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Economic Production Quantity Model with Scrapped Items and Limited Production Capacity

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Abstract. *In this paper, an Economic Production Quantity (EPQ) model is studied, in which the production defective-rate follows either a uniform or a normal probability distribution. Shortages are allowed and take a backorder state, and the existence of only one machine causes a limited production capacity for the common cycle length of all products. The aim of this research is to determine the optimal production quantity of each product, such that the expected total cost including holding, shortage, production, setup and defective items cost is minimized. The mathematical model of the problem is derived, for which the objective function is proved to be convex. Then, a derivative approach is utilized to obtain the optimal solution. At the end, two numerical examples are provided to illustrate the practical usage of the proposed method.*

Keywords: *Economic production quantity; Scrapped items; Limited production capacity.*

INTRODUCTION

The primary operation strategy and goal of most manufacturing firms is to seek a high satisfaction to customer's demands and to become a low-cost producer. To achieve these goals, the company must be able to effectively utilize resources and minimize costs.

In manufacturing companies, when items are internally produced instead of being obtained from an outside supplier, the Economic Production Quantity (EPQ) model is often employed to determine the optimal production lot size that minimizes overall production/inventory costs.

The classic EPQ model assumes that during a production run, a manufacturing facility functions perfectly. However, due to process deterioration or some other factors, the generation of imperfect quality items is inevitable. A considerable amount of research has been carried out by Cheng [1], Chiu et al. [2], Chung [3], Lee and Rosenblatt [4], Ben-Daya [5] and Chiu and Chiu [6] to address the imperfect quality EPQ problem. Furthermore, Chung [3] investigated bounds for production lot sizing with machine breakdown conditions. Rosenblatt and Lee [7] proposed an EPQ model that deals with imperfect quality. They assumed that at some random point in time, the process might shift from an in-control to an out-of-control state. Hayek and Salameh [8] derived an optimal operating policy for the finite production EPQ model, under the effect of reworking imperfect quality items. They assumed that all defective items were repairable and that backorders were allowed.

Numerous studies have been carried out to address the problems of an imperfect quality EPQ model with rework (see for example [8-12]). Chiu and Chiu [6] studied an optimal replenishment policy for an imperfect quality EPQ model with backlogging and failure in repair using a conventional approach.

In this paper, a multiproduct EPQ problem, in which the production defective-rates of all items are random variables and all defective items are assumed to be scrapped (rework is not allowed), is considered. Besides, the productions of all items are performed on

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a single machine, such that there is a limited capacity, also that shortages are allowed and are considered to be backorders.

The rest of the paper is organized as follows. the problem is first defined, and the mathematical model of the problem is developed. Then, the solution approach is proposed, and in order to illustrate the proposed method and its applications, two numerical examples are given. Following that, a sensitivity analysis on the model parameters is performed, and finally, the conclusion and some recommendations on future research are found.

PROBLEM DEFINITION AND MODELING

Imperfect production processes due to process deterioration or some other factors may randomly generate X percent of defective items at rate θ. The inspection cost per item is involved when all items are screened. All defective items are assumed to be scrapped, i.e. no rework is allowed. The annual constant production rate (P) is much larger than the annual constant demand rate (D) as the basic assumption of the finite production model. In other words, the expected production rate of the scrapped items, θ, can be expressed as $\theta = PE[X]$. Also, we assume that there is a real constant production capacity limitation on a single machine on which all products are produced, and that the setup cost is nonzero.

Since the problem at hand is of a multiproduct type, for products $j = 1, 2, \cdots, n$, the following notations are used in this research:

P_j: The annual constant production rate of the jth product,

X_j: The production percent of the jth products that are defective (a random variable),

$f_{X_j}(x_j)$: The probability density function of X_j,

θ_j: The expected annual production scrapped rate of the jth product,

D_j: The annual constant demand of the jth product,

Q_j^B: The production lot size per cycle of the jth product in which shortages are allowed as backorders (a decision variable),

B_j: The allowable backorder level of the jth product (a decision variable),

I_j^1: The maximum units of a on-hand inventory level when the regular production process stops,

N: The number of cycles per year,

T: The cycle length, $T = \frac{1}{N}$ (a decision variable),

t_j^1: The production uptime of the jth product,

t_j^2: The production downtime of the jth product,

t_j^3: The permitted shortage time of the jth product,

t_j^4: The time needed to satisfy all backorders in the next production of the jth product,

S_j: The setup time of the machine to produce the jth product,

A: The constant setup cost for all products (\$/setup),

C_j^P: The production cost per unit of the jth product (\$/item),

C_j^h: The holding cost per unit of the jth product per unit time (\$/item/unit time),

C_j^b: The backordering cost per unit of the jth product per unit time (\$/item/unit time),

C_j^s: The disposal cost per scrapped item of the jth product (\$/scrapped item),

C_A: The annual expected total setup cost,

C_P: The annual expected total production cost,

C_H: The annual expected total holding cost,

C_B: The annual expected total shortage cost,

C_S: The annual expected total scrapped items cost,

$E(.)$: Denotes the expected value,

Z: The annual expected total costs.

