He represented the link deformation and kinematics of revolute joints with a 4×4 matrix and used modal analysis for link deformations. This method of formulation had acceptable efficiency in comparison with other methods of that time. King applied Walker-Orin's method, based on Newton-Euler formulation, to improve Book's method [6]. But, his method still suffered from great computational complexity. Jin and Sankar also have a systematic approach for elastic links [7]. They obtained dynamical equations by using Lagrange formulation and the modes approach assumption. In this method 3×3 matrixes are used for computations and the results are simulated for a robot with one link. The computations, however, are massive.

Highly efficient multi-flexible-body methods have been previously presented by Anderson [10] and Banerjee [11] based on Kane's Method with many other comparably efficient multi-flexible-body routines developed

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by E. Haug, J. Angeles, R. Singh, R. Schwertassek, A. Jain, R. Wehage, J. Ambrosio and others [12]. Many of these methods are so-called $O(N)$ routines, being able to form equations of motion with an overall cost that increases only linearly with the number of system degrees of freedom N , (for rigid body systems). For flexible body systems, this overall cost (equations formation) is adjusted somewhat, being approximated as $O(n_f^2 \times m^2)$.

Dynamic equations of motion by the Gibbs-Appell formulation begin with a definition of Gibbs' function (acceleration energy) [13]. Then, a set of independent quasi velocities (linear combination of generalized velocities) should be selected. By taking the derivative of the Gibbs' function, with respect to quasi accelerations (time derivate of quasi velocities), and equalizing them with generalized forces, these equations will be obtained. But, this method has been the least used for resolution of the dynamic problem of manipulating robots. In the field of robotics, Popov proposed a method later developed by Vukobratovic and Potkonjak in which the G-A equations were used to develop a closed form representation of high computational complexity [14]. This method was used by Desoyer and Lugner to solve, by means of a recursive formulation, $O(n^2)$, the inverse dynamic problem, using the Jacobian matrix of the manipulator with the purpose of avoiding the explicit development of partial derivatives [15]. Another approach was suggested by Vereshchagin, which proposed manipulator motion equations from Gauss' principle and Gibbs' function [16]. This approach was used by Rudas and Toth to solve the inverse dynamic problem of robots [17]. Recently, Mata et al. presented a formulation of order $O(n)$ which solves the inverse dynamic problem and establishes recursive relations that involve a reduced number of algebraic operations [18].

In this article, a new systematic method for dynamic modeling of flexible robotic manipulators is developed using the Gibbs-Appell assumed modes method. In this method, the equation of motion for flexible robotic manipulators is written in the following form:

$$I(\Theta)\ddot{\vec{\Theta}} = \vec{Re}, \tag{1}$$

where $I(\Theta)$ is the inertia matrix of the whole system; $\vec{\Theta}$ denotes the vector of the generalized coordinate containing joint and deflection variables; and \vec{Re} is the vector composed of the strain, gravitational, Coriolis, centrifugal forces or torques and also the generalized forces or torques exerted to the joint and link variables. Also, a recursive algorithm is proposed that systematically derives the equation of motion of elastic robotic manipulators. Then, this method is compared with the recursive Lagrangian method and, as shown,

this method is computationally simpler and more efficient and it reduces a large amount of computational complexity. Finally, for verification of this method, a computational simulation for a manipulator with two elastic links is presented.

KINEMATICS OF FLEXIBLE LINK

In this section, the kinematics of a chain of n elastic links is taken into consideration. The coordinate system of every link is attached according to the rules developed by Denavit and Hartenberg. $X_0Y_0Z_0$ is the coordinate system that is attached to the base of the manipulator and can be considered as the reference coordinate system. Because of the elastic property of the links, two rotations occurred one of which is in the joints and the other of which is in the links. It is useful to separate the transformations due to the joints from the transformations which are due to the flexible links. So, we allocate two coordinates system to each link. $x_iy_iz_i$ is a coordinate system on link i whose origin is located at the beginning of this link, but $\hat{x}_i\hat{y}_i\hat{z}_i$ is the coordinate system that is attached to the end of this link. When link i has no deformation, the axes of $\hat{x}_i\hat{y}_i\hat{z}_i$ are parallel to the axes of $x_iy_iz_i$.

In Figure 1, the arbitrary point, Q , is shown. The position of this point with respect to the i th body's local reference system is expressed by ${}^i\vec{r}_{Q/O_i}$. To incorporate the deflection of the link, the approach of modal analysis is used. So:

$${}^i\vec{r}_{Q/O_i} = \vec{\eta} + \sum_{j=1}^{m_i} \delta_{ij}(t)\vec{r}_{ij}(\eta), \tag{2}$$

where $\vec{\eta} = \{\eta \ 0 \ 0\}^T$ and $\vec{r}_{ij} = \{x_{ij} \ y_{ij} \ z_{ij}\}^T$. Also, η is the undeformed distance between the origin, O_i , and the point, Q ; x_{ij}, y_{ij} and z_{ij} are the displacement components of j mode of the i th link; δ_{ij} is the time varying amplitude of mode j of link i ; and m_i is the number of modes used to describe the deflection of link i .

By using the rotation matrix, jR_i , we can express the arbitrary vector, ${}^i\vec{a}$, in every coordinate system, j , in the following form:

$${}^j\vec{a} = {}^jR_i {}^i\vec{a}. \tag{3}$$

As noted above, it is better to separate the rotations, due to joints from deflections. So, jR_i can be presented recursively as follows:

$${}^jR_i = {}^jR_{i-1}E_{i-1}A_i, \tag{4}$$

where A_i is the rotation matrix of the i th joint that shows the orientation of the $x_iy_iz_i$ coordinate system with respect to $\hat{x}_{i-1}\hat{y}_{i-1}\hat{z}_{i-1}$. The coefficients of this

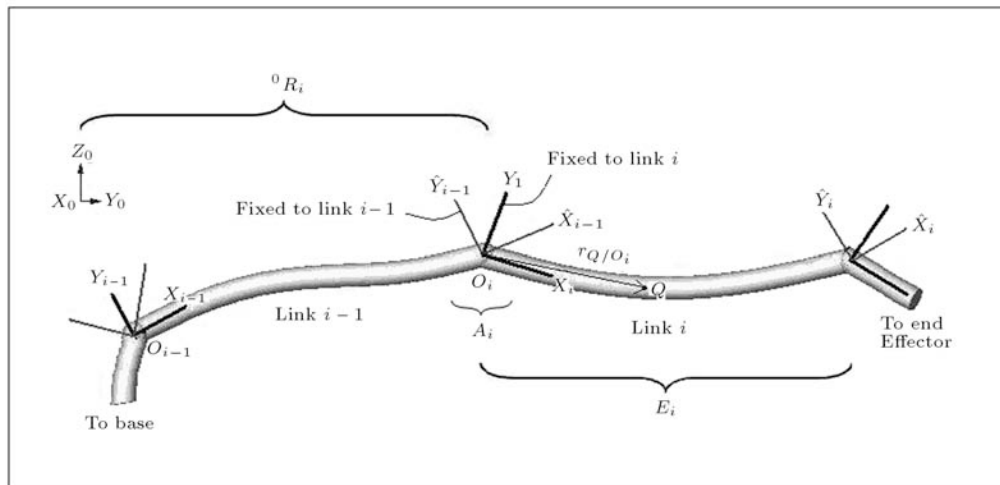


Figure 1. Manipulator with elastic links.

matrix can be presented by dot products of a pair of unit vectors as follow:

$$A_i = \begin{bmatrix} x_i \cdot \hat{x}_{i-1} & y_i \cdot \hat{x}_{i-1} & z_i \cdot \hat{x}_{i-1} \\ x_i \cdot \hat{y}_{i-1} & y_i \cdot \hat{y}_{i-1} & z_i \cdot \hat{y}_{i-1} \\ x_i \cdot \hat{z}_{i-1} & y_i \cdot \hat{z}_{i-1} & z_i \cdot \hat{z}_{i-1} \end{bmatrix}. \quad (5)$$

Also, E_i is the i th link rotation matrix that shows the orientation of $\hat{x}_i \hat{y}_i \hat{z}_i$ the coordinate system with respect to $x_i y_i z_i$. Like A_i , this matrix is also composed of dot products of a pair of unit vectors, but because of the small angles between these vectors, E_i is simplified in the following form:

$$E_i = \begin{bmatrix} 1 & -\theta_{zi} & \theta_{yi} \\ \theta_{zi} & 1 & -\theta_{xi} \\ -\theta_{yi} & \theta_{xi} & 1 \end{bmatrix}, \quad (6)$$

where θ_{xi}, θ_{yi} and θ_{zi} are infinitesimal rotations of $\hat{x}_i \hat{y}_i \hat{z}_i$, with respect to x_i, y_i and z_i axes, respectively. It should be noted that all the angles in E_i are evaluated at $\eta = l_i$, where l_i is the length of the i th link. Now, we define ${}^i\vec{\theta}_i$ as follows:

$${}^i\vec{\theta}_i = \{\theta_{xi} \quad \theta_{yi} \quad \theta_{zi}\}^T. \quad (7)$$

These small angles can be represented by truncated modal expansion as follows:

$${}^i\vec{\theta}_i = \sum_{j=1}^{m_i} \delta_{ij}(t) \vec{\theta}_{ij}, \quad (8)$$

where $\vec{\theta}_{ij} = \{\theta_{xij} \quad \theta_{yij} \quad \theta_{zij}\}^T$. By taking the time derivative of ${}^i\vec{\theta}_i$, the angular velocity and acceleration of $\hat{x}_i \hat{y}_i \hat{z}_i$ the coordinate system, with respect to $x_i y_i z_i$, will be obtained.

SYSTEM'S ACCELERATION ENERGY (GIBBS FUNCTION)

In this section, the expression for the system's acceleration energy is developed for use in Gibbs-Appell's equations. First, the acceleration energy for a differential element is written. Then, integration of this differential acceleration energy over the link gives the link's total contribution. Summation over all the links provides the total acceleration energy. The acceleration energy of a point on the i th link is:

$$ds_i = \frac{1}{2} dm ({}^i\ddot{\vec{r}}_Q^T \cdot {}^i\ddot{\vec{r}}_Q), \quad (9)$$

where dm is the differential mass of point Q and ${}^i\ddot{\vec{r}}_Q$ is the absolute acceleration of differential element Q that is expressed in the i th body's local reference system:

$$\begin{aligned} {}^i\ddot{\vec{r}}_Q &= {}^i\ddot{\vec{r}}_{O_i} + {}^i\ddot{\vec{r}}_{Q/O_i} + 2{}^i\vec{\omega}_i \times {}^i\dot{\vec{r}}_{Q/O_i} \\ &\quad + {}^i\dot{\vec{\omega}}_i \times {}^i\vec{r}_{Q/O_i} + {}^i\vec{\omega}_i \times ({}^i\vec{\omega}_i \times {}^i\vec{r}_{Q/O_i}). \end{aligned} \quad (10)$$

In the above expression, ${}^i\ddot{\vec{r}}_{O_i}$ is the absolute acceleration of the origin of the i th body's local reference system, ${}^i\vec{\omega}_i$ and ${}^i\dot{\vec{\omega}}_i$ are angular velocity and angular acceleration of the i th link, respectively and ${}^i\dot{\vec{r}}_{Q/O_i}$ and ${}^i\ddot{\vec{r}}_{Q/O_i}$ are the velocity and acceleration of differential element Q , with respect to the origin of the i th body's local reference system which will be obtained by taking the time derivative of Equation 2 as follows:

$${}^i\dot{\vec{r}}_{Q/O_i} = \sum_{j=1}^{m_i} \dot{\delta}_{ij}(t) \vec{r}_{ij}(\eta), \quad (11)$$

$${}^i\ddot{\vec{r}}_{Q/O_i} = \sum_{j=1}^{m_i} \ddot{\delta}_{ij}(t) \vec{r}_{ij}(\eta). \quad (12)$$

