Finite size correction on the surface width of random deposition with surface relaxation model

P. Azimi Anaraki *
Physics Department, Takestan Branch, Islamic Azad University, Qazvin, Iran

Abstract

In this research the Kardar-Parizi- Zhang equation for surface growth has been analyzed in the regime where the nonlinear coupling constant, λ, is small (Edwards-Wilkinson equation) and also by using Fourier Transformations the finite size correction on the mean-square width has been calculated. It has found that the calculated interface width for $\tilde{t}\to 0$ behaves as $W(L,t) \sim t^{\beta} + \frac{t^{\frac{2+d}{2}}}{L^{2d}}$ and for $\tilde{t}\to\infty$ behaves as $W(L,t) \sim L^{\alpha} - \frac{L^{\frac{6-d}{2}}}{t}$.

Keywords: Surface width, Random deposition, Surface relaxation, Kardar-Parizi-Zhang, Edwards- Wilkinson, Langevin

1 Introduction

Recently the scientists have become more interesting in studying surface and interface growth. They would like to investigate the dynamics of interface growth by introducing different mathematical equations and solving them with analytical methods to obtain the surfaces and interfaces parameters. Some examples of these interfaces are: liquid flow in a tissue that is suspended in water (1) , snowing on the ground surface (2) and surface growth in thin film technology (3,4). The surface width in one-dimensional substrate with length equal to L is calculated from the following equation (5):

$$W(L,t) = \sqrt{\frac{1}{L}\sum_{i=1}^{L}[h(i,t)-\bar{h}(t)]^2} \qquad (1)$$

Where $h(i,t)$ is the height of one site of the substrate and $\bar{h}$ (t) is mean height of the surface. For very short times the surface width behaves like (5):

$$W(L,t) = t^{\beta} \ [t << t_x] \qquad (2)$$

Where β is growth exponent, and t_x is cross overtime. Also, for very long time the surface width equation is (5):

$$W_{sat}(L) = L^{\alpha} \ [t >> t_x] \qquad (3)$$

* Corresponding author

In which, α is a roughening exponent and L is length of substrate. The surface width for times around the crossover time is equal to [6]:

$$W(L,t) \sim L^{\alpha} f(\frac{t}{L^{z}}) \quad (4)$$

$$z = \frac{\alpha}{\beta} \quad (5)$$

In which, z is a dynamic exponent. In order to hydrodynamical description of surfaces and interfaces roughening motion, a continuum Langevin equation proposed by Kardar, Parizi and Zhang (KPZ)[7]:

$$\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t) \quad (6)$$

Where, h is the height of one site of interface, v is the surface tension, λ is the velocity and η is the random function with Gaussian distribution which has the following mean value and correlation:

$$\langle \eta(x,t) \rangle = 0 \quad (7)$$

$$\langle \eta(x,t)\eta(x',t') \rangle = 2D\delta^{d}(x-x')\delta(t-t') \quad (8)$$

2 The liner Langevin equation

Considering the nonlinear coupling constant, λ, in equation (6) very small, [8] the continuum Langevin equation will be:

$$\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h + \eta(x,t) \quad [EW] \quad (9)$$

The noise, η, follows (7) and (8) equations. Edwards and Wilkinson derived this equation from a lattice model of sedimentation in the continuum limit for the first time [9]. This equation also describes the deposition dynamics of an equilibrium surface in the capillary regime by the evaporation method [10] that is described by Random deposition with surface relaxation model. While the characterizing exponents, α, β and z of this equation are well known by a dynamic scaling method as [5]:

$$\alpha = \frac{2-d}{2}, \quad z = 2 \quad , \quad \beta = \frac{2-d}{4} \quad (10)$$

It is sometimes desirable to determine the amplitude of various quantities in the scaling regime. These amplitudes are related to each other through the parameters ν and D of the continuum model. Consider a growth process initiated under periodic boundary conditions at $t \to -\infty$. Because of linear characteristics of equation (9), it can be solved by the Fourier transformation:

$$h(k,w) = \frac{\eta(k,w)}{\nu k^2 - iw} \quad (11)$$

Where $\eta(k,w)$ is the Fourier transformation of $\eta(x,t)$ (see Appendix A). Now to obtain the correlation function $\langle h(k,w)h(k',w') \rangle$, first the correlation relation of $\langle \eta(k,w)\eta(k',w') \rangle$ shall be obtained:

$$\langle \eta(k,w)\eta(k',w') \rangle = 2D(2\pi)^{d}\,\delta^{d}(k+k')\int_{0}^{\infty} d\tau\, e e^{i(w+w')\tau} \quad (12)$$

