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Research note

MHD flow of a Casson fluid over an exponentially shrinking sheet

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Abstract The magnetohydrodynamic (MHD) boundary layer flow of a Casson fluid over an exponentially permeable shrinking sheet has been investigated. The analytical solution arising differential system has been computed by the Adomian Decomposition Method (ADM). Variations of interesting parameters on the velocity are observed by plotting graphs.

1. Introduction

The boundary layer flow over a shrinking surface is encountered in several technological processes. Such situations occur in polymer processing, manufacturing of glass sheets, paper production, in textile industries and many others. Crane [1] initiated a study on the boundary layer flow of a viscous fluid towards a linear stretching sheet. An exact similarity solution for the dimensionless differential system was obtained. Carragher and Carane [2] discussed heat transfer on a continuous stretching sheet. Afterwards, many investigations were made to examine flow over a stretching/shrinking sheet under different aspects of MHD, suction/injection, heat and mass transfer etc. [3–10]. In these attempts, the boundary layer flow, due to stretching/shrinking, has been analyzed. Magyari and Keller [11] provided both analytical and numerical solutions for boundary layer flow over an exponentially stretching surface with an exponential temperature distribution. The combined effects of viscous dissipation and mixed convection on the flow of a viscous fluid over an exponentially stretching sheet were analyzed by Partha et al. [12]. Elbashbeshy [13] numerically studied flow and heat transfer over an exponentially stretching surface with wall mass suction. Sajid and Hayat [14] provided an analytical solution for boundary layer flow of a Jeffrey fluid over an exponentially stretching sheet. The numerical solution of the same problem was then given by Bidin and Nazar [15]. The effects of radiation on the MHD boundary layer flow of a viscous fluid over an exponentially stretching sheet were studied by Ishak [16].

To the best of our knowledge, no information is yet available for boundary layer flow induced by an exponentially shrinking sheet. The present work deals with the MHD flow of Casson fluid [17,18] induced by an exponentially shrinking sheet. The Adomian Decomposition Method (ADM) has been employed to obtain the analytical solution. This method has been successfully applied to various interesting problems [19–25]. The Padé approximation [26] is used to handle the boundary condition at infinity and to find the better convergence.

2. Mathematical model

Let us consider the two-dimensional flow of an incompressible Casson fluid over an exponentially shrinking sheet. The fluid is electrically conducting in the presence of a uniform magnetic field applied normal to the sheet, and the induced magnetic field is neglected under the approximation of small Reynolds number. We also assume the rheological equation of

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Casson fluid, reported by Mustafa et al. [19], is:

$$\tau^{1/n} = \tau_o^{1/n} + \mu\dot{\gamma}^{1/n}. \tag{1}$$

Or:

$$\tau_{ij} = \left[\mu_B + \left(\frac{P_y}{\sqrt{2\pi}}\right)^{1/n}\right]^n 2e_{ij}, \tag{2}$$

where μ is the dynamic viscosity, μ_B is plastic dynamic viscosity of the non-Newtonian fluid, P_y is the yield stress of fluid, π is the product of the component of deformation rate with itself, namely, $\pi = e_{ij}e_{ij}$, and e_{ij} is the (i, j)th component of the deformation rate. An anonymous referee has suggested considering the value of $n = 1$. However, in many application this value is $n \gg 1$.

The problem is governed by the following equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(1 + \frac{1}{\gamma}\right)\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho}u. \tag{4}$$

The boundary conditions are defined as:

$$u = U_w(x) = -U_0 \exp\left(\frac{x}{l}\right) \text{ and } v = V_w(x) = V_0 \exp\left(\frac{x}{2l}\right) \text{ at } y = 0,$$

$$u = 0 \quad \text{as} \quad y \to \infty. \tag{5}$$

In the above expression U_w is shrinking velocity with U_0 (shrinking constant), V_w is mass transfer velocity with ($V_0 > 0$ for mass injection and $V_0 < 0$ for mass suction), u and v are components of velocity in x and y-directions, respectively, and l is the characteristic length. It is assumed that the magnetic field $B(x)$ is of the form

$$B = B_0 \exp\left(\frac{x}{l}\right)$$

where B_0 is the constant magnetic field. We introduce stream function Ψ as:

$$u = \frac{\partial\Psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial\Psi}{\partial x}, \tag{6}$$

and define variables:

$$\Psi = \sqrt{2\nu l U_0}\, x f(\eta) \exp\left(\frac{x}{2l}\right),$$

$$\text{and} \quad \eta = y\sqrt{\frac{U_0}{2\nu l}} \exp\left(\frac{x}{2l}\right). \tag{7}$$