By substituting Equation 10 in Equation 9 and integrating over the link, one can obtain the link's total

acceleration energy. In this paper, it is assumed that the links are slender beams. For slender beams, $dm = \mu d\eta$ where μ is mass per unit length. So, one can integrate over η from 0 to l_i . Only the terms in ${}^i\vec{r}_{Q/O_i}$ and its derivatives (${}^i\dot{\vec{r}}_{Q/O_i}$, ${}^i\ddot{\vec{r}}_{Q/O_i}$) are functions of η for this link. Thus, the integration can be performed without knowledge of ${}^i\vec{\omega}_i$, ${}^i\dot{\vec{\omega}}_i$ and ${}^i\ddot{\vec{r}}_{O_i}$. Summing over all n links, one finds the system's acceleration energy to be:

$$\begin{aligned}
 S &= \sum_{i=1}^n \int_0^{l_i} ds_i \\
 S &= \sum_{i=1}^n \frac{1}{2} M_i {}^i\ddot{\vec{r}}_{O_i}^T \cdot {}^i\ddot{\vec{r}}_{O_i} + {}^i\ddot{\vec{r}}_{O_i}^T \cdot {}^i\vec{B}_{1i} \\
 &\quad - 2 {}^i\ddot{\vec{r}}_{O_i}^T B_{2i} {}^i\vec{\omega}_i - {}^i\ddot{\vec{r}}_{O_i}^T B_{3i} {}^i\dot{\vec{\omega}}_i - {}^i\ddot{\vec{r}}_{O_i}^T {}^i\vec{\omega}_i B_{3i} {}^i\vec{\omega}_i \\
 &\quad + \frac{1}{2} B_{4i} - 2 {}^i\dot{\vec{\omega}}_i^T \cdot {}^i\vec{B}_{5i} + {}^i\dot{\vec{\omega}}_i^T \cdot {}^i\vec{B}_{6i} \\
 &\quad - {}^i\vec{\omega}_i^T B_{7i} {}^i\vec{\omega}_i + 2 {}^i\dot{\vec{\omega}}_i^T B_{8i} {}^i\vec{\omega}_i + \frac{1}{2} {}^i\dot{\vec{\omega}}_i^T B_{9i} {}^i\dot{\vec{\omega}}_i \\
 &\quad + {}^i\dot{\vec{\omega}}_i^T {}^i\vec{\omega}_i B_{9i} {}^i\vec{\omega}_i + \text{irrelevant terms}, \tag{13}
 \end{aligned}$$

where:

$${}^i\vec{B}_{1i} = \int_0^{l_i} \mu {}^i\ddot{\vec{r}}_{Q/O_i} d\eta, \tag{14}$$

$$B_{2i} = \int_0^{l_i} \mu {}^i\dot{\vec{r}}_{Q/O_i} d\eta, \tag{15}$$

$$B_{3i} = \int_0^{l_i} \mu {}^i\ddot{\vec{r}}_{Q/O_i} d\eta, \tag{16}$$

$$B_{4i} = \int_0^{l_i} \mu {}^i\ddot{\vec{r}}_{Q/O_i} \cdot {}^i\ddot{\vec{r}}_{Q/O_i} d\eta, \tag{17}$$

$${}^i\vec{B}_{5i} = \int_0^{l_i} \mu {}^i\ddot{\vec{r}}_{Q/O_i} {}^i\dot{\vec{r}}_{Q/O_i} d\eta, \tag{18}$$

$${}^i\vec{B}_{6i} = \int_0^{l_i} \mu {}^i\ddot{\vec{r}}_{Q/O_i} {}^i\ddot{\vec{r}}_{Q/O_i} d\eta, \tag{19}$$

$$B_{7i} = \int_0^{l_i} \mu {}^i\ddot{\vec{r}}_{Q/O_i}^T {}^i\ddot{\vec{r}}_{Q/O_i} d\eta, \tag{20}$$

$$B_{8i} = \int_0^{l_i} \mu {}^i\ddot{\vec{r}}_{Q/O_i}^T {}^i\dot{\vec{r}}_{Q/O_i} d\eta, \tag{21}$$

$$B_{9i} = \int_0^{l_i} \mu {}^i\ddot{\vec{r}}_{Q/O_i}^T {}^i\ddot{\vec{r}}_{Q/O_i} d\eta. \tag{22}$$

In Equation 13, M_i is the total mass of the i th link. Also ${}^i\ddot{\vec{r}}_{Q/O_i}$, ${}^i\dot{\vec{r}}_{Q/O_i}$, ${}^i\ddot{\vec{r}}_{Q/O_i}$ and ${}^i\vec{\omega}_i$ are skew-symmetric

tensors representation of the ${}^i\ddot{\vec{r}}_{Q/O_i}$, ${}^i\dot{\vec{r}}_{Q/O_i}$, ${}^i\ddot{\vec{r}}_{Q/O_i}$ and ${}^i\vec{\omega}_i$ vectors. For developing an expression for S , these vector relations, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$, $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ and $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$ are frequently used. By interchanging the integration and summation in Equations 14 to 22, one obtains:

$${}^i\vec{B}_{1i} = \sum_{j=1}^{m_i} \ddot{\delta}_{ij} \vec{\varepsilon}_{ij}, \tag{23}$$

$$B_{2i} = \sum_{j=1}^{m_i} \dot{\delta}_{ij} \tilde{\varepsilon}_{ij}, \tag{24}$$

$$B_{3i} = \widetilde{M_i} r_{r_i} + \sum_{j=1}^{m_i} \delta_{ij} \tilde{\varepsilon}_{ij}, \tag{25}$$

$$B_{4i} = \sum_{j=1}^{m_i} \sum_{k=1}^{m_i} \ddot{\delta}_{ij} \delta_{ik} {}^s C_{ijk}, \tag{26}$$

$${}^i\vec{B}_{5i} = \sum_{j=1}^{m_i} \sum_{k=1}^{m_i} \ddot{\delta}_{ij} \dot{\delta}_{ik} {}^s \vec{C}_{ijk}, \tag{27}$$

$${}^i\vec{B}_{6i} = \sum_{j=1}^{m_i} \ddot{\delta}_{ij} \vec{\alpha}_{ij}, \tag{28}$$

$$B_{7i} = \sum_{j=1}^{m_i} \ddot{\delta}_{ij} \beta_{ij}, \tag{29}$$

$$B_{8i} = \sum_{j=1}^{m_i} \dot{\delta}_{ij} \beta_{ij}, \tag{30}$$

$$B_{9i} = c_i + \sum_{j=1}^{m_i} \delta_{ij} c_{ij}^T + \sum_{k=1}^{m_i} \delta_{ik} \beta_{ik}, \tag{31}$$

where:

$$\vec{\alpha}_{ij} = \vec{c}_{ij} + \sum_{k=1}^{m_i} \delta_{ik} \vec{c}_{ikj}, \tag{32}$$

$$\beta_{ij} = c_{ij} + \sum_{k=1}^{m_i} \delta_{ik} {}^m c_{ikj}. \tag{33}$$

By definition, $\vec{\eta}$ and \vec{r}_{ij} are skew-symmetric tensors associated with $\vec{\eta}$ and \vec{r}_{ij} vectors. The expressions of $\vec{\varepsilon}_{ij}$, $\tilde{\varepsilon}_{ij}$, $\widetilde{M_i} r_{r_i}$, ${}^s c_{ijk}$, \vec{c}_{ijk} , \vec{c}_{ij} , c_{ij} , ${}^m c_{ijk}$, c_i that appeared in Equations 23 to 33 can be written in the following form:

$$\vec{\varepsilon}_{ij} = \int_0^{l_i} \mu \vec{r}_{ij} d\eta, \tag{34}$$

$$\tilde{\varepsilon}_{ij} = \int_0^{l_i} \mu \tilde{r}_{ij} d\eta, \tag{35}$$

$$\widetilde{M_i} r_{r_i} = \int_0^{l_i} \mu \vec{\eta} d\eta, \tag{36}$$

$${}^s c_{ijk} = \int_0^{l_i} \mu \tilde{r}_{ij}^T \tilde{r}_{ik} d\eta, \quad (37)$$

$$\tilde{c}_{ijk} = \int_0^{l_i} \mu \tilde{r}_{ij} \tilde{r}_{ik} d\eta, \quad (38)$$

$$\tilde{c}_{ij} = \int_0^{l_i} \mu \tilde{\eta} \tilde{r}_{ij} d\eta, \quad (39)$$

$$c_{ij} = \int_0^{l_i} \mu \tilde{\eta}^T \tilde{r}_{ij} d\eta, \quad (40)$$

$${}^m c_{ijk} = \int_0^{l_i} \mu \tilde{r}_{ij}^T \tilde{r}_{ik} d\eta, \quad (41)$$

$$c_i = \int_0^{l_i} \mu \tilde{\eta}^T \tilde{\eta} d\eta. \quad (42)$$

Now, it should be noted that B_{9i} has a unit of inertia matrix. For example, its first term (c_i) represent rigid-body-inertia terms. It can also be shown that ${}^m c_{ijk} = {}^m c_{ikj}^T$. The terms defined in Equations 23 to 33 are easily simplified if one link in the system is considered rigid ($m_i = 0$). Furthermore, the expression for B_{9i} has a term of order δ^2 , which is small and a candidate for later elimination [5]. Finally, the integration of the modal shape products in Equations 34 to 42 can be done off-line one time for a given link structure.

Derivatives of Acceleration Energy

G-A equations are obtained by taking the derivative of Gibbs' function, with respect to generalized accelerations ($\ddot{q}_j, \ddot{\delta}_{jf}$):

$$\frac{\partial S}{\partial \ddot{q}_j}, \quad \frac{\partial S}{\partial \ddot{\delta}_{jf}}.$$

In Equation 13, there was a term named an irrelevant term. In fact, in Gibbs' function, the terms that are not functions of \ddot{q}_j and $\ddot{\delta}_{jf}$ can be eliminated, because they have no role in construction of the derivative of acceleration energy.