Then, the correlation relation of $\langle h(k,w)h(k',w') \rangle$ can be found:

$$\langle h(k,w)h(k',w') \rangle = \frac{1}{(\nu k^2 - iw)(\nu k'^2 - i'w)} 2D(2\pi)^{d}\,\delta^{d}(k+k')\int_{0}^{\infty} e^{i(w+w')\tau}\, d\tau \quad (13)$$

With reverse Fourier transformation on w:

$$\langle h(k,t)h(k',t)\rangle = 2D(2\pi)^d \delta^d(k+k')\int_0^\infty d\tau\left(\int \frac{e^{-iw(t-\tau)}}{(\nu k^2 - iw)}dw\right)^2 \quad (14)$$

By applying the Koushi integral theorem and residue calculation on equation (14) and performing the integral, it can be seen that previous equation will be resulted to:

$$\langle h(k,t)h(k',t)\rangle = \frac{D}{\nu k^2}(2\pi)^d \delta^d(k+k')(1-e^{-2\nu k^2 t}) \quad (15)$$

Let us first consider the mean-square width of a surface of liner size L:

$$W^2(L,t) = \left\langle \frac{1}{N}\sum_x \left(h(x,t) - \frac{1}{N}\sum_{x'} h(x',t)\right)^2 \right\rangle \quad (16)$$

It is equal to the performance of inverse Fourier transformation on k in equation (15):

$$W^2(L,t) = \int \frac{d^d k}{(2\pi)^d}\frac{d^d k'}{(2\pi)^d}\langle h(k,t)h(k',t)\rangle e^{ikx}e^{ik'x} \quad (17)$$

That N is the total number of substrate sites and the integral will be limited to $\frac{2\pi}{L} < k$ and $\frac{\pi}{a} > k'$ in required to the momentum space. By performing inverse Fourier transformation on k in equation (15):

$$\langle h(k,t)h(k',t)\rangle = W^2(L,t) = \int \frac{d^d k}{k^2}\frac{D}{\nu}(1-e^{-2\nu k^2 t})\int \frac{d^d k'}{(2\pi)^d}\delta^d(k+k')e^{ikx}e^{ik'x} \quad (18)$$

For performing integral (18) momentum integration shall be performed over a spherical shell. For this purpose, $\int d^d x, \delta^d(k+k')$ and $B(m+1,n+1)$ must be known. After performing integral (18):

$$W^2(L,t) = k_d \int_{\frac{2\pi}{L}}^{\frac{\pi}{a}} \frac{dk}{k^{3-d}}\frac{D}{\nu}(1-e^{-2\nu k^2 t}) \quad (19)$$

Where

$$k_d = \frac{1}{2^{d-1}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})} \quad (20)$$

Integrating by parts the right hand side of equation (19) and applying change of variable $y = 2\nu k^2 t$:

$$W^2(L,t) = \frac{k_d}{d-2}\frac{D}{\nu}(\frac{\pi}{a})^{d-2} + \frac{k_d}{(2-d)(2\pi)^{2-d}}\frac{D}{\nu}L^{2-d} - \frac{k_d}{d-2}\frac{D}{\nu}(\frac{\pi}{a})^{d-2}e^{\frac{-2\pi^2\nu t}{a^2}} - \frac{k_d}{(2-d)(2\pi)^{2-d}}\frac{D}{\nu}L^{2-d}e^{\frac{-8\pi^2\nu t}{L^2}}$$

$$-\frac{k_d}{(2-d)(2\pi)^{2-d}}\frac{D}{\nu}L^{2-d}\left(\frac{8\pi^2\nu t}{L^2}\right)^{1-\frac{d}{2}}\int_{\frac{8\pi^2\nu t}{L^2}}^{\infty} y^{\frac{d}{2}-1}e^{-y}dy \quad (21)$$

In this equation, because the lattice spacing is very small, the upper limit of the integral will be ∞. Finally the mean-square interface width will be:

$$W^2(L,t) = A + \frac{D}{\nu}L^{2-d}f_d(\frac{\nu t}{L^2}) + O\left(e^{\frac{-2\pi^2\nu t}{a^2}}\right) \quad (22)$$

$$A = \frac{k_d}{d-2}\frac{D}{v}(\frac{\pi}{a})^{d-2} \qquad (23)$$

$$O\left(e^{-\frac{2\pi^2 vt}{a^2}}\right) = \frac{k_d}{d-2}\frac{D}{v}\left(\frac{\pi}{a}\right)^{d-2} e^{\frac{-2\pi^2 vt}{a^2}} \qquad (24)$$