"With the help of the above transformation", equation of continuity (1) is identically satisfied and momentum Eq. (2) takes the form:

$$\left(1 + \frac{1}{\gamma}\right)f''' - M^2 f' + ff'' - 2(f')^2 = 0. \tag{8}$$

The corresponding boundary conditions are:

$$f = s, \qquad f' = -1 \quad \text{at } \eta = 0, \tag{9}$$

$$f' \to 0 \quad \text{as } \eta \to \infty, \tag{10}$$

where $M^2 = \frac{2\sigma B^2 l}{\rho U_0}$ is Hartmann number, and γ is Casson fluid parameter.

3. Solution by Adomian Decomposition Method (ADM)

To solve the above dimensionless equation with given boundary conditions from Eqs. (6) to (8) with the help of ADM, we first write Eq. (6) in operator form as:

$$f''' = \left(\frac{\gamma}{1+\gamma}\right)(M^2 f' + 2f'^2 - ff''), \tag{11}$$

$$Lf = \left(\frac{\gamma}{1+\gamma}\right)(M^2 f' + 2f'^2 - ff''), \tag{12}$$

where $L = \frac{d^3}{d\eta^3}$. Applying the inverse operator is defined as:

$$L^{-1}(*) = \int_0^\eta \int_0^\eta \int_0^\eta (*)\,dtdtdt. \tag{13}$$

Applying L^{-1}'' on both sides of Eq. (12), we obtain:

$$L^{-1}(Lf) = f = \left(\frac{\gamma}{1+\gamma}\right)L^{-1}(M^2 f' + 2f'^2 - ff''), \tag{14}$$

or:

$$= \left(\frac{\gamma}{1+\gamma}\right)\int_0^\eta \int_0^\eta \int_0^\eta (M^2 f' + 2f'^2 - ff'')\,dtdtdt. \tag{15}$$

With the help of boundary conditions (9) and (11), we can write Eq. (13) as:

$$f = s - \eta + \frac{\alpha}{2}\eta^2 + \left(\frac{\gamma}{1+\gamma}\right)\{(M^2)L^{-1}(f') + 2L^{-1}(f'^2) - L^{-1}(ff'')\}, \tag{16}$$

where $\alpha = f''(0)$ is to be determined, s is suction/injection at the wall. In ADM, the nonlinear terms in Eq. (16) can be decomposed as:

$$f'^2 = \sum_{k=0}^{\infty} A_k, \qquad ff'' = \sum_{k=0}^{\infty} B_k. \tag{17}$$

Adopting the algorithm for the Adomian polynomials proposed by Adomian [23], we defined:

$$A_i = \sum_{k=0}^{i} f_k' f_{i-k}' \qquad B_i = \sum_{k=0}^{i} f_k f_{i-k}'' \quad \forall i = 0 \cdots n. \tag{18}$$

Substituting Eq. (17) into Eq. (16) yields:

$$f = s - \eta + \frac{\alpha}{2}\eta^2 + \left(\frac{\gamma}{1+\gamma}\right) \times \left\{(M^2)L^{-1}(f') + 2L^{-1}\sum_{k=0}^{\infty} A_k - L^{-1}\sum_{k=0}^{\infty} B_k\right\}. \tag{19}$$

Hence, adopting the modified technique, we have a simple recursive Adomian Algorithm for generating the individual terms of the series solution of Eqs. (8)–(10):

$$f_0 = s - \eta, \tag{20}$$

$$f_1 = \frac{\alpha}{2}\eta^2 + \left(\frac{\gamma}{1+\gamma}\right)\{(M^2)L^{-1}(f_o') + 2L^{-1}A_o - L^{-1}B_o\}, \tag{21}$$

$$f_{k+1} = \left(\frac{\gamma}{1+\gamma}\right)\{(M^2)L^{-1}(f_k') + 2L^{-1}A_k - L^{-1}B_k\} \quad \forall k = 1 \cdots n. \tag{22}$$