The production rate, P_j, is always assumed to be greater than, or equal to, the demand rate, D_j. Furthermore, the production rate of the perfect quality items is assumed to be greater than, or equal to, the sum of the demand rate and the production rate of defective items; mathematically speaking: $P_j - \theta_j - D_j \geq 0$, or $1 - E[X_j] - \frac{D_j}{P_j} \geq 0$. Figure 1 depicts the on-hand inventory level and allowable backorder level of the EPQ model with backlogging permitted. To model the problem, a part of the modeling procedure is adopted from [8]. Since all products are manufactured on a single machine with a limited capacity, the cycle length is equal for all ($T_1 = T_2 = \cdots = T_n = T$). Then, based on Figure 1, for $j = 1, 2, \cdots, n$, we will have:

$$T = \sum_{i=1}^{4} t_j^i = \frac{Q_j^B E(1 - X_j)}{D_j}, \tag{1}$$

$$t_j^1 = \frac{I_j^1}{P_j - D_j - \theta_j}, \tag{2}$$

$$I_j^1 = (P_j - D_j - \theta_j)\frac{Q_j^B}{P_j} - B_j, \tag{3}$$

$$t_j^2 = \frac{I_j^1}{D_j}, \tag{4}$$

$$t_j^3 = \frac{B_j}{D_j}, \tag{5}$$

$$t_j^4 = \frac{B_j}{P_j - D_j - \theta_j}, \tag{6}$$

$$(t_j^1 + t_j^4) = \frac{Q_j^B}{P_j}. \tag{7}$$

The Objective Function

The objective function of the model is the summation of the expected annual production, holding, shortage, disposal and setup costs as:

$$Z = C_P + C_H + C_B + C_S + C_A. \tag{8}$$

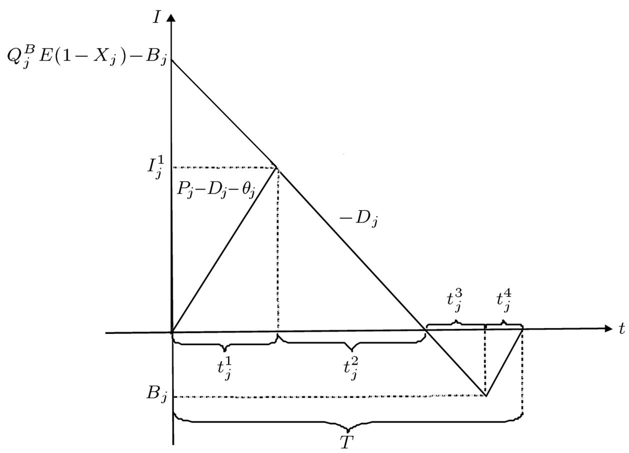

Figure 1. A production-inventory cycle.

In the following subsections, different parts of the objective function are described.

Production Cost

The production cost per unit and the production quantity per period of the jth product are C_j^P and Q_j^B, respectively. Hence, the production cost of the jth product per period is $C_j^P Q_j^B$. While the total annual production cost of the jth product in a disjoint production policy (each product is ordered separately) is $N \times C_j^P Q_j^P$, this cost for the joint policy (all products have a unique ordering cycle) is $\frac{C_j^P Q_j^B}{T}$. Furthermore, based on Equation 1 we have:

$$Q_j^B = \frac{T \times D_j}{E(1 - X_j)} = \frac{T \times D_j}{1 - E(X_j)}. \tag{9}$$

Hence, the expected annual production cost will be:

$$C_P = \sum_{j=1}^{n} \frac{C_j^P \left[\frac{T \times D_j}{1 - E(X_j)}\right]}{T} = \sum_{j=1}^{n} \frac{C_j^P D_j}{1 - E(X_j)}. \tag{10}$$

Holding Cost

The holding cost per unit of the jth product per unit time for both healthy and scrapped items is C_j^h. According to Figure 1, the total holding cost of healthy items per cycle and per year are shown in Equations 11 and 12, respectively:

$$\sum_{j=1}^{n} C_j^h \left[\frac{I_j^1}{2}(t_j^1 + t_j^2)\right], \tag{11}$$

$$N \sum_{j=1}^{n} C_j^h \left[\frac{I_j^1}{2}(t_j^1 + t_j^2)\right]. \tag{12}$$

However, Equation 12 for the joint production policy becomes:

$$\frac{1}{T} \sum_{j=1}^{n} C_j^h \left[\frac{I_j^1}{2}(t_j^1 + t_j^2)\right]. \tag{13}$$

Finally, the expected total annual holding cost of healthy items is (see Appendix A):

$$\sum_{j=1}^{n} C_j^h (P_j - \theta_j) \left[\frac{(P_j - D_j - \theta_j) T \times D_j}{2(P_j)^2 (1 - E(X_j))^2} - \frac{B_j}{P_j(1 - E(X_j))} + \frac{(B_j)^2}{2 D_j T (P_j - D_j - \theta_j)}\right]. \tag{14}$$

Since the scrap for each product is assumed to be held until the end of its production time, based on Figure 1, the total holding costs of the scrapped items per cycle and per year are shown in Equations 15 and 16, respectively:

$$\sum_{j=1}^{n} C_j^h \left[\frac{\theta_j(t_j^1+t_j^4)}{2}(t_j^1+t_j^4)\right], \tag{15}$$

$$N\sum_{j=1}^{n} C_j^h \left[\frac{\theta_j(t_j^1+t_j^4)}{2}(t_j^1+t_j^4)\right]. \tag{16}$$

Again, for the joint production policy, Equation 16 becomes:

$$\frac{1}{T}\sum_{j=1}^{n} C_j^h \left[\frac{\theta_j(t_j^1+t_j^4)}{2}(t_j^1+t_j^4)\right]. \tag{17}$$