$$f_d\left(\frac{vt}{L^2}\right) = \frac{k_d}{(2-d)(2\pi)^{2-d}}\left[1-e^{-8\pi^2\frac{vt}{L^2}} + \left(\frac{8\pi^2 vt}{L^2}\right)^{1-\frac{d}{2}} \int_{\frac{8\pi^2 vt}{L^2}}^{\infty} y^{\frac{d}{2}-1} e^{-y} dy\right] \quad (25)$$

$$f_d(x) = \frac{k_d}{(2-d)(2\pi)^{2-d}}\left[1-e^{-8\pi^2 x} + (8\pi^2 x)^{1-\frac{d}{2}} \int_{8\pi^2 x}^{\infty} y^{\frac{d}{2}-1} e^{-v} dy\right] \quad (26)$$

$$x = \frac{vt}{L^2}$$

By comparing equations (4) and (22) the critical exponents of EW equation can be obtained:

$$z = 2, \quad \alpha = \frac{2-d}{2}, \qquad \beta = \frac{2-d}{4} \qquad (27)$$

3 Result and Discussion

Now, it is desirable to understand the shape of EW equation interface width for $t \to 0 (x \to 0)$. Therefore, the limit of $f_d(x)$ and following that, the EW interface width shall be obtained for $x \to 0$ .For this purpose, the terms included e^{-z} will be expanded and the terms included in the first and more orders, *O(x)*, will be omitted, also the integral existing in $f_d(x)$ will move to $\Gamma(\frac{d}{2})$, and finally $f_d(x)$ will be as follows:

$$f_d(x) = \frac{1}{2(2-d)\pi^{2-\frac{d}{2}}\Gamma(\frac{d}{2})}\left[1-1+O(x) + (8\pi^2 x)^{1-\frac{d}{2}} \int_{8\pi^2 x \approx 0}^{\infty} y^{\frac{d}{2}-1} e^{-y} dy\right]$$

$$[x << 1] \qquad (28)$$

$$f_d(x) = \frac{2^{\frac{4-3d}{2}} \pi^{\frac{-d}{2}}}{(2-d)} x^{\frac{2-d}{2}} \qquad [x << 1] \qquad (29)$$

$$M = \frac{2^{\frac{4-3d}{2}} \pi^{\frac{-d}{2}}}{(2-d)} \qquad (30)$$

And for interface width:

$$w^2(L,t) = A + \frac{D}{v} L^{2-d} M x^{\frac{2-d}{2}} + O\left(e^{\frac{-2\pi^2 vt}{a^2}}\right) \quad [x << 1]$$

$$W(L,t) \sim t^{\frac{2-d}{4}} = t^{\beta} \qquad [x << 1] \qquad (31)$$

It can be seen that equation (31) is exactly the same as equation (2) that has been obtained from experimental results. Now, it is also desirable to understand the shape of EW equation interface width for $t \to \infty (x \to \infty)$. Therefore, the limit of $f_d(x)$ and following that, the EW interface width shall be obtained for $x \to \infty$ .For this purpose, the asymptotic relation for e^{-x}

is used and the terms included in the first and more orders will be omitted, also the integral existing $f_d(x)$ will move to zero and finally:

$$e^{-x} \rightarrow \sum a_n x^{-n} \rightarrow \frac{1}{x}+\frac{1}{x^2}+... \qquad (32)$$

$$e^{-8\pi^2 x} \sim 0 \qquad [x >> 1] \qquad (33)$$

$$f_d(x)=\frac{k_d}{(2-d)(2\pi)^{2-d}}=Cons\tan t = P \qquad [x>>1] \quad (34)$$

And for the interface width:

$$W^2(L,t)=A+\frac{D}{v}L^{2-d}P+O\left(e^{\frac{-2\pi^2 vt}{a^2}}\right) \qquad [x>>1]$$

$$W(L,t)\approx L^{\frac{2-d}{2}}=L^{\alpha} \qquad [x>>1] \qquad (35)$$

It can be seen that equation (35) is exactly the same as equation (3) that has been obtained from experimental results. Again it is desirable to know behavior of the interface width function for $t \xrightarrow{\sim} 0 \ (x \xrightarrow{\sim} 0)$. Therefore, the limit of $f_d(x)$ and following that, the EW interface width shall be obtained for $x \xrightarrow{\sim} 0$. For this purpose, the terms including e^{-x} will be expanded and the terms including the second and more orders, $O(x^2)$, will be omitted, also the integral existing in $f_d(x)$ will move to $\Gamma(\frac{d}{2})$ with acceptable approximation, and finally:

$$f_d(x)=\frac{1}{2(2-d)\pi^{2-\frac{d}{2}}\Gamma(\frac{d}{2})}\left[1-1+8\pi^2 x-O(x^2)+(8\pi^2 x)^{\frac{2-d}{2}}\int_{8\pi^2 x\approx 0}^{\infty} y^{\frac{d}{2}-1}e^{-y}dy\right] \qquad (36)$$