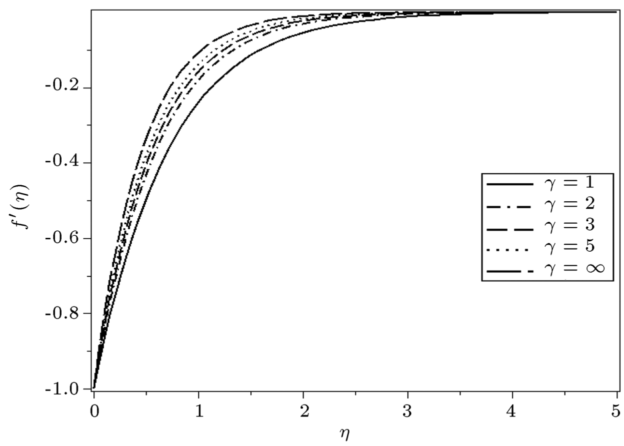

Figure 1: Shows the variation of velocity and boundary layer thickness for various values of Casson fluid parameter γ for $s = 1$ and $M = 2$.

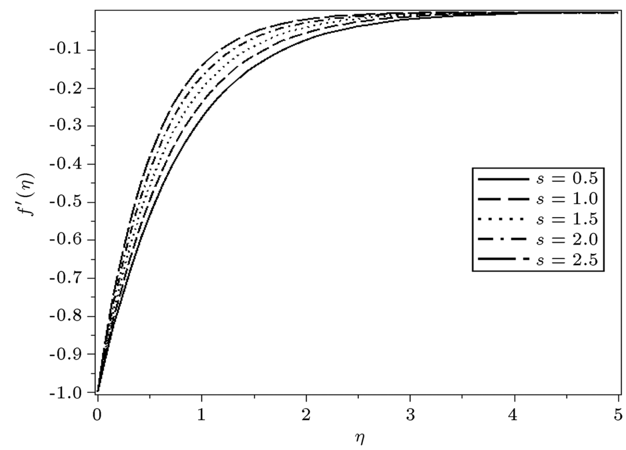

Figure 2: Shows the variation of velocity and boundary layer thickness for various values of shrinking parameter s for $\gamma = 1$ and $M = 2$.

For practical numerical computations, we shall use the finite j-term approximation of $f(\eta)$:

$$\phi_j(\eta) = \sum_{i=0}^{j-1} f_i. \tag{23}$$

The algorithm (20)–(22) is coded in the computer algebra package, Maple, and we employ Maple's built-in Padé approximants procedure. To achieve reasonable accuracy, we obtain the 41-term approximation of $f(\eta)$, i.e. $\phi_{41}(\eta) = \sum_{i=0}^{40} f_i$, where the first four terms are given as follows:

$$f_o = s - \eta, \tag{24}$$

$$f_1 = \frac{1}{2}\alpha\eta^2 + \frac{1}{6}\frac{\gamma(-M^2+2)}{1+\gamma}\eta^3, \tag{25}$$

$$\begin{aligned} f_2 = & -\frac{1}{6}\frac{\gamma s\alpha\eta^3}{1+\gamma} + \left(-\frac{1}{12}\frac{\gamma^2 s}{(1+\gamma)^2} - \frac{1}{8}\frac{\gamma\alpha}{1+\gamma} + \frac{1}{24}\frac{\gamma M^2\alpha}{1+\gamma} \right. \\ & \left. + \frac{1}{24}\frac{\gamma^2 s M^2}{(1+\gamma)^2}\right)\eta^4 + \left(\frac{1}{30}\frac{\gamma^2M^2}{(1+\gamma)^2} - \frac{1}{30}\frac{\gamma^2}{(1+\gamma)^2} \right. \\ & \left. - \frac{1}{120}\frac{\gamma^2M^4}{(1+\gamma)^2}\right)\eta^5, \end{aligned} \tag{26}$$