In Gibbs' function, only ${}^i \ddot{r}_{O_i}$ and ${}^i \dot{\omega}_i$ are functions of \ddot{q}_j . So, the partial derivative of Gibbs' function with respect to \ddot{q}_j becomes:

$$\begin{aligned} \frac{\partial S}{\partial \ddot{q}_j} = & \sum_{i=j+1}^n \frac{\partial {}^i \ddot{r}_{O_i}^T}{\partial \ddot{q}_j} \left(M_i {}^i \ddot{r}_{O_i} + {}^i \tilde{B}_{1i} \right. \\ & \left. - 2B_{2i} {}^i \dot{\omega}_i - B_{3i} {}^i \dot{\omega}_i - {}^i \dot{\omega}_i B_{3i} {}^i \dot{\omega}_i \right) \\ & + \sum_{i=j}^n \frac{\partial {}^i \dot{\omega}_i^T}{\partial \ddot{q}_j} \left(B_{3i} {}^i \ddot{r}_{O_i} + {}^i \tilde{B}_{6i} \right. \\ & \left. + 2B_{8i} {}^i \dot{\omega}_i + B_{9i} {}^i \dot{\omega}_i + {}^i \dot{\omega}_i B_{9i} {}^i \dot{\omega}_i \right). \end{aligned} \quad (43)$$

Here, it should be noted that, in the above expression, this property of the skew-symmetric matrix, in which $a^T = -a$ is used. The partial derivative of Gibbs' function with respect to $\ddot{\delta}_{jf}$ is more complex, because in addition to ${}^i \ddot{r}_{O_i}$ and ${}^i \dot{\omega}_i$, the expressions of ${}^i \tilde{B}_{1i}, B_{4i}, {}^i \tilde{B}_{5i}, {}^i \tilde{B}_{6i}$ and B_{7i} are also functions of deflection variables. So, the expression of $\frac{\partial S}{\partial \ddot{\delta}_{jf}}$ can be presented as follows:

$$\begin{aligned} \frac{\partial S}{\partial \ddot{\delta}_{jf}} = & \sum_{i=j+1}^n \frac{\partial {}^i \ddot{r}_{O_i}^T}{\partial \ddot{\delta}_{jf}} \left(M_i {}^i \ddot{r}_{O_i} + {}^i \tilde{B}_{1i} \right. \\ & \left. - 2B_{2i} {}^i \dot{\omega}_i - B_{3i} {}^i \dot{\omega}_i - {}^i \dot{\omega}_i B_{3i} {}^i \dot{\omega}_i \right) \\ & + \sum_{i=j+1}^n \frac{\partial {}^i \dot{\omega}_i^T}{\partial \ddot{\delta}_{jf}} \left(B_{3i} {}^i \ddot{r}_{O_i} + {}^i \tilde{B}_{6i} \right. \\ & \left. + 2B_{8i} {}^i \dot{\omega}_i + B_{9i} {}^i \dot{\omega}_i - {}^i \dot{\omega}_i B_{9i} {}^i \dot{\omega}_i \right) \\ & + \sum_{k=1}^{m_j} \ddot{\delta}_{jk} {}^s c_{jfk} - 2 {}^j \dot{\omega}_j^T \sum_{k=1}^{m_j} \dot{\delta}_{jk} \tilde{c}_{jfk} \\ & - {}^j \dot{\omega}_j^T \beta_{jf} {}^j \dot{\omega}_j + {}^j \ddot{r}_{O_j}^T \tilde{e}_{jf} + {}^j \dot{\omega}_j^T \tilde{\alpha}_{jf}. \end{aligned} \quad (44)$$

An additional simplification of $\frac{\partial S}{\partial \ddot{\delta}_{jf}}$ arises, due to the fact that ${}^s c_{jfk} = {}^s c_{jkf}$.

SYSTEM'S POTENTIAL ENERGY

The potential energy of the system arises from two sources:

1. Potential energy due to gravity,
2. Potential energy due to elastic deformations.

The effect of gravity on manipulators can be considered simply by putting ${}^0 \ddot{r}_{O_0} = \vec{g}$, where \vec{g} is the acceleration of gravity. Under these circumstances, we can assume that the base of the manipulator has an acceleration of 1 g to the top. So, the effect of gravity has been considered without additional computations.

To express the strain potential energy stored in the i th link, let us assume that the assumptions of the classical beam (Euler-Bernoulli) hold. So, the strain potential energy will be expressed in terms of deflections and rotations as follows:

$$\begin{aligned} V_{ei} = & \frac{1}{2} \int_0^{l_i} \left[EA \left(\frac{\partial u_i}{\partial \eta} \right)^2 + EI_y \left(\frac{\partial^2 w_i}{\partial \eta^2} \right)^2 \right. \\ & \left. + EI_z \left(\frac{\partial^2 v_i}{\partial \eta^2} \right)^2 + GI_x \left(\frac{\partial \theta_{xi}}{\partial \eta} \right)^2 \right] d\eta, \end{aligned} \quad (45)$$

where EI_y and EI_z are the bending stiffness in the OY and OZ directions, respectively; EA is the extensional

stiffness; GI_x is the torsional stiffness; u_i, v_i and w_i are the deflections in the OX, OY and OZ directions, respectively; and θ_{xi} is the rotation in the OX direction as shown in Figure 2.

It is easy to show that the following relations between the component of deflections and rotations exist:

$$\theta_{zi} = \frac{\partial v_i}{\partial \eta} = \sum_{j=1}^{m_i} \frac{\partial y_{ij}}{\partial \eta}, \tag{46}$$

$$\theta_{yi} = -\frac{\partial w_i}{\partial \eta} = -\sum_{j=1}^{m_i} \frac{\partial z_{ij}}{\partial \eta}, \tag{47}$$

where θ_{yi} and θ_{zi} are the rotations in OY and OZ directions, respectively.

By substituting Equations 46 and 47 in Equation 45 and ignoring the strain potential energy due to axial deformation, in comparison with the strain potential energy due to bending and torsion [5], the expression for V_{ei} is simplified as follows:

$$V_{ei} = \frac{1}{2} \int_0^{l_i} \left[EI_y \left(\frac{\partial \theta_{yi}}{\partial \eta} \right)^2 + EI_z \left(\frac{\partial \theta_{zi}}{\partial \eta} \right)^2 + GI_x \left(\frac{\partial \theta_{xi}}{\partial \eta} \right)^2 \right] d\eta. \tag{48}$$

As noted previously, angles θ_{xi}, θ_{yi} and θ_{zi} can be presented with a truncated modal approximation. For example the rotation about the OX axis is presented as follows:

$$\theta_{xi}(\eta) = \sum_{k=1}^{m_i} \delta_{ik}(t) \theta_{xik}(\eta), \tag{49}$$

where θ_{xik} is the angle corresponding to the k th mode of link i at point η . By substituting the achieved expressions of θ_{xi}, θ_{yi} and θ_{zi} in Equation 48, the strain potential energy for the whole system will be obtained as follows:

$$V_e = \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^{m_i} \sum_{l=1}^{m_i} \delta_{ik} \delta_{il} K_{ikl}, \tag{50}$$

where:

$$K_{ikl} = K_{xikl} + K_{yikl} + K_{zikl}. \tag{51}$$

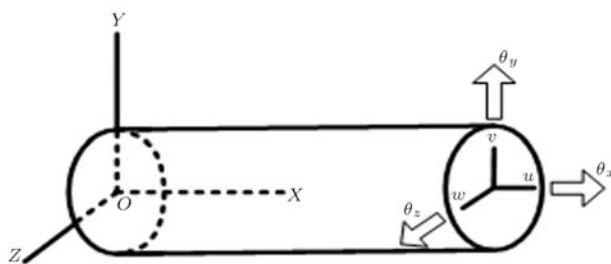


Figure 2. Deflections and rotations of a link.

Also, K_{xikl}, K_{yikl} and K_{zikl} are defined as follows:

$$K_{xikl} = \int_0^{l_i} GI_x \frac{\partial \theta_{xil}(\eta)}{\partial \eta} \frac{\partial \theta_{xik}(\eta)}{\partial \eta} d\eta, \tag{52}$$

$$K_{yikl} = \int_0^{l_i} EI_y \frac{\partial \theta_{yil}(\eta)}{\partial \eta} \frac{\partial \theta_{yik}(\eta)}{\partial \eta} d\eta, \tag{53}$$

$$K_{zikl} = \int_0^{l_i} EI_z \frac{\partial \theta_{zil}(\eta)}{\partial \eta} \frac{\partial \theta_{zik}(\eta)}{\partial \eta} d\eta. \tag{54}$$

It should be noted that $K_{ikl} = K_{ilk}$. For deriving the dynamic equation of motion, the partial derivatives of strain potential energy with respect to the generalized coordinate is needed. Upon taking the partial derivative with respect to q_j , one obtains:

$$\frac{\partial V_e}{\partial q_j} = 0. \tag{55}$$

But taking partial derivatives with respect to δ_{jf} results in:

$$\frac{\partial V_e}{\partial \delta_{jf}} = \sum_{k=1}^{m_j} \delta_{jk} K_{jkf}, \tag{56}$$

where K_{jkf} can analytically or numerically be determined.

DERIVATION OF DYNAMIC EQUATIONS OF MOTION USING G-A'S FORMULATION

The components of the complete equations of motion in G-A's formulation, except for the external forcing terms, have been evaluated in Equations 43 and 55 for the joint equations and in Equations 44 and 56 for deflection equations. The generalized force in joint equations is the torque, τ_j , that applies to joints. But, in deflection equations, the corresponding generalized force will be zero, if the corresponding modal deflections or rotations have no displacement at those locations where external forces are applied [5]. So, with this assumption, the dynamic equation of motion in G-A's formulation will be completed as follows:

1. The joint equations of motion:

$$\frac{\partial S}{\partial \ddot{q}_j} = \tau_j. \tag{57}$$

2. The deflection equations of motion:

$$\frac{\partial S}{\partial \ddot{\delta}_{jf}} + \frac{\partial V_e}{\partial \delta_{jf}} = 0. \tag{58}$$

The above equations are in the form of inverse dynamic. In this type of dynamic, the forces exerted by the actuators are obtained algebraically for certain configurations of the manipulator (position, velocity and

acceleration). On the other hand, the forward dynamic problem computes the acceleration of the joints of the manipulator, once the forces exerted by the actuators are given. This problem is part of the process that must be followed to perform the simulation of the dynamic behavior of the manipulator. This process is completed after it calculates the velocity and position of the joints by means of a process of numerical integration in which the acceleration of the joints and the initial configuration are data input to the problem [15].

FORWARD DYNAMIC EQUATIONS OF MOTION

In this section, the first step will extend the equations of ${}^i\ddot{\mathbf{r}}_{O_i}$ and also ${}^i\ddot{\boldsymbol{\omega}}_i$. These equations are used to separate the second derivatives of joint variables and deflection variables from the dynamic equations of motion.

The absolute acceleration of the origin of the i th body's local reference system in recursive form can be presented as follows:

$${}^i\ddot{\mathbf{r}}_{O_i} = {}^iR_{i-1} \left({}^{i-1}\ddot{\mathbf{r}}_{O_{i-1}} + {}^{i-1}\ddot{\mathbf{r}}_{O_i/O_{i-1}} + 2^{i-1}\dot{\boldsymbol{\omega}}_{i-1} \times {}^{i-1}\dot{\mathbf{r}}_{O_i/O_{i-1}} + {}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} \times {}^{i-1}\dot{\mathbf{r}}_{O_i/O_{i-1}} + {}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} \times ({}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} \times {}^{i-1}\dot{\mathbf{r}}_{O_i/O_{i-1}}) \right), \quad (59)$$

where:

$${}^i\dot{\mathbf{r}}_{O_{i+1}/O_i} = \dot{\mathbf{l}}_i + \sum_{j=1}^{m_i} \delta_{ij}(t) \dot{\mathbf{r}}_{ij}(l_i), \quad (60)$$

$${}^i\dot{\mathbf{r}}_{O_{i+1}/O_i} = \sum_{j=1}^{m_i} \dot{\delta}_{ij}(t) \dot{\mathbf{r}}_{ij}(l_i), \quad (61)$$

$${}^i\ddot{\mathbf{r}}_{O_{i+1}/O_i} = \sum_{j=1}^{m_i} \ddot{\delta}_{ij}(t) \dot{\mathbf{r}}_{ij}(l_i), \quad (62)$$

and also $\dot{\mathbf{l}}_i = \{l_i \ 0 \ 0\}^T$. Before developing an expression for angular acceleration, we should present angular velocity, because by taking its time derivative, angular acceleration will be obtained. The angular velocity of the i th link is the same as the $i - 1$ th link plus two new components, one of which (${}^i\dot{\mathbf{z}}_i\dot{q}_i$), comes from the angular velocity of the i th link and the other (${}^{i-1}\dot{\boldsymbol{\theta}}_{i-1}$) is produced due to the elasticity of the $i - 1$ th link. So, the expression of angular velocity can be presented as follows:

$${}^i\dot{\boldsymbol{\omega}}_i = {}^iR_{i-1} \left({}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} + {}^{i-1}\dot{\boldsymbol{\theta}}_{i-1}(l_{i-1}) \right) + {}^i\dot{\mathbf{z}}_i\dot{q}_i, \quad (63)$$

where:

$${}^i\dot{\boldsymbol{\theta}}_i(l_i) = \sum_{j=1}^{m_i} \dot{\delta}_{ij}(t) \dot{\boldsymbol{\theta}}_{ij}(l_i), \quad (64)$$

and ${}^i\dot{\mathbf{z}}_i = \{0 \ 0 \ 1\}^T$. By taking the time derivative of Equation 63, the expression of angular acceleration will be obtained:

$${}^i\ddot{\boldsymbol{\omega}}_i = {}^iR_{i-1} \left({}^{i-1}\ddot{\boldsymbol{\omega}}_{i-1} + {}^{i-1}\ddot{\boldsymbol{\theta}}_{i-1} + {}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} \times {}^{i-1}\dot{\boldsymbol{\theta}}_{i-1} \right) + {}^iR_{i-1} \left({}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} + {}^{i-1}\dot{\boldsymbol{\theta}}_{i-1} \right) \times {}^i\dot{\mathbf{z}}_i\dot{q}_i + {}^i\dot{\mathbf{z}}_i\ddot{q}_i. \quad (65)$$