Hence, the expected total annual holding cost of scrapped items is (see Appendix B):

$$\sum_{j=1}^{n} C_j^h \theta_j \left[\frac{(D_j)^2\times T}{2(1-E(X_j))^2(P_j)^2}\right]. \tag{18}$$

Finally, the expected total annual holding cost of healthy and scrapped items is:

$$C_H=\sum_{j=1}^{n} C_j^h(P_j-\theta_j)\left[\frac{(P_j-D_j-\theta_j)T\times D_j}{2(P_j)^2(1-E(X_j))^2} -\frac{B_j}{P_j(1-E(X_j))}+\frac{(B_j)^2}{2D_jT(P_j-D_j-\theta_j)}\right] +\sum_{j=1}^{n} C_j^h\left[\frac{(D_j)^2T}{2(1-E(X_j))^2(P_j)^2}\right]. \tag{19}$$

Shortage Cost

The shortage cost per unit of the jth product is C_j^b. Based on Figure 1, the total shortage costs per cycle and per year are shown in Equations 20 and 21, respectively.

$$\sum_{j=1}^{n} C_j^b\left[\frac{B_j}{2}(t_j^3+t_j^4)\right], \tag{20}$$

$$N\times\sum_{j=1}^{n} C_j^b\left[\frac{B_j}{2}(t_j^3+t_j^4)\right]. \tag{21}$$

Because of the joint production policy, Equation 21 becomes:

$$\frac{1}{T}\sum_{j=1}^{n} C_j^b\left[\frac{B_j}{2}(t_j^3+t_j^4)\right]. \tag{22}$$

Finally, the expected total annual shortage cost is (see Appendix C):

$$C_B=\frac{1}{T}\sum_{j=1}^{n} C_j^b\left[\frac{(P_j-\theta_j)(B_j)^2}{2D_j(P_j-D_j-\theta_j)}\right]. \tag{23}$$

Disposal Cost

The disposal cost per unit of the scrapped item of the jth product is C_j^s and the quantity of scrapped items is $E(X_j)Q_j^B$. Hence, the expected total disposal cost per cycle is $\sum_{j=1}^{n} C_j^s E(X_j)Q_j^B$. This quantity per year becomes:

$$N\times\sum_{j=1}^{n} C_j^s E(X_j)Q_j^B. \tag{24}$$

Because of the joint production policy, Equation 24 becomes:

$$\frac{1}{T}\times\sum_{j=1}^{n} C_j^s E(X_j)Q_j^B. \tag{25}$$

Since $Q_j^B=\frac{T\times D_j}{E(1-X_j)}=\frac{T\times D_j}{1-E(X_j)}$, the annual expected total scrapped items cost is:

$$C_S=\frac{1}{T}\times\sum_{j=1}^{n} C_j^s E(X_j)\left[\frac{T\times D_j}{1-E(X_j)}\right] =\sum_{j=1}^{n}\frac{C_j^s E(X_j)D_j}{1-E(X_j)}. \tag{26}$$

Setup Cost

The cost of a setup is A, which occurs N times per year. So, the annual setup cost will be:

$$C_A=N\times A=\frac{A}{T}. \tag{27}$$

As a result, the objective function of the model becomes:

$$Z=C_P+C_H+C_B+C_S+C_A=\sum_{j=1}^{n}\frac{C_j^P D_j}{1-E(X_j)} +\sum_{j=1}^{n} C_j^h(P_j-\theta_j)\times\left[\frac{(P_j-D_j-\theta_j)T\times D_j}{2(P_j)^2(1-E(X_j))^2} -\frac{B_j}{P_j(1-E(X_j))}+\frac{(B_j)^2}{2D_jT(P_j-D_j-\theta_j)}\right] +\sum_{j=1}^{n} C_j^h\left[\frac{(D_j)^2T}{2(1-E(X_j))^2(P_j)^2}\right] +\frac{1}{T}\sum_{j=1}^{n} C_j^b\left[\frac{(P_j-\theta_j)(B_j)^2}{2D_j(P_j-D_j-\theta_j)}\right] +\sum_{j=1}^{n}\frac{C_j^s E(X_j)D_j}{1-E(X_j)}+\frac{A}{T}$$

$$= \sum_{j=1}^{n}\left[\frac{(C_j^b+C_j^h)(P_j-\theta_j)}{2D_j(P_j-D_j-\theta_j)}\right]\frac{(B_j)^2}{T}$$

$$-\sum_{j=1}^{n}\left[\frac{C_j^h(P_j-\theta_j)}{P_j(1-E(X_j))}\right]B_j$$

$$+\sum_{j=1}^{n}C_j^h\left[\frac{D_j[(P_j-\theta_j)(P_j-D_j-\theta_j)+D_j]}{2(P_j)^2(1-E(X_j))^2}\right]T$$

$$+\sum_{j=1}^{n}\left[\frac{(C_j^P+C_j^sE(X_j))D_j}{1-E(X_j)}\right]+\frac{A}{T}. \quad (28)$$

To make sure that all of the n products will be produced by a single machine, we need to have a capacity limitation that is explained in the next subsection.