$$f_d(x)=\frac{2^{\frac{4-3d}{2}}\pi^{\frac{-d}{2}}}{(2-d)}x^{\frac{2-d}{2}}+\frac{4}{(2-d)\pi^{\frac{-d}{2}}\Gamma(\frac{d}{2})}x-O(x^2) \qquad (37)$$

$$W^2(L,t)=A+\frac{D}{v}L^{2-d}\left[\frac{2^{\frac{4-3d}{2}}\pi^{\frac{-d}{2}}}{(2-d)}x^{\frac{2-d}{2}}+\frac{4}{(2-d)\pi^{\frac{-d}{2}}\Gamma(\frac{d}{2})}x-O(x^2)\right]+O\left(e^{\frac{-2\pi^2 vt}{a^2}}\right) \qquad (38)$$

$$W^2(L,t)=A+\frac{D}{v}L^{2-d}\left[Mx^{\frac{2-d}{2}}+Nx\right] \qquad (39)$$

Where

$$M=\frac{2^{\frac{4-3d}{2}}\pi^{\frac{-d}{2}}}{(2-d)}, \qquad N=\frac{4}{(2-d)\pi^{\frac{-d}{2}}\Gamma(\frac{d}{2})}$$

$$W(L,t)\approx\left(\frac{D}{v}\right)^{\frac{1}{2}}M^{\frac{1}{2}}v^{\frac{2-d}{4}}t^{\frac{2-d}{4}}+\left(\frac{D}{v}\right)^{\frac{1}{2}}M^{\frac{-1}{2}}Nv^{\frac{2+d}{4}}\frac{t^{\frac{2+d}{4}}}{L^{2d}} \qquad (40)$$

$$W(L,t)\approx At^{\beta}+B\frac{t^{\frac{2+d}{4}}}{L^{2d}} \qquad (41)$$

A and *B* have constant values, which are defined as follows:

$$A=\left(\frac{D}{\nu}\right)^{\frac{1}{2}} M^{\frac{1}{2}} \nu^{\frac{2-d}{4}} \qquad , \qquad B=\left(\frac{D}{\nu}\right)^{\frac{1}{2}} \frac{1}{2} M^{\frac{-1}{2}} N \nu^{\frac{2+d}{4}}$$

It can be seen that in this case, the interface width is not only a function of t but also a correction term which is a function of L and added to the interface width equation. Again it is desirable to know behavior of the interface width function for $t \xrightarrow{\sim} \infty (x \xrightarrow{\sim} \infty)$.Therefore the limit of $f_d(x)$ and following that, the EW interface width shall be obtained for $x \xrightarrow{\sim} \infty$. For this purpose, in the asymptotic relation for e^{-x}, all of the terms except for the first one $(i.e. \frac{1}{x})$ shall be omitted:

$$e^{-x} \to \sum a_n x^{-n} \to \frac{1}{x} + \frac{1}{x^2} + ...$$

$$e^{-z} \to \frac{1}{x} \qquad (42)$$

Then, relations of $f_d(x)$ and interface width can be written as follows:

$$f_d(x) = \frac{1}{2(2-d)\pi^{2-\frac{d}{2}}\Gamma(\frac{d}{2})}\left[1 - \frac{1}{8\pi^2 x} + (8\pi^2 x)^{1-\frac{d}{2}} \int_{8\pi^2 x}^{\infty} y^{\frac{d}{2}-1} e^{-y} dy\right] \qquad (43)$$

$$f_d(x) = \frac{1}{2(2-d)\pi^{2-\frac{d}{2}}\Gamma(\frac{d}{2})} - \frac{1}{2^4(2-d)\pi^{4-\frac{d}{2}}\Gamma(\frac{d}{2})} x^{-1} \qquad (44)$$

$$f_d(x) = H - Ix^{-1} \qquad (45)$$

Where H and I have constant values:

$$H = \frac{1}{2(2-d)\pi^{2-\frac{d}{2}}\Gamma(\frac{d}{2})}, \qquad I = \frac{1}{2^4(2-d)\pi^{4-\frac{d}{2}}\Gamma(\frac{d}{2})} \qquad (46)$$

and for the interface width:

$$W^2(L,t) = A + \frac{D}{\nu} L^{2-d} [H - Ix^{-1}] + O\left(e^{\frac{-2\pi^2 \nu t}{a^2}}\right)$$

$$W(L,t) \approx \left(\frac{D}{\nu}\right)^{\frac{1}{2}} L^{\frac{2-d}{2}} H^{\frac{1}{2}} \left[1 - \frac{I}{H} x^{-1}\right]^{\frac{1}{2}} \qquad (47)$$