In the above expression, ${}^{i-1}\ddot{\boldsymbol{\theta}}_{i-1}(l_{i-1})$ is the angular acceleration that is produced because of the elasticity of the $i - 1$ th link:

$${}^{i-1}\ddot{\boldsymbol{\theta}}_i(l_i) = \sum_{j=1}^{m_i} \ddot{\delta}_{ij}(t) \dot{\boldsymbol{\theta}}_{ij}(l_i). \quad (66)$$

Now, by having ${}^i\ddot{\mathbf{r}}_{O_i}$ and ${}^i\ddot{\boldsymbol{\omega}}_i$ in recursive form, we can convert them in summation form as follows:

$${}^i\ddot{\mathbf{r}}_{O_i} = \sum_{k=1}^{i-1} {}^iR_k {}^k\ddot{\mathbf{r}}_{O_{k+1}/O_k} + \sum_{k=1}^{i-1} {}^iR_k \left({}^k\dot{\boldsymbol{\omega}}_k \times {}^k\dot{\mathbf{r}}_{O_{k+1}/O_k} \right) + {}^i\ddot{\mathbf{r}}_{O_{v,i}}, \quad (67)$$

$${}^i\ddot{\boldsymbol{\omega}}_i = \sum_{k=1}^i {}^iR_k {}^k\ddot{\boldsymbol{\omega}}_k + \sum_{k=1}^{i-1} {}^iR_k {}^k\ddot{\boldsymbol{\theta}}_k(l_k) + {}^i\ddot{\boldsymbol{\omega}}_{v,i}, \quad (68)$$

where:

$${}^i\ddot{\mathbf{r}}_{O_{v,i}} = 2 \sum_{k=1}^{i-1} {}^iR_k \left({}^k\dot{\boldsymbol{\omega}}_k \times {}^k\dot{\mathbf{r}}_{O_{k+1}/O_k} \right) + \sum_{k=1}^{i-1} {}^iR_k \left({}^k\dot{\boldsymbol{\omega}}_k \times ({}^k\dot{\boldsymbol{\omega}}_k \times {}^k\dot{\mathbf{r}}_{O_{k+1}/O_k}) \right), \quad (69)$$

$${}^i\ddot{\boldsymbol{\omega}}_{v,i} = \sum_{k=1}^{i-1} {}^iR_k {}^k\dot{\boldsymbol{\omega}}_k \times {}^iR_{k+1} {}^{k+1}\dot{\mathbf{z}}_{k+1}\dot{q}_{k+1} + \sum_{k=1}^{i-1} {}^iR_k {}^k\dot{\boldsymbol{\theta}}_k \times {}^iR_{k+1} {}^{k+1}\dot{\mathbf{z}}_{k+1}\dot{q}_{k+1}. \quad (70)$$

In fact, ${}^i\ddot{\mathbf{r}}_{O_{v,i}}$ and ${}^i\ddot{\boldsymbol{\omega}}_{v,i}$ are those constructive terms of ${}^i\ddot{\mathbf{r}}_{O_i}$ and ${}^i\ddot{\boldsymbol{\omega}}_i$ that do not contain the second derivatives of joint variables and deflection variables. By having ${}^i\ddot{\mathbf{r}}_{O_i}$ and ${}^i\ddot{\boldsymbol{\omega}}_i$ in summation form, the calculation of partial derivatives that appeared in the dynamic equations of motion can be done as follows:

$$\frac{\partial {}^i\ddot{\boldsymbol{\omega}}_i}{\partial \ddot{q}_j} = {}^iR_j {}^j\dot{\mathbf{z}}_j, \quad (71)$$

$$\frac{\partial {}^i\ddot{\boldsymbol{\omega}}_i}{\partial \ddot{\delta}_{jf}} = {}^iR_j \dot{\boldsymbol{\theta}}_{jf}(l_j), \quad (72)$$

$$\frac{\partial {}^i\ddot{\mathbf{r}}_{O_i}}{\partial \ddot{q}_j} = {}^iR_j {}^j\dot{\mathbf{z}}_j \times {}^i\dot{\mathbf{r}}_{O_i/O_j}, \quad (73)$$

$$\frac{\partial^i \ddot{\vec{r}}_{O_i}}{\partial \delta_{j f}^2} = {}^i R_j \ddot{\vec{r}}_{j f}(l_j) + {}^i R_j \ddot{\theta}_{j f}(l_j) \times {}^i \vec{r}_{O_i/O_{j+1}}, \quad (74)$$

where ${}^i \vec{r}_{O_i/O_j}$ is a position vector drawn from the j th body's local reference system to the i th body's local reference system ($j < i$).

Inertia Coefficients

For construction of the inertia coefficients that multiply the second derivatives, we substitute Equations 71 to 74 and also the summation form of ${}^i \ddot{\vec{r}}_{O_i}$ and ${}^i \ddot{\omega}_i$ (Equations 67 to 68) into the relevant parts of Equations 43 and 44. By collecting the terms that contain \ddot{q}_j and $\ddot{\delta}_{j f}$ and by arranging them, we obtain expressions that should be written in matrix form. By assembling these matrixes, the inertia matrix of the whole system will be obtained. In continuation the details of the above steps are brought.

Inertia Coefficients of Joint Variable in Joint Equations

All occurrences of \ddot{q}_j in Equation 43 are in the expressions of ${}^i \ddot{\omega}_i$ and ${}^i \ddot{\vec{r}}_{O_i}$; by isolating these terms and interchanging the order of summations as follows:

$$\sum_{i=j}^n \sum_{k=1}^i = \sum_{k=1}^{n-1} \sum_{i=\max(k+1,j)}^n$$

$$\sum_{i=j+1}^n \sum_{k=1}^i = \sum_{k=1}^n \sum_{i=\max(k,j+1)}^n$$

$$\sum_{i=j}^n \sum_{k=1}^{i-1} = \sum_{k=1}^{n-1} \sum_{i=\max(k+1,j)}^n$$

$$\sum_{i=j+1}^n \sum_{k=1}^{i-1} = \sum_{k=1}^{n-1} \sum_{i=\max(k+1,j+1)}^n$$

$$\sum_{k=1}^{n-1} \sum_{t=1}^k = \sum_{t=1}^{n-1} \sum_{k=t}^{n-1}$$

the below expression for the terms that contain \ddot{q}_j is obtained:

$$\left(\sum_{k=1}^n {}^j \vec{z}_j^T ({}^j \sigma_k - {}^j \psi_k)^k \vec{z}_k - \sum_{k=1}^{n-1} {}^j \vec{z}_j^T {}^j U_k^k \vec{z}_k \right) \ddot{q}_k, \quad (75)$$

where:

$${}^j \sigma_k = \sum_{i=\max(k,j)}^n {}^j R_i B_{9i} {}^i R_k, \quad (76)$$

$${}^j \psi_k = \sum_{i=\max(k,j+1)}^n {}^j \vec{r}_{O_i/O_j} {}^j R_i B_{3i} {}^i R_k, \quad (77)$$

$${}^j U_k = \sum_{t=k}^{n-1} ({}^j \gamma_t + {}^j \xi_{t+}) {}^t \vec{r}_{O_{t+1}/O_t} {}^t R_k. \quad (78)$$

Also ${}^j \gamma_t$ and ${}^j \xi_{t+}$ are defined as follows:

$${}^j \gamma_t = \sum_{i=\max(t+1,j+1)}^n {}^j \vec{r}_{O_i/O_j} M_i {}^j R_t, \quad (79)$$

$${}^j \xi_{t+} = \sum_{i=\max(t+1,j)}^n {}^j R_i B_{3i} {}^i R_t. \quad (80)$$

In the next section, Expression 75 will be written in matrix form that makes the inertia matrix of the joint variable in the joint equations. As will be shown, this matrix is symmetric and this fact reduces the necessary computations. Also, the expressions appeared in summation form (${}^j \xi_{t+}$, ${}^j \gamma_t$, ${}^j U_k$, ${}^j \psi_k$, ${}^j \sigma_k$) can be calculated recursively. This is an important issue that causes the reduction of necessary computations and will be considered in detail in the next section.

Inertia Coefficients of Deflection Variables in Joint Equations

In consideration of Equation 43, we observe that the deflection variables $\delta_{j f}$ appear not only in ${}^i \ddot{\vec{r}}_{O_i}$ and ${}^i \ddot{\omega}_i$, but also in ${}^i \vec{B}_{1i}$ and ${}^i \vec{B}_{6i}$. By isolating these terms, the expression for the terms that contain $\delta_{j f}$ is obtained as below:

$$\begin{aligned} & \left(\sum_{k=1}^{n-1} \sum_{t=1}^{m_k} {}^j \vec{z}_j^T ({}^j \sigma_{k+} - {}^j \psi_{k+}) \vec{\theta}_{kt} \right. \\ & + \sum_{k=1}^{n-1} \sum_{t=1}^{m_k} {}^j \vec{z}_j^T ({}^j \xi_{k+} + {}^j \gamma_k) \vec{r}_{kt} \\ & - \sum_{k=1}^{n-2} \sum_{t=1}^{m_k} {}^j \vec{z}_j^T {}^j U_k + \vec{\theta}_{kt} \\ & + \sum_{k=j+1}^n \sum_{t=1}^{m_k} {}^j \vec{z}_j^T {}^j \vec{r}_{O_k/O_j} {}^j R_k \vec{\varepsilon}_{kt} \\ & \left. + \sum_{k=j}^n \sum_{t=1}^{m_k} {}^j \vec{z}_j^T {}^j R_k \vec{\alpha}_{kt} \right) \delta_{kt}, \quad (81) \end{aligned}$$

where:

$${}^j \sigma_{k+} = \sum_{i=\max(k+1,j)}^n {}^j R_i B_{9i} {}^i R_k, \quad (82)$$

$${}^j \psi_{k+} = \sum_{i=\max(k+1,j+1)}^n {}^j \vec{r}_{O_i/O_j} {}^j R_i B_{3i} {}^i R_k, \quad (83)$$

$${}^jU_{k+} = \sum_{t=k+1}^{n-1} ({}^j\gamma_t + {}^j\xi_{t+}) {}^t\tilde{r}_{O_{t+1}/O_t} {}^tR_k. \quad (84)$$

By writing Expression 81 in matrix form, the inertia coefficients of deflection variables in the joint equations will be obtained.