The Constraint

Since $t_j^1+t_j^4$ and S_j are the production time and setup time of the jth product, respectively, the summation of the total production and setup time (for all products) will be $\sum_{j=1}^{n}(t_j^1+t_j^4)+\sum_{j=1}^{n}S_j$ in which it should be smaller than, or equal to, the period length (T). So, the constraint of the model will be:

$$\sum_{j=1}^{n}(t_j^1+t_j^4)+\sum_{j=1}^{n}S_j\le T. \quad (29)$$

Then, based on the derivation in Appendix D, we will have:

$$T\ge\frac{\sum_{j=1}^{n}S_j}{1-\sum_{j=1}^{n}\frac{D_j}{P_j(1-E(X_j))}}. \quad (30)$$

The Final Model

According to Equations 28 and 30, the final model becomes:

$$\min:\ Z=\sum_{j=1}^{n}\left[\frac{(C_j^b+C_j^h)(P_j-\theta_j)}{2D_j(P_j-D_j-\theta_j)}\right]\frac{(B_j)^2}{T}$$

$$-\sum_{j=1}^{n}\left[\frac{C_j^h(P_j-\theta_j)}{P_j(1-E(X_j))}\right]B_j$$

$$+\sum_{j=1}^{n}\left[\frac{(C_j^P+C_j^sE(X_j))D_j}{1-E(X_j)}\right]$$

$$+\sum_{j=1}^{n}C_j^h\left[\frac{D_j\,[(P_j-\theta_j)(P_j-D_j-\theta_j)+D_j]}{2(P_j)^2(1-E(X_j))^2}\right]T$$

$$+\frac{A}{T},$$

s.t. :

$$T\ge\frac{\sum_{j=1}^{n}S_j}{1-\sum_{j=1}^{n}\frac{D_j}{P_j(1-E(X_j))}},$$

$$B_j\ge 0 \qquad \forall j, j=1,2,\cdots,n. \quad (31)$$

In the next section, a solution method to the model Equation 31 is proposed.

A SOLUTION METHOD

In order to find the optimal solution of model (Relation 31), we first provide proof of the convexity of the objective function. Then, we use the derivative approach to find the optimal point of the objective function. For a feasibility requirement, the production rate of perfect quality items is assumed to be greater than, or equal to, the sum of the demand rate and the production rate of defective items. Mathematically speaking, in a single product model, $P_j-\theta_j-D_j\ge 0$, or $1-E[X_j]-\frac{D_j}{P_j}\ge 0$. In a multi-products model, we need:

$$\sum_{j=1}^{n}\frac{D_j}{P_j(1-E[X_j])}\le 1.$$

To handle the constraint, we will check the optimal solution in Equation 30. If the constraint is not satisfied, then $T_{\min}$, as the minimum value of T, will be considered as the optimal point.

To prove the convexity of the objective function, let us rewrite the objective function as:

$$Z=\sum_{j=1}^{n}\alpha_j\frac{(B_j)^2}{T}-\sum_{j=1}^{n}\beta_jB_j+\sum_{j=1}^{n}\gamma_jT+\sum_{j=1}^{n}\lambda_j+\frac{A}{T}, \quad (32)$$

in which:

$$\alpha_j=\left[\frac{(C_j^b+C_j^h)(P_j-\theta_j)}{2D_j(P_j-D_j-\theta_j)}\right]>0, \quad (33)$$

$$\beta_j=\left[\frac{C_j^h(P_j-\theta_j)}{P_j(1-E(X_j))}\right]>0, \quad (34)$$

$$\gamma_j=\left[\frac{C_j^hD_j\,[(P_j-\theta_j)(P_j-D_j-\theta_j)+D_j]}{2(P_j)^2(1-E(X_j))^2}\right]>0, \quad (35)$$

$$\lambda_j=\left[\frac{(C_j^P+C_j^sE(X_j))D_j}{1-E(X_j)}\right]>0. \quad (36)$$

Theorem 1

The objective function Z in Equation 32 is convex.

Proof

To prove the convexity of Z, one can utilize the Hessian matrix equation as:

$$\mathbf{H}=\begin{bmatrix} \frac{\partial^2 Z}{\partial T^2} & \frac{\partial^2 Z}{\partial T\partial B_1} & \frac{\partial^2 Z}{\partial T\partial B_2} & \cdots & \frac{\partial^2 Z}{\partial T\partial B_n} \\ \frac{\partial^2 Z}{\partial B_1\partial T} & \frac{\partial^2 Z}{\partial B_1^2} & \frac{\partial^2 Z}{\partial B_1\partial B_2} & \cdots & \frac{\partial^2 Z}{\partial B_1\partial B_n} \\ \frac{\partial^2 Z}{\partial B_2\partial T} & \frac{\partial^2 Z}{\partial B_2\partial B_1} & \frac{\partial^2 Z}{\partial B_2^2} & \cdots & \frac{\partial^2 Z}{\partial B_2\partial B_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 Z}{\partial B_n\partial T} & \frac{\partial^2 Z}{\partial B_n\partial B_1} & \frac{\partial^2 Z}{\partial B_n\partial B_2} & \cdots & \frac{\partial^2 Z}{\partial B_n^2} \end{bmatrix}. \quad (37)$$

Then, according to Appendix E, the objective function is strictly convex.