Because the second term inside the bracket is small, it can be expanded, and finally the interface width will be found as follows:

$$W(L,t) = \left(\frac{D}{\nu}\right)^{\frac{1}{2}} H^{\frac{1}{2}} L^{\frac{2-d}{2}} - \left(\frac{D}{\nu}\right)^{\frac{1}{2}} \frac{1}{2} H^{-\frac{1}{2}} I \frac{1}{\nu} \frac{L^{\frac{6-d}{2}}}{t} \qquad (48)$$

$$W(L,t) \approx CL^{\alpha} - D\frac{L^{\frac{6-D}{2}}}{t} \qquad (49)$$

Where

$$C = \left(\frac{D}{\nu}\right)^{\frac{1}{2}} H^{\frac{1}{2}} \qquad , \qquad D = \left(\frac{D}{\nu}\right)^{\frac{1}{2}} H^{\frac{-1}{2}} I \frac{1}{\nu}$$

It can be seen that in this case, the interface width is not only a function of L but also a correction term that is function of t and added to the interface width equation.

4 Conclusion

By exact solution of Edwards- Wilkinson equation, the interface width, for very long and very short times, has been calculated: $W(L,t) \sim t^{\beta}$ and $W_{sat}(L) \sim L^{\alpha}$ respectively. These relations are exactly the same as those, which had been mentioned before in part I of this article. In addition, the Edwards- Wilkinson interface width for times a bit more than zero was calculated: $W(L,t) \sim t^{\beta} + \frac{t^{\frac{2+d}{2}}}{L^{2d}}$. It can be seen that in this relation, the interface width function is not only the function of t but also the term $\frac{t^{\frac{2+d}{2}}}{L^{2d}}$ which is a function of L, and is added to the relation. Also the Edwards- Wilkinson interface width for times a few less than ∞ was calculated $W(L,t) \sim L^{\alpha} - \frac{L^{\frac{6-d}{2}}}{t}$. It can be seen that in this relation, the interface width function is not only the function of L but also the term $\frac{L^{\frac{6-d}{2}}}{t}$, which is a function of t, and added to the relation.

Appendix A

In this section the detail calculation of the Edwards Wilkinson interface width has been written. The Fourier transformation and inverse Fourier transformation can be written as follows:

$$h(k,w) = \int_{-\infty}^{+\infty} dt \int d^d x h(x,t) e^{-i(kx-wt)} \qquad (50)$$

$$h(x,t) = \int_{-\infty}^{+\infty} dw \int d^d k h(k,w) e^{i(kx-wt)} \qquad (51)$$

By performing Fourier transformation on EWequation (9):

$$\int_{-\infty}^{+\infty} \frac{\partial h(x,t)}{\partial t} e^{iwt} dt \int_{-\infty}^{+\infty} e^{-ikx} dx = \int_{-\infty}^{+\infty} \frac{\partial^2 h(x,t)}{\partial x^2} e^{-ikx} dx \int_{-\infty}^{+\infty} e^{iwt} dt + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \eta(x,t) e^{-i(kx-wt)} dt dx \qquad (52)$$

By part integration of equation (52) it can be found:

$$\int_{-\infty}^{+\infty} (-iw) h(x,w) e^{-ikx} dx = \int_{-\infty}^{+\infty} v(ik)^2 h(k,t) e^{iwt} dt + \eta(k,w)$$

$$h(k,w) = \frac{\eta(k,w)}{vk^2 - iw} \qquad (53)$$

where $\eta(k,w)$ is the Fourier transform of $\eta(x,t)$. Before calculating the correlation function of $h(k,w)$, the correlation function of $\eta(k,w)$ shall be obtained. For this purpose the following figures of delta function and equation (7) and (8) shall be used:

$$\delta^d(k+k') = \frac{1}{(2\pi)^d} \int e^{ix(k+k')} d^d x$$

$$\delta(w+w') = \frac{1}{(2\pi)} \int e^{it(w+w')} dt$$

$$\left\langle \eta(k,w)\eta(k',w') \right\rangle = \left\langle \int \eta(x,t) e^{-i(kx-wt)} d^d x dt \int \eta(x',t') e^{-i(k'x'-w't')} d^d x' dt' \right\rangle \qquad (54)$$

$$\langle \eta(k,w)\eta(k',w')\rangle = \int d^d x dt e^{-i(kx-wt)} \int d^d x' dt' e^{-i(k'x'-w't')} \langle \eta(x,t)\eta(x',t')\rangle \quad (55)$$

The bracket shall act over the noise function, therefore the other terms will be outside the bracket.