It can be shown that the inertia coefficients for joint variables in the deflection equations are the same as the coefficients of deflection variables in the joint equations. This issue implies the symmetry of the inertia matrix of the whole system and can be used for reduction of necessary computations.

Inertia Coefficients of Deflection Variables in Deflection Equations

In a manner much the same as the previous two steps, the below expression is obtained by isolating the terms that contain $\ddot{\delta}_{jf}$ in deflection equations:

$$\begin{aligned} & \left(\sum_{k=1}^{n-1} \sum_{t=1}^{m_k} \vec{\theta}_{jf}^T \left({}^{j+}\sigma_{k+} - {}^{j+}\psi_{k+} \right) \vec{\theta}_{kt} \right. \\ & - \sum_{k=1}^{n-2} \sum_{t=1}^{m_k} \vec{\theta}_{jf}^T {}^{j+}U_{k+} \vec{\theta}_{kt} \\ & - \sum_{k=1}^{n-2} \sum_{t=1}^{m_k} \vec{r}_{jf}^T {}^jV_k \vec{\theta}_{kt} \\ & - \sum_{k=1}^{n-1} \sum_{t=1}^{m_k} \vec{r}_{jf}^T {}^{j+}\xi_{k+} \vec{\theta}_{kt} \\ & - \sum_{k=1}^{j-2} \sum_{t=1}^{m_k} \vec{e}_{jf}^T {}^jW_k \vec{\theta}_{kt} \\ & + \sum_{k=1}^{j-1} \sum_{t=1}^{m_k} \vec{\alpha}_{jf}^T {}^jR_k \vec{\theta}_{kt} + \sum_{t=1}^{m_j} {}^s c_{jft} \\ & + \sum_{k=1}^{n-1} \sum_{t=1}^{m_k} \vec{\theta}_{jf}^T \left({}^{j+}\gamma_k + {}^{j+}\xi_{k+} \right) \vec{r}_{kt} \\ & + \sum_{k=1}^{n-1} \sum_{t=1}^{m_k} \vec{r}_{jf}^T {}^j\lambda_k \vec{r}_{kt} \\ & + \sum_{k=1}^{j-1} \sum_{t=1}^{m_k} \vec{e}_{jf}^T {}^jR_k \vec{r}_{kt} \\ & + \sum_{k=j+1}^n \sum_{t=1}^{m_k} \vec{r}_{jf}^T {}^jR_k \vec{e}_{kt} \\ & + \sum_{k=j+2}^n \sum_{t=1}^{m_k} \vec{\theta}_{jf}^T {}^j\tilde{r}_{O_k/O_{j+1}} {}^jR_k \vec{e}_{kt} \\ & + \sum_{k=j+1}^n \sum_{t=1}^{m_k} \vec{\theta}_{jf}^T {}^jR_k \vec{\alpha}_{kt} \Big) \ddot{\delta}_{kt}, \quad (85) \end{aligned}$$

where:

$${}^{j+}\sigma_{k+} = \sum_{i=\max(k+1,j+1)}^n {}^jR_i B_{3i} {}^iR_k, \quad (86)$$

$${}^{j+}\xi_{k+} = \sum_{i=\max(k+1,j+1)}^n {}^jR_i B_{3i} {}^iR_k, \quad (87)$$

$${}^jV_k = \sum_{t=k+1}^{n-1} {}^j\lambda_t {}^t\tilde{r}_{O_{t+1}/O_t} {}^tR_k, \quad (88)$$

$${}^{j+}\psi_{k+} = \sum_{i=\max(k+1,j+2)}^n {}^j\tilde{r}_{O_i/O_{j+1}} {}^jR_i B_{3i} {}^iR_k, \quad (89)$$

$${}^{j+}\gamma_k = \sum_{i=\max(k+1,j+2)}^n {}^j\tilde{r}_{O_i/O_{j+1}} M_i {}^jR_k, \quad (90)$$

$${}^jW_k = \sum_{t=k+1}^{j-1} {}^jR_t {}^t\tilde{r}_{O_{t+1}/O_t} {}^tR_k, \quad (91)$$

$${}^{j+}U_{k+} = \sum_{t=k+1}^{n-1} \left({}^{j+}\gamma_t + {}^{j+}\xi_{t+} \right) {}^t\tilde{r}_{O_{t+1}/O_t} {}^tR_k, \quad (92)$$

$${}^j\lambda_k = \sum_{i=\max(k+1,j+1)}^n M_i {}^jR_k. \quad (93)$$

Like the previous two steps, the above expression is written in matrix form. The symmetry of this matrix can be shown by expanding its coefficients. On the other hand, all the expressions in summation form can be calculated recursively.

Final Form of Forward Dynamic Equations

The complete simulation equations have now been derived. It remains to assemble them in final form and point out some remaining recursions that can be used to reduce the number of calculations. The second derivatives of the joint and deflection are desired on the “left hand side” of the equation as unknowns, and the remaining dynamic effects and the inputs are desired on the “right hand side”. To carry out this process completely, one would take the inverse of the inertia matrix, $I(\theta)$, and premultiply the vector of other dynamic effects, \vec{Re} . Because of its complexity, this inverse can only be evaluated numerically. Thus, for the purpose of this paper, the equations will be considered in the following form:

$$I(\Theta) \ddot{\Theta} = \vec{Re}, \quad (94)$$

where:

$I(\Theta)$ The inertia matrix consisting of coefficients will be obtained in the next section;

$\vec{\Theta}$	The vector of generalized coordinate;
$\vec{\Theta}$	$\{q_1 \ \delta_{11} \ \delta_{12} \ \dots \ \delta_{1m_1} \ q_2 \ \delta_{21} \ \dots$ $\delta_{2m_2} \ \dots \ q_k \ \delta_{k1} \ \dots \ \delta_{km_k} \ \dots \ \delta_{nm_n}\}^T$;
q_k	The joint variable for the k th joint;
δ_{kt}	The deflection variable (amplitude) of the t th mode of link k ;
\vec{Re}	Vectors of remaining dynamics and external forcing terms, $\{\text{Re}_1 \ \text{Re}_{11} \ \dots \ \text{Re}_{1m_1} \ \text{Re}_2 \ \text{Re}_{21} \ \dots \ \text{Re}_{2m_2} \ \dots \ \text{Re}_j \ \text{Re}_{j1} \ \dots \ \text{Re}_{jm_j} \ \dots \ \text{Re}_{nm_n}\}^T$;
Re_j	Dynamics from the joint equations j (Equation 43), excluding second derivatives of the generalized coordinate;
Re_{jf}	Dynamics from the deflection equations jf (Equation 44), excluding second derivatives of the generalized coordinate.

At first, consider Re_j . In joint equations, by collecting the terms that do not contain \ddot{q}_j and $\ddot{\delta}_{jf}$, the expression is obtained as below:

$$\text{Re}_j = \tau_j - \sum_{i=j+1}^n \frac{\partial^i \vec{r}_{O_i}^T}{\partial \ddot{q}_j} \cdot {}^i \vec{S}_i - \sum_{i=j}^n \frac{\partial^i \vec{\omega}_i^T}{\partial \ddot{q}_j} \cdot {}^i \vec{T}_i, \quad (95)$$

where:

$${}^i \vec{S}_i = M_i {}^i \ddot{r}_{O_{v,i}} - 2B_{2i} {}^i \vec{\omega}_i - B_{3i} {}^i \dot{\vec{\omega}}_{v,i} - {}^i \vec{\omega}_i B_{3i} {}^i \vec{\omega}_i, \quad (96)$$

$${}^i \vec{T}_i = B_{3i} {}^i \ddot{r}_{O_{v,i}} + 2B_{8i} {}^i \vec{\omega}_i + B_{9i} {}^i \dot{\vec{\omega}}_{v,i} + {}^i \vec{\omega}_i B_{9i} {}^i \vec{\omega}_i. \quad (97)$$

By substituting Equations 71 and 73 in Equation 95 and changing it to a recursive expression, a new equation for Re_j is obtained:

$$\text{Re}_j = \tau_j - {}^j \vec{z}_j^T {}^j \vec{\chi}_j, \quad (98)$$

where:

$${}^j \vec{\chi}_j = {}^j \vec{T}_j + {}^j \vec{r}_{O_{j+1}/O_j} {}^j \vec{\phi}_j + {}^j R_{j+1} {}^{j+1} \vec{\chi}_{j+1}, \quad (99)$$

and:

$${}^j \vec{\phi}_j = {}^j R_{j+1} \left({}^{j+1} \vec{S}_{j+1} + {}^{j+1} \vec{\phi}_{j+1} \right). \quad (100)$$

Now, consider Re_{jf} . If in the deflection equation the terms that do not contain \ddot{q}_j and $\ddot{\delta}_{jf}$ are collected, the following expression will be obtained:

$$\begin{aligned} \text{Re}_{jf} = & - \sum_{k=1}^{m_j} \delta_{jk} K_{j k f} - \sum_{i=j+1}^n \frac{\partial^i \vec{r}_{O_i}^T}{\partial \ddot{\delta}_{jf}} \cdot {}^i \vec{S}_i \\ & - \sum_{i=j+1}^n \frac{\partial^i \vec{\omega}_i^T}{\partial \ddot{\delta}_{jf}} \cdot {}^i \vec{T}_i + Q_{jf}, \end{aligned} \quad (101)$$

where:

$$\begin{aligned} Q_{jf} = & 2^j \omega_j^T \cdot \sum_{k=1}^{m_j} \delta_{jk} \vec{c}_{j k} + {}^j \vec{\omega}_j^T \beta_{j f} {}^j \vec{\omega}_j \\ & - {}^j \ddot{r}_{O_{v,j}}^T \cdot \vec{\varepsilon}_{j f} - {}^j \dot{\vec{\omega}}_{v,j}^T \cdot \vec{\alpha}_{j f}. \end{aligned} \quad (102)$$

Like the previous step, the following recursive equation for Re_{jf} is obtained:

$$\begin{aligned} \text{Re}_{jf} = & - \sum_{k=1}^{m_j} \delta_{jk} K_{j k f} + Q_{jf} \\ & - \vec{r}_{j f}^T \cdot {}^j \vec{\phi}_j - \vec{\theta}_{j f}^T {}^j R_{j+1} {}^{j+1} \vec{\chi}_{j+1}. \end{aligned} \quad (103)$$

Equations 98 and 103 are used to construct the right hand side equations of motion.

PROPOSED ALGORITHM

Now, we shall present an algorithm that results from the expressions developed in previous sections. In this algorithm, all cross products are done in tensor notation. And, also, each specific algorithmic expression is accompanied by information that indicates the number of algebraic operations that are involved, showing separately products M and A sums. The calculations are done in a step by step process, as follows:

Step 1: The rotation matrix will be calculated by this algorithm.

for $i = 2 : 1 : n$

$${}^{i-1} R_i = E_{i-1} A_i \quad \& \quad {}^i R_{i-1} = {}^{i-1} R_i^T; \quad 15M6A$$

Step 2: The vectors of ${}^i \vec{\omega}_i$, ${}^i \dot{\vec{\omega}}_{v,i}$ and ${}^i \ddot{r}_{O_{v,i}}$ can be calculated recursively, as follows.