The expected cost function Z is strictly convex for all nonzero T and B_j. Hence, it follows that to find the optimal production period length and the optimal level of backorder B_j, one can partially differentiate Z with respect to T and B_j, and solve the resulted system of equations obtained by equating the partial derivatives with zero. According to what we derived in Appendix F, the system of equations becomes:

$$\frac{\partial Z}{\partial T}=0\rightarrow T=\sqrt{\frac{A}{\sum_{j=1}^{n}\gamma_j-\sum_{j=1}^{n}\left[\frac{(\beta_j)^2}{4\alpha_j}\right]}}, \quad (38a)$$

$$\frac{\partial Z}{\partial B_j}=0\rightarrow B_j^*=\frac{\beta_j T^*}{2\alpha_j}=\frac{\beta_j}{2\alpha_j}T^*. \quad (38b)$$

Then, we will have:

$$Q_j^{B^*}=\frac{D_j T^*}{1-E(X_j)}$$

$$=\frac{D_j}{1-E(X_j)}\sqrt{\frac{A}{\sum_{j=1}^{n}\gamma_j-\sum_{j=1}^{n}\left[\frac{(\beta_j)^2}{4\alpha_j}\right]}}. \quad (39)$$

To handle the constraint, we will check the optimal solution in Equation 30. If the constraint is not satisfied, then $T_{\min}$, as the minimum value of T, will be considered as the optimal point. Based on Appendix D:

$$T_{\min}=\frac{\sum_{j=1}^{n}S_j}{1-\sum_{j=1}^{n}\frac{D_j}{P_j(1-E[X_j])}}. \quad (40)$$

To ensure the possibility and acceptability of production of all products by a machine, the steps involved in the algorithm of finding the optimal and possible values of T^*, B_j^* and $Q_j^{B^*}$ must be performed as follows:

Step 0: If $\sum_{j=1}^{n}\frac{D_j}{P_j(1-E[X_j])}\leq 1$, then go to Step 1. Otherwise, the problem will be infeasible.

Step 1: Calculate T by Equation 38a.

Step 2: Calculate $T_{\min}$ by Equation 40.

Step 3: If $T\geq T_{\min}$, then $T^*=T$, else $T^*=T_{\min}$.

Step 4: Calculate B_j by Equation 38b.

Step 5: Calculate $Q_j^{B^*}$ by Equation 39.

In the next section, two numerical examples are given to illustrate the applications of the proposed method in cases of uniform and normal distribution functions for $f_{X_j}(x_j)$.

NUMERICAL EXAMPLES

Consider a multiproduct inventory control problem with five products, the general and specific data of which are given in Tables 1 and 2, respectively. We consider two numerical examples with uniform and normal probability distributions for X_j. The set up cost is considered $A=\$450$, and Tables 3 and 4 show the best results for the two numerical examples.

SENSITIVITY ANALYSIS

To study the effects of parameter changes on the optimal result derived by the proposed method, a sensitivity analysis is performed by changing (increasing or decreasing) the parameters by 20% and 50%, taking one parameter at a time and keeping the remaining parameters at their original values. This analysis is performed on the two numerical examples given in the previous section. The results of the sensitivity analysis for the uniform and normal distribution cases are shown in Tables 5 and 6, respectively.

A careful study of Table 5 reveals the following:

- T and T^* are moderately sensitive, $T_{\min}$ is insensitive, and Z is slightly sensitive to the changes in the values of parameter A,

Table 1. General data for examples.

Product	D_j	P_j	S_j	C_j^P	C_j^h	C_j^b	C_j^s
1	200	1800	0.001	15	5	10	1
2	300	2500	0.002	12	4	8	0.8
3	400	3000	0.003	10	3	6	0.6
4	500	3500	0.004	8	2	4	0.4
5	600	4500	0.005	6	1	2	0.2

Table 2. Specific data for examples.

Product	$X_j \sim U[a_j, b_j]$				$X_j \sim N[\mu_j, \sigma_j^2]$		
	a_j	b_j	$E[X_j]$	θ_j	$\mu_j = E[X_j]$	σ_j^2	θ_j
1	0	0.1	0.05	90	0. 25	0.01	450
2	0	0.15	0.075	187.5	0.28	0.02	700
3	0	0.2	0.1	300	0.33	0.03	990
4	0	0.25	0.125	437.5	0.38	0.04	1330
5	0	0.3	0.15	675	0.42	0.05	1890

Table 3. The best results for Example 1 (uniform distribution).

Product	Uniform					
	$T_{\min}$	T	T^*	B_j	Q_j^B	Z
1	0.0526	0.5608	0.5608	33.02	118.06	21614
2				48.80	181.88	
3				63.70	249.24	
4				78.21	320.46	
5				94.57	395.86	

Table 4. The best results for Example 2 (normal distribution).

Product	Normal					
	$T_{\min}$	T	T^*	B_j	Q_j^B	Z
1	0.5796	0.5777	0.5796	32.91	154.56	29286
2				48.30	241.50	
3				61.90	346.02	
4				74.34	467.41	
5				89.27	599.57	

Table 5. Effects of parameter changes for the uniform distribution case.

% Changes in Parameters and Their Values		% Changes in			
		$T_{\min}$	T	T^*	Z
A	+50	0	+22.48	+22.48	+1.34
	+20	0	+9.54	+9.54	+0.57
	-20	0	-10.54	-10.54	-0.63
	-50	0	-29.28	-29.28	-1.75
$E[X_j]$	+50	+18.82	-3.93	-3.93	+6.4
	+20	+6.46	-1.5	-1.5	+2.45
	-20	-5.3	+1.52	+1.52	-2.33
	-50	-12.17	3.72	3.72	-5.61
S_j	+50	+50	0	0	0
	+20	+20.15	0	0	0
	-20	-19.96	0	0	0
	-50	-50	0	0	0

Table 6. Effects of parameter changes for the normal distribution case.