$$\langle \eta(k,w)\eta(k',w')\rangle = 2D\int d^d x e^{-ik(x+x')}\delta^d(x-x')d^d x' \int dt e^{it(w+w')}\delta(t-t') \quad (56)$$

$$\langle \eta(k,w)\eta(k',w')\rangle = 2D(2\pi)^d \delta^d(k+k')(2\pi)\delta(w+w') \quad (57)$$

$$\langle \eta(k,w)\eta(k',w')\rangle = 2D(2\pi)^d \delta^d(k+k')\int_0^\infty d\tau e^{i(w+w')\tau} \quad (58)$$

Most of the quantities below can be expressed in terms of the amplitude of modes $h(k,t)$ at time t,therefore this amplitude can be obtained after performing the w integral. By performing the inverse Fourier integral over w :

$$\langle h(k,w)h(k',w')\rangle = \frac{1}{(vk^2-iw)(vk'^2-iw')}2D(2\pi)^d \delta^d(k+k')\int_0^\infty d\tau e^{i(w+w')\tau} \quad (59)$$

$$\left\langle \int dw e^{-iwt}h(k,w)\int dw' e^{-iw't}h(k',w')\right\rangle = 2D(2\pi)^d \delta^d(k+k')$$

$$\int \frac{e^{-iw't}}{(vk'^2-iw')}dw' \int \frac{e^{-iwt}}{(vk^2-iw)}dw \int_0^\infty e^{i(w+w')\tau}d\tau \quad (60)$$

$$\langle h(k,t)h(k',t)\rangle = 2D(2\pi)^d \delta^d(k+k')\int_0^\infty d\tau \int \frac{e^{-iw'(t-\tau)}}{(vk'^2-iw')}dw' \int \frac{e^{-iw(t-\tau)}}{(vk^2-iw)}dw \quad (61)$$

$$\langle h(k,t)h(k',t)\rangle = 2D(2\pi)^d \delta^d(k+k')\int_0^\infty d\tau \left(\int \frac{e^{-iw(t-\tau)}}{(vk^2-iw)}dw\right)^2 \quad (62)$$

In equation (62) it shall be known that the $(t-\tau)$ value is less or more than zero. For this purpose, $\int_0^\infty d\tau$ is broken into two parts, $\int_0^t d\tau + \int_t^\infty d\tau$:

$$\langle h(k,t)h(k',t)\rangle = 2D(2\pi)^d \delta^d(k+k')\left[\int_0^t d\tau \left(\int \frac{e^{-iw(t-\tau)}}{(vk^2-iw)}dw\right)^2 + \int_t^\infty d\tau \left(\int \frac{e^{-iw(t-\tau)}}{(vk^2-iw)}dw\right)^2\right] \quad (63)$$

So, it can be clearly found that the value of $(t-\tau)$ in the second integral is more than zero and in the fourth integral is less than zero, by applying the Koushi integral theorem that is expressed as follows:

$$\oint f(w)dw = 2\pi i \sum a_{-1} \quad (64)$$

Where a_{-1} is residue, and is calculated:

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dw^{m-1}}[(w-w_0)^m f(w)]_{w=w_0} \quad (65)$$

In the above equation w_0 is unique point and equal to $w_0 = -ivk^2$ therefore:

$$a_{-1} = ie^{-vk(t-\tau)} \quad (66)$$

Now tow contour shall be assumed for these two integrals:

$$I_1 = \int dw \frac{e^{-iw(t-\tau)}}{(vk^2-iw)}$$

$$I_2 = \int dw \frac{e^{iw(t-\tau)}}{(vk^2 - iw)} \qquad (67)$$

It can be understand that the amount of I_2 integral is zero, and by the use of Koushi integral theorem, I_1 can be found:

$$I_1 = \int dw \frac{e^{-iw(t-\tau)}}{(vk^2 - iw)} = -2\pi e^{-vk^2(t-\tau)} \qquad (68)$$

Whit the use of equations (63) and (68):

$$\langle h(k,t)h(k',t)\rangle = \frac{D}{vk^2}(2\pi)^d \delta^d(k+k')(1-e^{-2vk^2 t}) \qquad (69)$$

Like equation (16), the mean-square width of surface is:

$$W^2(L,t) = \left\langle \frac{1}{N}\sum_x \left[h(x,t) - \frac{1}{N}\sum_{x'} h(x',t) \right]^2 \right\rangle \qquad (70)$$

$$W^2(L,t) = \int \frac{d^d k}{(2\pi)^d}\frac{d^d k'}{(2\pi)^d}\langle h(k,t)h(k',t)\rangle e^{ikx} e^{ik'x} \qquad (71)$$