$$\begin{aligned} f_3 = & \frac{1}{24}\frac{s^2\gamma^2\alpha\eta^4}{(1+\gamma)^2} + \left(\frac{1}{60}\frac{\gamma^3 s^2}{(1+\gamma)^3} - \frac{1}{120}\frac{\gamma^3 s^2 M^2}{(1+\gamma)^3} \right. \\ & \left. + \frac{1}{40}\frac{\gamma\alpha^2}{1+\gamma} + \frac{1}{24}\frac{\gamma^2 s\alpha}{(1+\gamma)^2} - \frac{1}{60}\frac{\gamma^2 M^2 s\alpha}{(1+\gamma)^2}\right)\eta^5 \\ & + \left(\frac{19}{720}\frac{\gamma\alpha^2}{(1+\gamma)^2} - \frac{7}{20}\frac{\gamma^3 s M^2}{(1+\gamma)^3} + \frac{1}{120}\frac{\gamma^3 s}{(1+\gamma)^3} \right. \\ & \left. + \frac{1}{360}\frac{\gamma^3 s M^4}{(1+\gamma)^3} - \frac{1}{60}\frac{\gamma^2 M^2\alpha}{(1+\gamma)^2} + \frac{1}{720}\frac{\gamma M^4\alpha}{(1+\gamma)^2}\right)\eta^6 \\ & + \left(-\frac{1}{5040}\frac{\gamma^3M^4}{(1+\gamma)^3} + \frac{1}{420}\frac{\gamma^3M^2}{(1+\gamma)^3} \right. \\ & \left. - \frac{1}{140}\frac{\gamma^3M^2}{(1+\gamma)^3} + \frac{2}{315}\frac{\gamma^3}{(1+\gamma)^3}\right)\eta^7. \end{aligned} \tag{27}$$

4. Results and discussion

In this section, we discuss the different physical parameters, such as fluid parameter γ, Hartmann number, M, and suction injection parameter s. Here, we use the diagonal Padè approximant for the valid convergent. Figure 1 shows that the influence of fluid parameter γ. It shows that the magnitude of velocity and boundary layer thickness decreases with an increase in fluid parameter, γ. It is noticed that when the fluid parameter approaches infinity, the problem in the given case reduces to a Newtonian case. Figures 2 and 3 shows that the boundary layer thickens and the magnitude of the velocity decreases with an increase in Hartmann number, M, and suction injection parameter, s. Tables 1–3 gives the numerical values of $\alpha = f''(0)$ for different order of approximations (Pade).

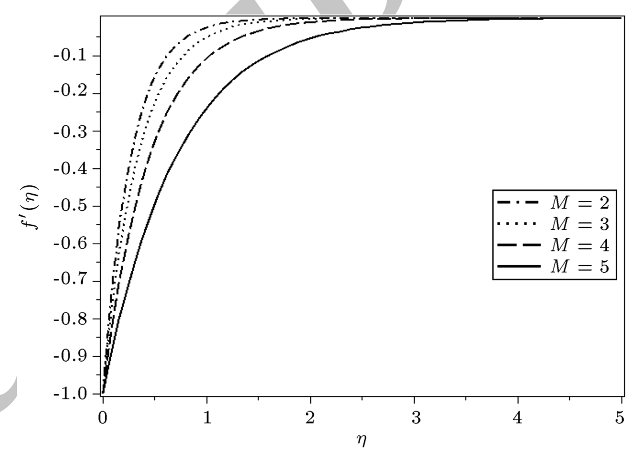

Figure 3: Shows the variation of velocity and boundary layer thickness for various values of Hartmann number M for $s = 1$ and $\gamma = 3$.

Table 1: Numerical values of $\alpha = f''(0)$ for $s = 1,\ M = 2$ and $\gamma = 1$.

Pade	α
[5/5]	1.36569
[10/10]	1.36668
[15/15]	1.36668
[20/20]	1.36668
[25/25]	1.36668

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Table 2: Numerical values of $\alpha = f''(0)$ for $s = 0.5$, $M = 2$ and $\gamma = 1$.

Pade	α
[5/5]	1.215482
[10/10]	1.215503
[15/15]	1.215503
[20/20]	1.215503
[25/25]	1.215503

Table 3: Numerical values of $\alpha = f''(0)$ for $s = 1$, $M = 3$ and $\gamma = 1$.

Pade	α
[5/5]	2.185549
[10/10]	2.184183
[15/15]	2.184183
[20/20]	2.184183
[25/25]	2.184183

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