% Changes in Parameters and Their Values		% Changes in $T_{\min}$	T	T^*	Z
A	+50	0	+22.47	+22.07	+0.9
	+20	0	+9.53	+9.18	+0.39
	-20	0	-10.56	0	-0.53
	-50	0	-21.25	0	-1.32
$E[X_j]$	+50	Infeasible	-	-	-
	+20	Infeasible	-	-	-
	-20	-78.3	-1.07	-1.4	-9.41
	-50	-88.68	-2.28	-2.61	-20.46
S_j	+50	+50	0	+50	-0.22
	+20	+20	0	+20	-0.17
	-20	-20	0	-0.33	0
	-50	-50	0	-0.33	0

- While T, Z and T^* are slightly sensitive, $T_{\min}$ is moderately sensitive to the changes in the values of parameter $E[X_j]$.
- While $T_{\min}$ is highly sensitive, T, Z and T^* are insensitive to changes in the value of parameter S_j.

Furthermore, a careful study of Table 6 reveals the following:

- T and T^* are moderately sensitive, $T_{\min}$ is insensitive, and Z is slightly sensitive to the changes in the values of parameter A.
- T and T^* are slightly sensitive to the decreasing changes in the values of parameter $E[X_j]$.
  Moreover, $T_{\min}$ and Z are highly and moderately sensitive to the decreasing values of parameter $E[X_j]$, respectively. However, when $E[X_j]$ increases, the problem becomes infeasible.
- While $T_{\min}$ is highly sensitive, T is insensitive to the changes in the value of parameter S_j. T^* is highly sensitive to the changes in the values of parameter S_j when it increases and T^* is slightly sensitive to the changes in the values of S_j when it decreases. Z is insensitive when S_j decreases and is slightly sensitive when S_j increases.

CONCLUSION AND RECOMMENDATIONS FOR FUTURE RESEARCH

In this paper, an EPQ model with a production capacity limitation and a random defective production rate was developed. The aim of this paper was to determine an optimal solution for period lengths, backordered quantities and order quantities of several products, based on minimizing the expected total annual cost including holding, shortage, production, setup and waste item costs. We proved the objective function of the proposed mathematical model to be convex and employed the derivative approach to find the solution. At the end, two numerical examples based on uniform and normal distribution functions for X_j were solved, in order to demonstrate the applicability of the proposed methodology and a sensitivity analysis was performed on the parameter changes.

Some recommendations for future research are:

- Some uncertain variables may be considered for other factors.
- Defective items can be reworked.
- Shortage can occur in a combination of backorders and lost-sales.
- Multiproduct multi-constraint problems can be developed and solved.

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APPENDIX A

Expected Total Holding Cost of the Healthy Items

According to Equation 14:

$$\frac{1}{T}\sum_{j=1}^{n} C_j^h\left[\frac{I_j^1}{2}(t_j^1+t_j^2)\right]$$

$$=\frac{1}{T}\sum_{j=1}^{n} C_j^h\left[\frac{I_j^1}{2}\left(\frac{I_j^1}{P_j-D_j-\theta_j}+\frac{I_j^1}{D_j}\right)\right]$$

$$=\frac{1}{T}\sum_{j=1}^{n} C_j^h\left[\frac{(I_j^1)^2(P_j-\theta_j)}{2D_j(P_j-D_j-\theta_j)}\right]$$

$$=\frac{1}{T}\sum_{j=1}^{n} C_j^h\left[(P_j-D_j-\theta_j)\frac{Q_j^B}{P_j}-B_j\right]^2$$

$$\times\left[\frac{(P_j-\theta_j)}{2D_j(P_j-D_j-\theta_j)}\right]$$

$$=\frac{1}{T}\sum_{j=1}^{n} C_j^h\left[(P_j-D_j-\theta_j)^2\frac{(Q_j^B)^2}{(P_j)^2}\right.$$

$$\left.-2(P_j-D_j-\theta_j)\frac{Q_j^B B_j}{P_j}+(B_j)^2\right]$$

$$\times\left[\frac{(P_j-\theta_j)}{2D_j(P_j-D_j-\theta_j)}\right]$$

$$=\frac{1}{T}\sum_{j=1}^{n} C_j^h(P_j-\theta_j)\left[\frac{(P_j-D_j-\theta_j)(Q_j^B)^2}{2(P_j)^2D_j}\right.$$

$$\left.-\frac{Q_j^B B_j}{P_jD_j}+\frac{(B_j)^2}{2D_j(P_j-D_j-\theta_j)}\right].$$

Knowing that $Q_j^B=\frac{TD_j}{E(1-X_j)}=\frac{TD_j}{1-E(X_j)}$, we have:

$$\frac{1}{T}\sum_{j=1}^{n} C_j^h\left[\frac{I_j^1}{2}(t_j^1+t_j^2)\right]$$

$$=\sum_{j=1}^{n} C_j^h(P_j-\theta_j)\left[\frac{(P_j-D_j-\theta_j)TD_j}{2(P_j)^2(1-E(X_j))^2}\right.$$

$$\left.-\frac{B_j}{P_j(1-E(X_j))}+\frac{(B_j)^2}{2D_jT(P_j-D_j-\theta_j)}\right].$$