To obtain equation (71) the inverse Fourier transformation in *d* dimensions on of *k* is required, therefore:

$$\langle h(x,t)h(x',t)\rangle = W^2(L,t) = \int \frac{d^d k}{k^2}\frac{D}{v}(1-e^{-2vk^2 t})\int \frac{d^d k'}{(2\pi)^d}\delta^d(k+k')e^{ikx}e^{ik'x} \qquad (72)$$

Integral(72)shall be performed in spherical coordinates system, for this purpose $\delta^d(k+k')$ and $d^d k$ in spherical coordinate system shall be known:

$$\int d^d k = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\int dk k^{d-1}\int_0^{\pi} \sin^{d-2}\theta d\theta \qquad (73)$$

$$\delta^d(k+k') = \frac{\delta(k+k')\delta(\theta)\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d-1}{2}}(-k)^{d-1}\sin^{d-2}\theta} \qquad (74)$$

In addition the *B* function is required:

$$B(m+1,n+1) = 2\int_0^{\frac{\pi}{2}} \cos^{2m+1}\theta \sin^{2n+1}\theta d\theta \qquad (75)$$

$$B(m+1,n+1) = \frac{m!n!}{(m+n+1)} = \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)} \qquad (76)$$

For calculating the second integral in right hand of equation (73), the equation (76) shall be used:

$$m = -\frac{1}{2} \quad , \quad n = \frac{d-3}{2} \qquad (77)$$

With using equations (77) and (76):

$$\int_0^{\pi} \sin^{d-2}\theta d\theta = \frac{\pi^{\frac{1}{2}}\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} \qquad (78)$$

Now the mean-square width of the surface can be expressed:

$$W^2(L,t)=\frac{1}{2^{d-1}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int\frac{dk}{k^{3-d}}\frac{D}{v}(1-e^{-2vk^2t})\int\delta^d(k+k')d^dk'e^{ix(k+k')} \qquad (79)$$

Now it can be shown that the amount of integral $\int\delta^d(k+k')d^dk'e^{ix(k+k')}$ is equal to 1 for this intention the equations (73),(74) and (78) shall be interest:

$$\int\delta^d(k+k')d^dk'e^{ix(k+k')}=\frac{\delta(k+k')\delta(\theta)\Gamma(\frac{d-1}{2})2\pi^{\frac{d-1}{2}}}{2\pi^{\frac{d-1}{2}}(-k)^{d-1}\sin^{d-2}\theta\Gamma(\frac{d-1}{2})}dk'k'^{d-1}$$

$$\int_0^{\pi}\sin^{d-2}\theta d\theta e^{ix(k+k')}=1 \qquad (80)$$

Therefore:

$$W^2(L,t)=k_d\int_{\frac{2\pi}{L}}^{\frac{\pi}{a}}\frac{dk}{k^{3-d}}\frac{D}{v}(1-e^{-2vk^2t}) \qquad (81)$$

Where:

$$k_d=\frac{1}{2^{d-1}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})} \qquad (82)$$

For comparison with a lattice model, a is identified with a lattice spacing. This identification, as well as replacing a discrete set of modes by a continuum, is not exact. In this sense the numerical factors of terms involving a and L depend on the substrate lattice structure.performing the integration by part (81) can be written:

$$W^2(L,t)=k_d\frac{D}{v}\left[\frac{k^{d-2}}{d-2}\right]_{\frac{2\pi}{L}}^{\frac{\pi}{a}}-k_d\frac{D}{v}\left[\frac{k^{d-2}}{d-2}e^{-2vk^2t}\Bigg|_{\frac{2\pi}{L}}^{\frac{\pi}{a}}+\int_{\frac{2\pi}{L}}^{\frac{\pi}{a}}\frac{k^{d-2}}{d-2}4vkte^{-2vk^2t}dk\right] \qquad (83)$$

The integral $\int k^{d-3}e^{-2vk^2t}$ could be explained by part integration:

$$\int k^{d-3}e^{-2vk^2t}dt=\frac{k^{d-2}}{d-2}e^{-2vk^2t}+\int\frac{k^{d-2}}{d-2}4vk^2te^{-2vk^2t}dk \qquad (84)$$

$$W^2(L,t)=\frac{k_d}{(d-2)}\frac{D}{v}(\frac{\pi}{a})^{d-2}+\frac{k_d}{(2-d)(2\pi)^{2-d}}\frac{D}{v}L^{2-d}-\frac{k_d}{d-2}\frac{D}{v}\left(\frac{\pi}{a}\right)^{d-2}e^{-\frac{2\pi^2vt}{a^2}}$$

$$-\frac{k_d}{(2-d)(2\pi)^{2-d}}\frac{D}{v}L^{2-d}e^{-\frac{8\pi^2vt}{L^2}}+4k_dDt\int_{\frac{2\pi}{L}}^{\frac{\pi}{a}}\frac{k^{d-1}}{d-2}e^{-2vk^2t}dk \qquad (85)$$