Initialize:

$${}^1 \vec{\omega}_1 = {}^1 \vec{z}_1 \dot{q}_1; \quad \& \quad {}^1 \dot{\vec{\omega}}_{v,1} = \{0 \ 0 \ 0\}^T; \quad \&$$

$${}^1 \ddot{r}_{O_{v,1}} = A_1^T \{g_x \ g_y \ g_z\}^T;$$

for $i = 2 : 1 : n$

Equation 63 9M10A

$$\begin{aligned} {}^i \dot{\vec{\omega}}_{v,i} = & {}^i R_{i-1} \left(\left({}^{i-1} \vec{\omega}_{i-1} + {}^{i-1} \dot{\vec{\theta}}_{i-1} \right) {}^{i-1} R_i {}^i \vec{z}_i \dot{q}_i \right. \\ & \left. + {}^{i-1} \dot{\vec{\omega}}_{v,i-1} \right); \quad 18M18A \end{aligned}$$

$$\begin{aligned} {}^i\ddot{\vec{r}}_{O_n,i} &= {}^iR_{i-1} \left(2^{i-1}\tilde{\omega}_{i-1}{}^{i-1}\dot{\vec{r}}_{O_i/O_{i-1}} \right. \\ &\quad + {}^{i-1}\dot{\tilde{\omega}}_{v,i-1}{}^{i-1}\vec{r}_{O_i/O_{i-1}} \\ &\quad + {}^{i-1}\tilde{\omega}_{i-1} \left({}^{i-1}\tilde{\omega}_{i-1}{}^{i-1}\vec{r}_{O_i/O_{i-1}} \right) \\ &\quad \left. + {}^{i-1}\ddot{\vec{r}}_{O_n,i-1} \right); \quad 33M27A \end{aligned}$$

Step 3: In this step, the vectors of ${}^i\vec{S}_i$ and ${}^i\vec{T}_i$ are calculated.

for $i = 2 : 1 : n$

Equation 96; 27M 21A

for $i = 1 : 1 : n$

Equation 97; 39M 33A

Step 4: The vectors of ${}^i\vec{\phi}_i$ and ${}^i\vec{\chi}_i$ can be calculated by the following algorithm:

Initialize:

$${}^n\vec{\phi}_n = \{0 \ 0 \ 0\}^T; \quad \& \quad {}^n\vec{\chi}_n = {}^n\vec{T}_n$$

for $j = n - 1 : -1 : 1$

Equation 100; 9M 9A

Equation 99; 15M 15A

Step 5: Calculation of Q_{jf} ,

for $j = 1 : 1 : n; f = 1 : 1 : m_j$

Equation 102; 21M 17A

Step 6: In this step, Equations 98 and 103 are used to calculate Re_j and Re_{jf} .

for $j = 1 : 1 : n$

Equation 98; 0M 1A

for $f = 1 : 1 : m_n$

$$Re_{nf} = - \sum_{k=1}^{m_n} \delta_{nk} K_{nkf} + Q_{nf}; \quad 0M \ 1A$$

for $j = 1 : 1 : n - 1; f = 1 : 1 : m_j$

Equation 103; 15M 13A

At the end of this step, the right hand side of the equations of motion is completely evaluated. In continuation, a recursive algorithm is presented that evaluates the left hand side of the equations of motion

and, also, the inertia matrix of the whole system.

Step 7: Calculation of the compound rotation matrix,

for $j = 1 : 1 : n$

$${}^jR_j = I_{3 \times 3};$$

for $j = 1 : 1 : n - 2; k = j + 2 : 1 : n$

$${}^jR_k = {}^jR_{k-1}{}^{k-1}R_k; \quad \&$$

$${}^kR_j = {}^jR_k^T; \quad 27M \ 18A$$

Step 8: The following algorithm evaluates the vector of ${}^i\vec{r}_{O_j/O_i}$:

for $j = 2 : 1 : n - 1; j = i - 1 : -1 : 1$

$${}^j\vec{r}_{O_{i+1}/O_i} = {}^jR_{j+1}{}^{j+1}\vec{r}_{O_{i+1}/O_i}; \quad 9M \ 6A$$

for $i = 1 : 1 : n - 2; j = i + 2 : 1 : n$

$${}^i\vec{r}_{O_j/O_i} = {}^i\vec{r}_{O_{j-1}/O_i} + {}^i\vec{r}_{O_j/O_{j-1}}; \quad 0M \ 3A$$

Step 9: In this step, the variables that have been appeared in summation form in the inertia matrix are evaluated.

- Calculation of ${}^j\sigma_k$:

for $k = n : -1 : 1; j = k : -1 : 1$

if ($k = j$)

$$\text{if } (k = n) \quad {}^n\sigma_n = B_{9n};$$

else

$${}^k\sigma_k = B_{9k} + {}^k\sigma_{k+1}{}^{k+1}R_k; \quad 27M \ 27A$$

else

$${}^j\sigma_k = {}^jR_{j+1}{}^{j+1}\sigma_k; \quad \& \quad {}^k\sigma_j = {}^j\sigma_k^T; \quad 27M \ 18A$$

A recursive algorithm, like the one mentioned above, for calculation of ${}^j\xi_k$, can be used. However, it should be noticed that, instead of B_{9i} , we have B_{3i} and, also, at the last line, we have:

$${}^k\xi_j = -{}^j\xi_k^T;$$

- Calculation of ${}^j\psi_k$:

for $j = n - 1 : -1 : 1$

$${}^j\psi_n = {}^j\vec{r}_{O_n/O_j}{}^j\xi_n; \quad 18M \ 9A$$

for $j = n - 1 : -1 : 1; k = n - 1 : -1 : 1$

if ($k > j$)

$${}^j\psi_k = {}^j\psi_{k+1}{}^{k+1}R_k + {}^j\tilde{r}_{O_k/O_j}{}^jR_k B_{3k}; \quad 63M \ 45A$$

else ${}^j\psi_k = {}^j\psi_{k+1}{}^{k+1}R_k; \quad 27M \ 18A$

- Calculation of ${}^j\lambda_k$:

for $k = n - 1 : -1 : 1; j = k : -1 : 1$

if ($k = j$)

if ($k = n - 1$) ${}^{n-1}\lambda_{n-1} = M_n I_{3 \times 3}; \quad 3M \ 0A$

else

$${}^k\lambda_k = M_{k+1} I_{3 \times 3} + {}^{k+1}\lambda_{k+1}; \quad 3M \ 3A$$

else

$${}^j\lambda_k = {}^jR_{j+1}{}^{j+1}\lambda_k; \ \& \quad {}^k\lambda_j = {}^j\lambda_k^T; \quad 27M \ 18A$$

- Calculation of ${}^j\gamma_k$:

for $j = n - 1 : -1 : 1$

$${}^j\gamma_{n-1} = {}^j\tilde{r}_{O_n/O_j}{}^j\lambda_{n-1}; \quad 18M \ 9A$$

for $j = n - 1 : -1 : 1; k = n - 2 : -1 : 1$

if ($k < j$) ${}^j\gamma_k = {}^j\gamma_{k+1}{}^{k+1}R_k; \quad 27M \ 18A$

else

$${}^j\gamma_k = {}^j\gamma_{k+1}{}^{k+1}R_k + M_{k+1}{}^j\tilde{r}_{O_{k+1}/O_j}{}^jR_k; \quad 51M \ 36A$$

- Calculation of jU_k :

for $j = 1 : 1 : n$

$${}^jU_{n-1} = ({}^j\gamma_{n-1} + {}^j\xi_{n-1}){}^{n-1}\tilde{r}_{O_n/O_{n-1}};$$

18M 18A

for $j = 1 : 1 : n; k = n - 2 : -1 : 1$

$${}^jU_k = ({}^j\gamma_k + {}^j\xi_{k+1}){}^k\tilde{r}_{O_{k+1}/O_k} + {}^jU_{k+1}{}^{k+1}R_k; \quad 45M \ 45A$$

- Calculation of jV_k :

for $j = 1 : 1 : n - 1$

$${}^jV_{n-2} = {}^j\lambda_{n-1}{}^{n-1}\tilde{r}_{O_n/O_{n-1}}{}^{n-1}R_{n-2}; \quad 18M \ 9A$$

for $j = 1 : 1 : n - 1; k = n - 3 : -1 : 1$

$${}^jV_k = {}^j\lambda_{k+1}{}^{k+1}\tilde{r}_{O_{k+2}/O_{k+1}}{}^{k+1}R_k + {}^jV_{k+1}{}^{k+1}R_k; \quad 45M \ 36A$$

- Calculation of ${}^j\sigma_{k+}$:

for $k = 1 : 1 : n - 1; j = 1 : 1 : n$

$${}^j\sigma_{k+} = {}^j\sigma_{k+1}{}^{k+1}R_k; \quad 27M \ 18A$$

- Calculation of ${}^{j+}\sigma_{k+}$:

for $j = 1 : 1 : n - 1; k = 1 : 1 : n - 1$

$${}^{j+}\sigma_{k+} = {}^jR_{j+1}{}^{j+1}\sigma_{k+}; \quad 27M \ 18A$$

For calculation of ${}^j\xi_{k+}, {}^j\psi_{k+}$ and ${}^jU_{k+}$ we use the algorithm like the one that was used for calculation of ${}^j\sigma_{k+}$. Also, ${}^{j+}\xi_{k+}, {}^{j+}\psi_{k+}, {}^{j+}\gamma_k$ and ${}^{j+}U_{k+}$ can be calculated by the algorithm like the one that was used for calculation of ${}^{j+}\sigma_{k+}$.

Step 10: Finally, calculation of the inertia matrix for the whole system is considered.

- Calculation of the inertia matrix for joint variables in joint equations:

for $j = 1 : 1 : n - 1; k = j : 1 : n - 1$

$$I_{jk} = {}^jz_j^T ({}^j\sigma_k - {}^j\psi_k - {}^jU_k) {}^kz_k; \ \&$$

$$I_{kj} = I_{jk}; \quad 0M \ 18A$$

for $j = 1 : 1 : n$

if ($j \neq n$) $I_{jn} = {}^jz_j^T ({}^j\sigma_n - {}^j\psi_n) {}^nz_n; \ \&$

$$I_{nj} = I_{jn}; \quad 0M \ 9A$$

else $I_{nn} = {}^nz_n^T {}^n\sigma_n {}^nz_n; \quad 0M \ 0A$

- Calculation of the inertia matrix for deflection variables in joint equations:

for $j = 2 : 1 : n; k = 1 : 1 : j - 1; t = 1 : 1 : m_k$

$$I_{jkt} = {}^j \bar{z}_j^T \left(({}^j \sigma_{k+} - {}^j \psi_{k+} - {}^j U_{k+}) \bar{\theta}_{kt} + ({}^j \xi_{k+} + {}^j \gamma_k) \bar{r}_{kt} \right); \quad 18M \ 42A$$

for $j = 1 : 1 : n - 1; k = j; t = 1 : 1 : m_k$

$$I_{jkt} = {}^j \bar{z}_j^T \left(({}^j \sigma_{k+} - {}^j \psi_{k+} - {}^j U_{k+}) \bar{\theta}_{kt} + ({}^j \xi_{k+} + {}^j \gamma_k) \bar{r}_{kt} + \bar{\alpha}_{kt} \right); \quad 18M \ 45A$$

for $j = 1 : 1 : n - 2; k = j + 1 : 1 : n - 1; t = 1 : 1 : m_k$

$$I_{jkt} = {}^j \bar{z}_j^T \left(({}^j \sigma_{k+} - {}^j \psi_{k+} - {}^j U_{k+}) \bar{\theta}_{kt} + ({}^j \xi_{k+} + {}^j \gamma_k) \bar{r}_{kt} + {}^j R_k \bar{\alpha}_{kt} + {}^j \tilde{r}_{O_k/O_j} {}^j R_k \bar{\varepsilon}_{kt} \right); \quad 42M \ 63A$$