APPENDIX B

Expected Total Holding Cost of the Scrapped Items

According to Equation 18:

$$\frac{1}{T}\sum_{j=1}^{n} C_j^h\left[\frac{\theta_j(t_j^1+t_j^4)}{2}(t_j^1+t_j^4)\right]$$

$$=\frac{1}{T}\sum_{j=1}^{n} C_j^h\left[\frac{\theta_j\left[\frac{I_j^1}{P_j-D_j-\theta_j}+\frac{B_j}{P_j-D_j-\theta_j}\right]}{2}\right.$$

$$\left.\times\left[\frac{I_j^1}{P_j-D_j-\theta_j}+\frac{B_j}{P_j-D_j-\theta_j}\right]\right]$$

$$=\frac{1}{T}\sum_{j=1}^{n} C_j^h\left[\frac{\theta_j\left[\frac{I_j^1+B_j}{P_j-D_j-\theta_j}\right]}{2}\left[\frac{I_j^1+B_j}{P_j-D_j-\theta_j}\right]\right]$$

$$= \frac{1}{T}\sum_{j=1}^{n} C_j^h \left[\frac{\theta_j (I_j^1 + B_j)^2}{2(P_j - D_j - \theta_j)^2}\right]$$

$$= \frac{1}{T}\sum_{j=1}^{n} C_j^h \theta_j \left[\frac{(I_j^1)^2 + 2I_j^1 B_j + (B_j)^2}{2(P_j - D_j - \theta_j)^2}\right].$$

Knowing that $I_j^1 = (P_j - D_j - \theta_j)\frac{Q_j^B}{P_j} - B_j$ and $Q_j^b = \frac{TD_j}{E(1-X_j)} = \frac{TD_j}{1-E(X_j)}$, we have:

$$\frac{1}{T}\sum_{j=1}^{n} C_j^h \left[\frac{\theta_j (t_j^1 + t_j^4)}{2}(t_j^1 + t_j^4)\right]$$

$$= \frac{1}{T}\sum_{j=1}^{n} \frac{C_j^h \theta_j}{2(P_j - D_j - \theta_j)^2}\left[\left[(P_j - D_j - \theta_j)\frac{Q_j^B}{P_j} - B_j\right]^2\right.$$

$$\left. + 2\left[(P_j - D_j - \theta_j)\frac{Q_j^B}{P_j} - B_j\right] B_j + (B_j)^2\right]$$

$$= \frac{1}{T}\sum_{j=1}^{n} \frac{C_j^h \theta_j}{2(P_j - D_j - \theta_j)^2}\left[\left[(P_j - D_j - \theta_j)^2 \frac{(Q_j^B)^2}{(P_j)^2}\right.\right.$$

$$\left. -2(P_j - D_j - \theta_j)\frac{B_j Q_j^B}{P_j} + (B_j)^2\right] + (B_j)^2$$

$$\left. + 2\left[(P_j - D_j - \theta_j)\frac{Q_j^B}{P_j} - B_j\right] B_j\right]$$

$$= \frac{1}{T}\sum_{j=1}^{n} C_j^h \theta_j \left[\frac{(Q_j^B)^2}{2(P_j)^2}\right]$$

$$= \frac{1}{T}\sum_{j=1}^{n} C_j^h \theta_j \left[\frac{(\frac{TD_j}{1-E(X_j)})^2}{2(P_j)^2}\right]$$

$$= \sum_{j=1}^{n} C_j^h \theta_j \left[\frac{(D_j)^2 T}{2(1 - E(X_j))^2 (P_j)^2}\right].$$

APPENDIX C

Expected Total Shortage Cost

According to Equation 23:

$$\frac{1}{T}\sum_{j=1}^{n} C_j^b \left[\frac{B_j}{2}(t_j^3 + t_j^4)\right]$$

$$= \frac{1}{T}\sum_{j=1}^{n} C_j^b \left[\frac{B_j}{2}\left(\frac{B_j}{D_j} + \frac{B_j}{P_j - D_j - \theta_j}\right)\right]$$

$$= \frac{1}{T}\sum_{j=1}^{n} C_j^b \left[\frac{(P_j - \theta_j)(B_j)^2}{2D_j(P_j - D_j - \theta_j)}\right].$$

APPENDIX D

Possibility and Acceptability of T

The summation of the total production and setup time (for all products) should be smaller than, or equal to, the period length (T). So:

$$\sum_{j=1}^{n} t_j^1 + t_j^4 + \sum_{j=1}^{n} S_j \le T.$$

Moreover, based on Equation 7, we have:

$$\sum_{j=1}^{n} \frac{Q_j^B}{P_j} + \sum_{j=1}^{n} S_j \le T$$

$$\Rightarrow \sum_{j=1}^{n} \frac{\frac{TD_j}{1-E(X_j)}}{P_j} + \sum_{j=1}^{n} S_j \le T$$

$$\Rightarrow \sum_{j=1}^{n} S_j \le T - \sum_{j=1}^{n} \frac{D_j}{P_j(1 - E(X_j))} T$$

$$\Rightarrow T \ge \frac{\sum_{j=1}^{n} S_j}{1 - \sum_{j=1}^{n} \frac{D_j}{P_j(1-E(X_j))}} = T_{\min}.$$