By applying the following change of variable in the remaining integral in equation (85):

$$y=2vk^2t\Rightarrow dk=(8vty)^{-\frac{1}{2}}dy$$

$$k=\frac{2\pi}{L}\Rightarrow y=\frac{8\pi^2vt}{L^2}$$

$$k=\frac{\pi}{a}\Rightarrow y=\frac{2\pi^2 vt}{a^2}\approx\infty$$

$$k=\left(\frac{y}{2vt}\right)^{\frac{1}{2}} \qquad (86)$$

By placing equations (86) the remaining integral of equation (85) it can be expressed:

$$\frac{4k_d Dt}{d-2}\int_{\frac{2\pi}{L}}^{\frac{\pi}{2}} k^{d-1}e^{-2vk^2 t}dk=\frac{2k_d Dt}{(2vt)^{\frac{d}{2}}(d-2)}\int_{\frac{8\pi^2 vt}{L^2}}^{\infty} y^{\frac{d}{2}-1}e^{-y}dy \qquad (87)$$

By multiplying and then dividing the term $vL^{2-d}(2\pi)^{2-d}$ to equation (87):

$$\frac{4k_d Dt}{d-2}\int_{\frac{2\pi}{L}}^{\frac{\pi}{2}} k^{d-1}e^{-2vk^2 t}dk=\left(\frac{8\pi^2 vt}{L^2}\right)^{1-\frac{d}{2}}\frac{k_d}{(d-2)(2\pi)^{2-d}}\frac{D}{v}L^{2-d}\int_{\frac{8\pi^2 vt}{L^2}}^{\infty} y^{\frac{d}{2}-1}e^{-y}dy \qquad (88)$$

and finally:

$$W^2(L,t)=\frac{k_d}{d-2}\frac{D}{v}\left(\frac{\pi}{a}\right)^{d-2}+\frac{k_d}{(2-d)(2\pi)^{2-d}}\frac{D}{v}L^{2-d}-\frac{k_d}{d-2}\frac{D}{v}\left(\frac{\pi}{a}\right)^{d-2}e^{\frac{-2\pi^2 vt}{a^2}}$$

$$-\frac{k_d}{(2-d)(2\pi)^{2-d}}\frac{D}{v}L^{2-d}e^{\frac{-8\pi^2 vt}{L^2}}-\frac{k_d}{(2-d)(2\pi)^{2-d}}\frac{D}{v}L^{2-d}\left(\frac{8\pi^2 vt}{L^2}\right)^{1-\frac{d}{2}}\int_{\frac{8\pi^2 vt}{L^2}}^{\infty} y^{\frac{d}{2}-1}e^{-y}dy \qquad (89)$$

References

1- Barabasi, A. L., Buldyerv, S. V., Havlin, S., Huber, G., Stanly, H. E., and Vicsek,T., *Imbibition in porous media: Experimentand Theory, in Surface Disordering: Growth Roughening and Phase Transitions* , Nova Science, New York, 19 (1992).

2- Halpin-Healy, T., and Zhang, C. Y., *Growth Directed Polymers and all that*, *Phys.Rep*., **254**, 215 (1995).

3- Brune, H., Romainczyk, C., Roder, H., and Kern, K., *Fractal and Dendritic Growth Patterns in Two Dimensions Nature*, **369**, 469 (1994).

4- Zhang, J., Zhang, C. Y., Alstrom, P., and. Levinsen, M. T., *Modeling Forest Fire by a Paper-Burning Experiment, Arealization of the Interface Growth Mechanism*, *Physica A* ,**189**, 383 (1992).

5- Barabasi, A. L., and Stanley, H. E., *Fractal Concepts in Surface Growth*,Cambridge University Press, New York, (1995).

6- Family, F., and Vicsek, T., *Scaling of the Active Zone in the Eden Process on Percolation Networks and Ballistic Deposition Model*, *J. Phys. A* ,**18**, L75 (1985)

7- Kardar, M., Parisi, J., and Zhang, C. Y., *Dynamic Scalling of Growing Interfaces,Phys.Rev. Lett.*, **56**, 889 (1986).

8- Natterman, T., and Tang, L. H., *Kinetic Surface Roughning: I. The Kardar-Parisi-Zhang Equation in the Weak Coupling*, *Phys.Rev. A*, **45**, 7156 (1992).

9- Edwards, S. F., and Wilkinson, D. R., *The Surface Statistical of a Granular Aggregate*, *Proc. R. Soc. London A*, **381**, 17 (1982).

10- Hohenberg, P. C., and Halperin, B. I.,*Rev .Mod.Phys*., **79**, 435 (1977)