for $j = 1 : 1 : n - 1; k = n; t = 1 : 1 : m_k$

$$I_{jnt} = {}^j \bar{z}_j^T \left({}^j R_k \bar{\alpha}_{kt} + {}^j \tilde{r}_{O_k/O_j} {}^j R_k \bar{\varepsilon}_{kt} \right); \quad 24M \ 18A$$

for $j = k = n; t = 1 : 1 : m_k$

$$I_{jkt} = {}^j \bar{z}_k^T \bar{\alpha}_{kt}; \quad 0M \ 0A$$

- Calculation of deflection variables in deflection equations:

for $j = 1 : 1 : n - 1; k = j; t = 1 : 1 : m_k; f = 1 : 1 : m_j$

$$I_{jkft} = \bar{\theta}_{jf}^T \left(({}^j \gamma_k + {}^j \xi_{k+}) \bar{r}_{kt} + ({}^j \sigma_{k+} - {}^j \psi_{k+} - {}^j U_{k+}) \bar{\theta}_{kt} - \bar{r}_{jf}^T \left(({}^j V_k + {}^j \xi_{k+}) \bar{\theta}_{kt} - {}^j \lambda_k \bar{r}_{kt} \right) \right) + {}^s c_{jft}; \quad 42M \ 72A$$

for $j = 1 : 1 : n - 2; k = j + 1 : 1 : n - 1; t = 1 : 1 : m_k;$

$f = 1 : 1 : m_j$

$$I_{jkft} = \bar{\theta}_{jf}^T \left(({}^j \gamma_k + {}^j \xi_{k+}) \bar{r}_{kt} + ({}^j \sigma_{k+} - {}^j \psi_{k+} - {}^j U_{k+}) \bar{\theta}_{kt} + {}^j R_k \bar{\alpha}_{kt} + {}^j R_{j+1} {}^{j+1} \tilde{r}_{O_k/O_{j+1}} {}^{j+1} R_k \bar{\varepsilon}_{kt} \right) - \bar{r}_{jf}^T \left(({}^j V_k + {}^j \xi_{k+}) \bar{\theta}_{kt} - {}^j \lambda_k \bar{r}_{kt} - {}^j R_k \bar{\varepsilon}_{kt} \right); \quad 84M \ 107A$$

for $j = 1 : 1 : n - 1; k = n; t = 1 : 1 : m_k; f = 1 : 1 : m_j$

$$I_{jkft} = \bar{\theta}_{jf}^T \left({}^j R_k \bar{\alpha}_{kt} + {}^j R_{j+1} {}^{j+1} \tilde{r}_{O_k/O_{j+1}} {}^{j+1} R_k \bar{\varepsilon}_{kt} \right) + \bar{r}_{jf}^T {}^j R_k \bar{\varepsilon}_{kt}; \quad 48M \ 35A$$

for $j = n; k = n; t = 1 : 1 : m_k; f = 1 : 1 : m_j$

$$I_{jkft} = {}^s c_{jft}; \quad 0M \ 0A$$

The required mathematical operations for calculating the above steps are listed in Table 1, where n is the total number of links; n_f is the number of flexible links and m is the number of modes describing each flexible link, the same for all flexible links.

In Table 2, the computational complexity of this method compared with the ones of [5], are shown. Also, Table 2 shows the number of operations for two typical cases.

As a general comparison, the number of mathematical operations of the method proposed in this article for the dynamic modeling of flexible manipulators is less than the recursive Lagrangian method in [5].

COMPUTATIONAL SIMULATION

In this section, we verify the proposed method for the dynamic modeling of flexible robotic manipulators in the preceding sections by means of computational simulation for a manipulator with two elastic links. The first mode shape of clamped-free beams is used to model the elastic deformation of each link. All necessary parameters of flexible links for this computational simulation are shown in Table 3. These parameters are the same as in [4].

To clearly explain computational procedures for the simulation, we rewrite Equation 94 in state form.

$$\dot{\bar{\Theta}}_1 = \bar{\Theta}_2,$$

$$\dot{\bar{\Theta}}_2 = I^{-1}(\bar{\Theta}_1) \bar{R}e.$$

The initial conditions are also the same as in [4] as shown in Figure 3.

Table 1. The required mathematical operation for deriving the equation of motion in G-A’s formulation.

Sums	Products	Step
$6n - 6$	$15n - 15$	1
$55n - 55$	$60n - 60$	2
$54n - 21$	$66n - 27$	3
$24n - 24$	$24n - 24$	4
$17n_fm$	$21n_fm$	5
$n - 12m + 13mn_f$	$15mn_f - 15m$	6
$9n^2 - 27n + 18$	$13.5n^2 - 40.5n + 27$	7
$4.5n^2 - 13.5n + 9$	$4.5n^2 - 13.5n + 9$	8
$276 - 652.5n + 328.5n^2$	$390 - 901.5n + 457.5n^2$	9
$9n^2 - 9 + 52.5mn_f^2 - 79.5mn_f + 18m - 89.5m^2n_f + 18m^2 + 53.5m^2n_f^2$	$30mn_f^2 - 30mn_f - 60m^2n_f + 18m^2 + 42m^2n_f^2$	10

Table 2. The comparison of computational complexity.

Sums		Products		Principle	Authors
$329n + 115.5mn_f^2 + 19m^2n_f + 123mn_fm + 85n^2 + 68nn_fm + 6.5m^2n_f^2 - 91 + 80nn_fm + 111n_f$		$279n + 118mn_f^2 + 17.5m^2n_f + 137.5mn_fm + 84n^2 + 74nn_fm + 6m^2n_f^2 - 57 + 86nn_fm + 126n_f$		I-E	Book
$6m - 553n - 49.5mn_fm + 351n^2 + 18m^2 + 53.5m^2n_f^2 + 52.5mn_f^2 - 89.5m^2n_fm + 188$		$-15m - 790.5n + 6mn_fm + 475.5n^2 + 18m^2 + 42m^2n_f^2 + 30mn_f^2 - 60m^2n_fm + 300$		G-A	This work
$n = 6, n_f = 6, m = 3$		$n = 3, n_f = 2, m = 2$			
24569A	25251M	4851A	4922M		Book
18332A	19557M	2134M	2706M		This work

$\theta_1 = -90 \text{ deg}, \theta_2 = 5 \text{ deg},$

$\dot{\theta}_1(0) = \dot{\theta}_2(0) = \delta_{11}(0) = \delta_{12}(0) = \delta_{21}(0) = \delta_{22}(0) = 0.$

Then, by using a numerical method, such as Runge-Kutta, a set of differential equations will be solved. By solving this set of differential equations, the time response of the system will be obtained, $(q_1, \dot{q}_1, \delta_{11}, \delta_{11}, q_2, \dot{q}_2, \delta_{21}, \delta_{21})$. In [4], which uses FEM for simulation, the results of flexural displacement and angular displacement in the middle and at the end of each link are shown. So, for comparison, we present the same results in Figures 4 to 15.

Variables $u_3, u_4, u_5, u_6, w_3, w_4, w_5$ and w_6 in Fig-

Table 3. The necessary parameters for simulation.

Parameters	Value	Unit
The length of the links	$L_1 = L_2 = 1$	m
Module of elasticity	$E_1 = E_2 = 2.0 \times 10^{11}$	N/m ²
Moment of inertia	$I_{1z} = I_{2z} = 5.0 \times 10^{-9}$	m ⁴
Mass per unit length	$\rho_1 = \rho_2 = 5$	kg/m

ures 4 to 15 represent the flexural responses of the system. The response of these variables portrays the vibration modes of the system response and their influence on the quality of the system response. Simulations results show that the response of the flexible manipulator is highly undesirable and, in order to get

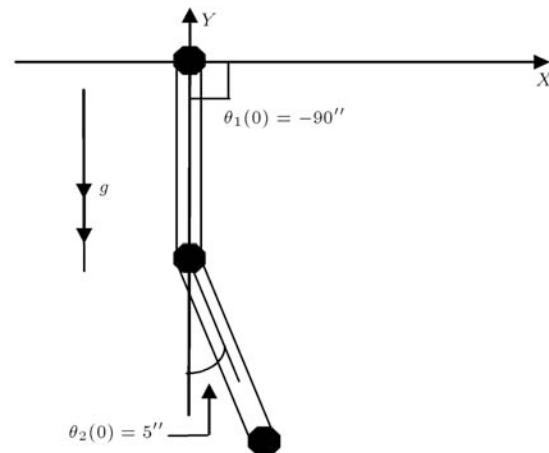


Figure 3. Initial condition for simulation.

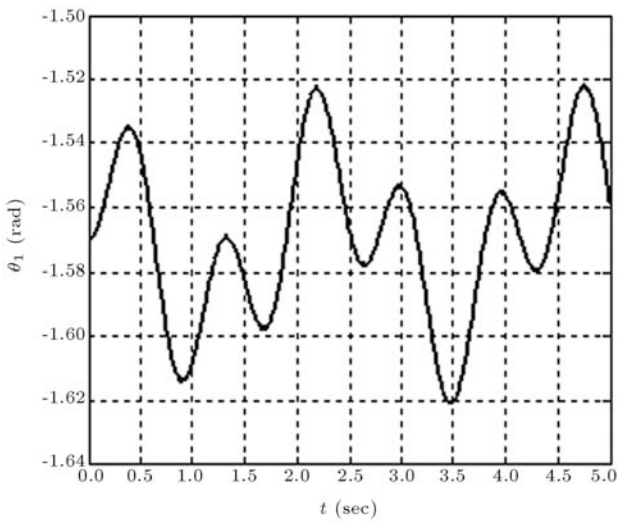


Figure 4. Angular displacement of the first joint.

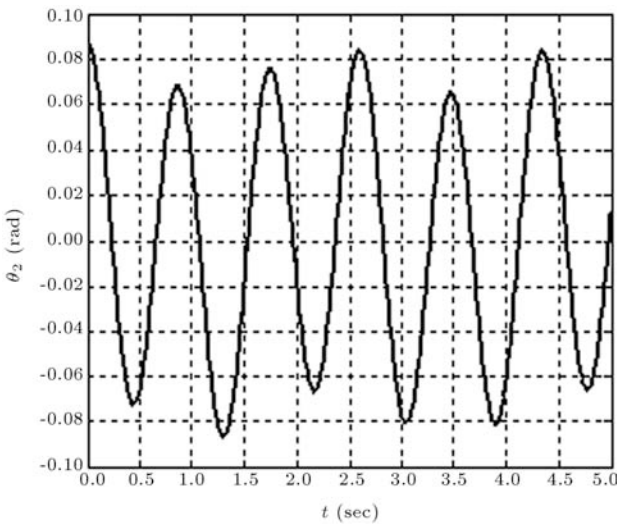


Figure 5. Angular displacement of the second joint.