APPENDIX E

The Convexity Proof of Z

Let $\mathbf{a^T} = [T, B_1, B_2, \cdots, B_n]$ be an arbitrary vector and $\mathbf{H}$ the hessian. Then, in order to prove that Z is convex, we need to show the quadratic form $\mathbf{a^T H a} \ge 0$. Or:

$$\mathbf{a^T H a} = [T, B_1, B_2, \cdots, B_n]$$

$$\times \begin{bmatrix} \frac{\partial^2 Z}{\partial^2 T} & \frac{\partial^2 Z}{\partial T \partial B_1} & \frac{\partial^2 Z}{\partial T \partial B_2} & \cdots & \frac{\partial^2 Z}{\partial T \partial B_n} \\ \frac{\partial^2 Z}{\partial B_1 \partial T} & \frac{\partial^2 Z}{\partial^2 B_1} & \frac{\partial^2 Z}{\partial B_1 \partial B_2} & \cdots & \frac{\partial^2 Z}{\partial B_1 \partial B_n} \\ \frac{\partial^2 Z}{\partial B_2 \partial T} & \frac{\partial^2 Z}{\partial B_2 \partial B_1} & \frac{\partial^2 Z}{\partial^2 B_2} & \cdots & \frac{\partial^2 Z}{\partial B_2 \partial B_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 Z}{\partial B_n \partial T} & \frac{\partial^2 Z}{\partial B_n \partial B_1} & \frac{\partial^2 Z}{\partial B_n \partial B_2} & \cdots & \frac{\partial^2 Z}{\partial^2 B_n} \end{bmatrix}$$

$$\times \begin{bmatrix} T \\ B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}.$$

But:

$$\frac{\partial Z}{\partial T} = \frac{-\sum_{j=1}^{n} \alpha_j B_j^2 - A}{T^2} + \sum_{j=1}^{n} \gamma_j$$

$$\Rightarrow \frac{\partial^2 Z}{\partial T^2} = \frac{2\sum_{j=1}^{n} \alpha_j B_j^2 + 2A}{T^3},$$

and:

$$\frac{\partial Z}{\partial B_j} = \frac{2\alpha_j B_j}{T} - \beta_j \Rightarrow \frac{\partial^2 Z}{\partial B_j^2} = \frac{2\alpha_j}{T};$$

$$j = 1, 2, ..., n,$$

$$\Rightarrow \frac{\partial^2 Z}{\partial B_j \partial T} = \frac{\partial^2 Z}{\partial T \partial B_j} = \frac{-2\alpha_j B_j}{T}.$$

Hence:

$$[T, B_1, B_2, \cdots, B_n]$$

$$\times \begin{bmatrix} \frac{2\sum_{j=1}^{n}\alpha_j B_j^2 + 2A}{T^3} & -\frac{2\alpha_1 B_1}{T^2} & -\frac{2\alpha_2 B_2}{T^2} & \cdots & -\frac{2\alpha_n B_n}{T^2} \\ -\frac{2\alpha_1 B_1}{T^2} & \frac{2\alpha_1}{T} & 0 & \cdots & 0 \\ -\frac{2\alpha_2 B_2}{T^2} & 0 & \frac{2\alpha_2}{T} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{2\alpha_n B_n}{T^2} & 0 & 0 & \cdots & \frac{2\alpha_n}{T} \end{bmatrix}$$

$$\times \begin{bmatrix} T \\ B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} = \left[\frac{2A}{T^2}, 0, 0, \cdots, 0\right] \begin{bmatrix} T \\ B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} = \frac{2A}{T} \geq 0.$$

APPENDIX F

Finding Roots of Derivative of Z, with Respect to B_j and T

$$\frac{\partial Z}{\partial T} = \frac{\partial \left[\sum_{j=1}^{n} \alpha_j \frac{(B_j)^2}{T} - \sum_{j=1}^{n} \beta_j B_j + \sum_{j=1}^{n} \gamma_j T + \sum_{j=1}^{n} \lambda_j + \frac{A}{T}\right]}{\partial T}$$

$$= \frac{-\sum_{j=1}^{n} \alpha_j (B_j)^2 - A}{T^2} + \sum_{j=1}^{n} \gamma_j = 0$$

$$\rightarrow T^2 = \frac{\sum_{j=1}^{n} \alpha_j (B_j)^2 + A}{\sum_{j=1}^{n} \gamma_j},$$

$$\frac{\partial Z}{\partial B_j} = \frac{\partial \begin{bmatrix} \sum_{j=1}^{n} \alpha_j \frac{(B_j)^2}{T} - \sum_{j=1}^{n} \beta_j B_j \\ + \sum_{j=1}^{n} \gamma_j T + \sum_{j=1}^{n} \lambda_j + \frac{A}{T} \end{bmatrix}}{\partial B_j}$$

$$= \frac{2\alpha_j B_j}{T} - \beta_j = 0 \rightarrow B_j^* = \frac{\beta_j}{2\alpha_j} T,$$

$$T^2 = \frac{\sum_{j=1}^{n} \alpha_j \left[\frac{(\beta_j)^2}{(2\alpha_j)^2}\right] T^2 + A}{\sum_{j=1}^{n} \gamma_j}$$

$$\rightarrow \sum_{j=1}^{n} \gamma_j T^2 = \sum_{j=1}^{n} \left[\frac{(\beta_j)^2}{4\alpha_j}\right] T^2 + A$$

$$\rightarrow T^2 = \frac{A}{\sum_{j=1}^{n} \gamma_j - \sum_{j=1}^{n} \left[\frac{(\beta_j)^2}{4\alpha_j}\right]},$$

$$\Rightarrow T = \sqrt{\frac{A}{\sum_{j=1}^{n} \gamma_j - \sum_{j=1}^{n} \left[\frac{(\beta_j)^2}{4\alpha_j}\right]}}.$$

BIOGRAPHIES

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