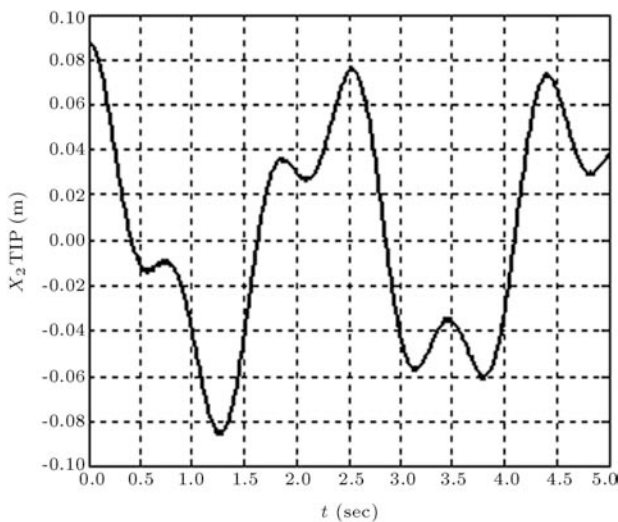


Figure 6. X position of end effector.

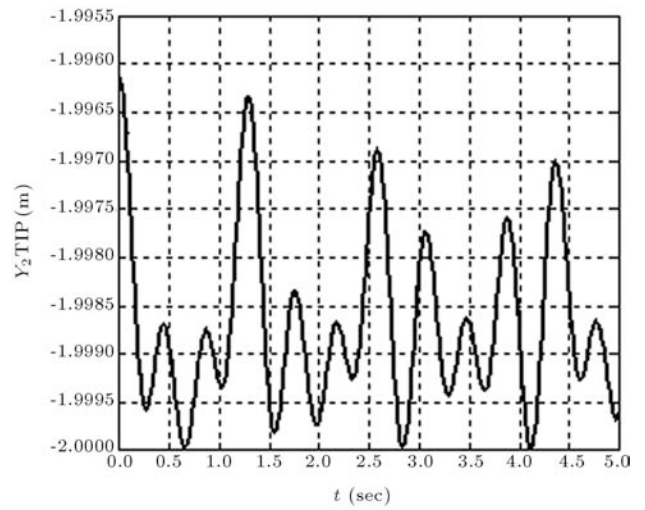


Figure 7. Y position of end effector.

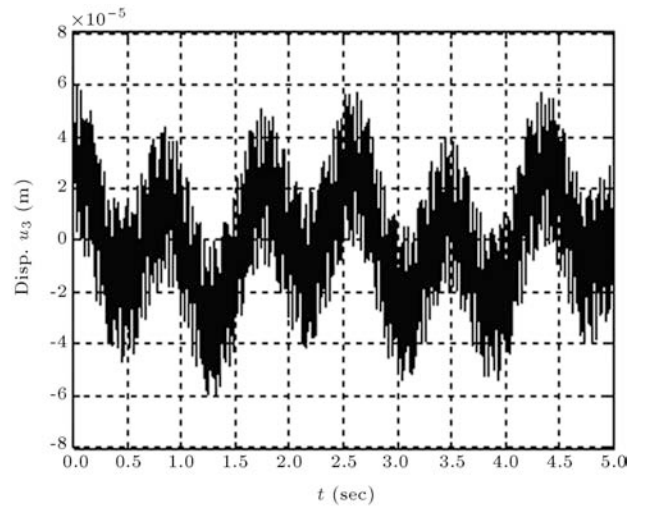


Figure 8. Flexural displacement in the middle of the first link.

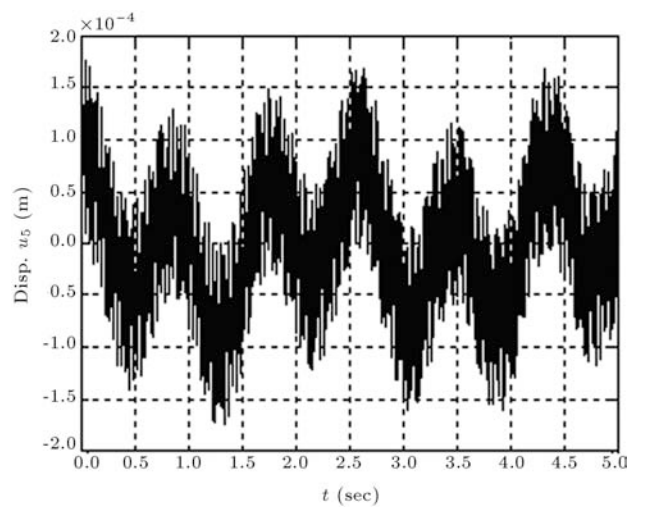


Figure 9. Flexural displacement at the end of the first link.

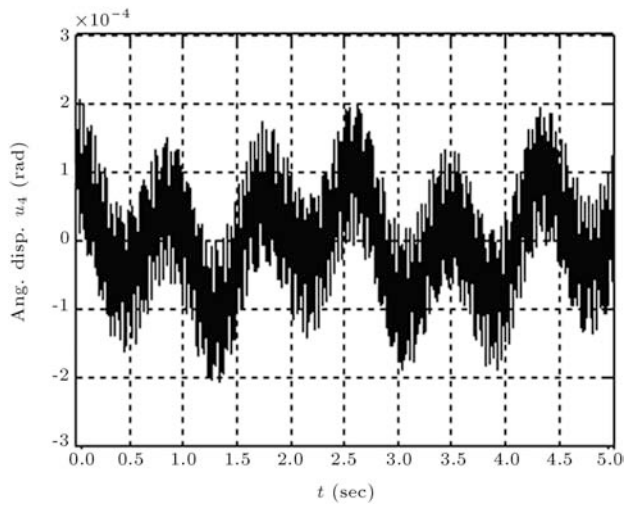


Figure 10. Angular displacement in the middle of the first link.

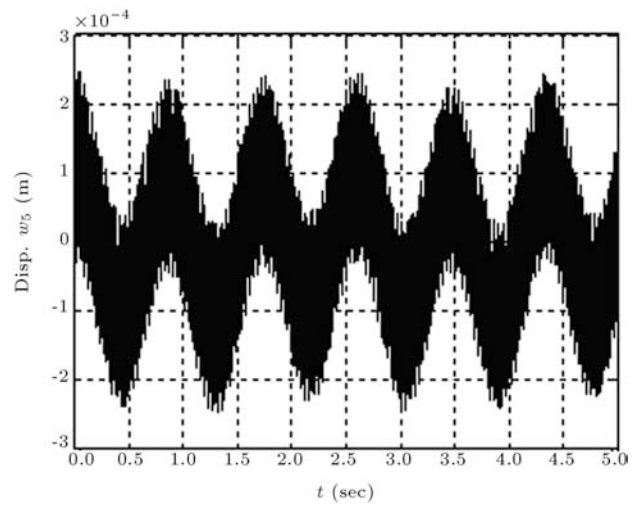


Figure 13. Flexural displacement at the end of the second link.

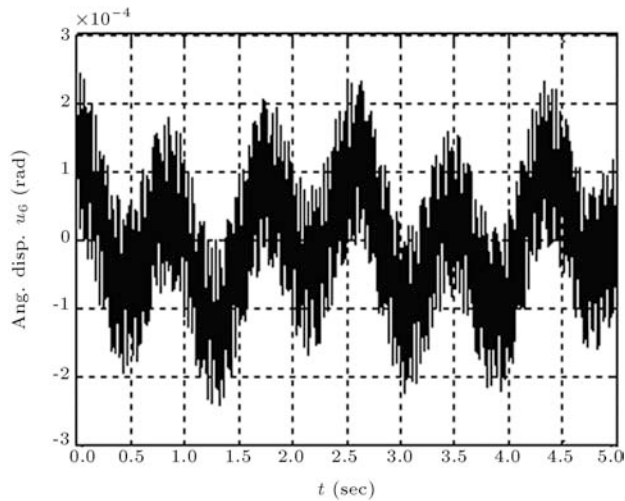


Figure 11. Angular displacement at the end of the first link.

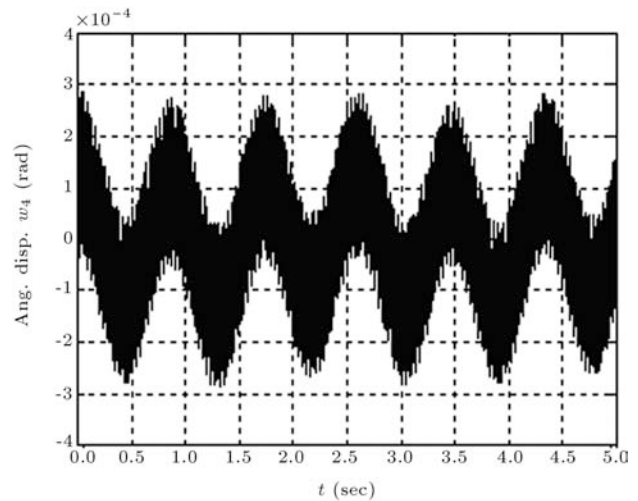


Figure 14. Angular displacement in the middle of the second link.

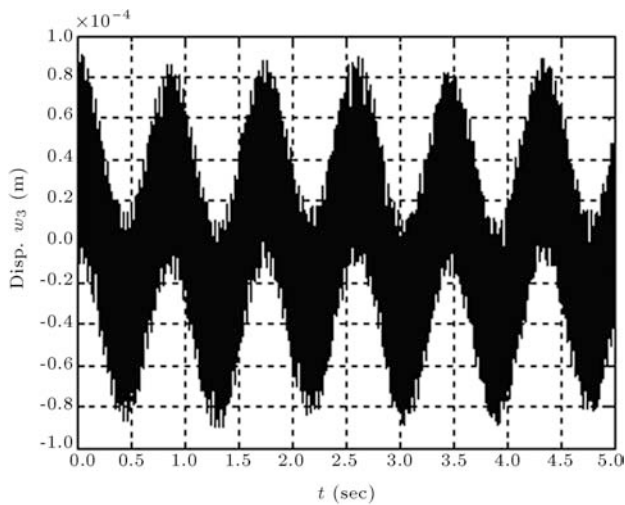


Figure 12. Flexural displacement in the middle of the second link.

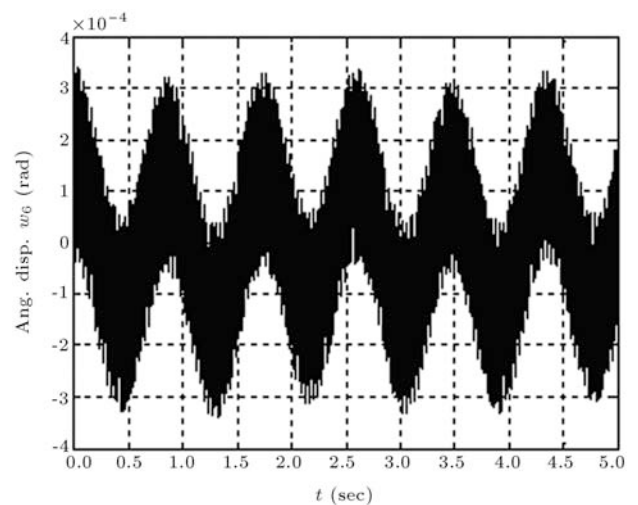


Figure 15. Angular displacement at the end of the second link.

the dynamics of the system to be acceptable for most practical purposes, very effective controls are needed to control the vibration modes. On the other hand, as seen, the results are in good concordance with ones in [4]. It should be noted that the simulation is done by using only one mode shape. More accurate results will be obtained by using more mode shapes.

CONCLUSION

This article has presented an efficient and systematic method for the dynamic modeling of flexible robotic manipulators. The proposed method can be applied to the design of control systems and the dynamic simulation of flexible manipulators. The advantages of this method in comparison with others are as follows:

1. A reduction in computations by using only 3×3 and 3×1 matrices.
2. Increase in the speed of generating the equations of motion by reducing the number of additions and multiplications, as shown in Table 2.
3. Ease of understanding, as it uses primitive dynamic concepts.

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