Numerical solution of fuzzy differential inclusion by Euler method

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Abstract

In this paper we introduce Euler method for solving one dimensional fuzzy differential inclusions. Fuzzy reachable set can be approximated by Euler method with complete analysis. The method is illustrated by numerical example.

***Keywords*:** Fuzzy differential inclusion, Fuzzy reachable set, Euler method.

Introduction

Knowledge about the behavior of differential equation is often incomplete or vague. For example, values of parameter, functional relationships, or initial conditions, may not be known precisely. The initial value problem for fuzzy differential equation (FDE) is studied in (1,2) with *possibilistic irreversibility property*. See(3-7) for further information. Recently Hüllermeier (8) suggested a different formulation of fuzzy initial value problem FIVPs based on a family of differential inclusions at each r-level,

$0 \le r \le 1$,

$$x'(t) \in \left[f(t, x(t))\right]_r, \quad x(0) \in \left[x_0\right]_r,$$

where $[f(.,.)]_r : [o,T] \times R^n \to \kappa_c^n$, and κ_c^n is the space of nonempty convex compact subsets of R^n, also it is shown that the solution has the property that $diam\,(supp\, x(t)) \to 0 \; as\, t \to \infty$, (9,10,8).

This paper is organized as follows. In Section 2, the definition of fuzzy differential inclusion is given. The numerical method for fuzzy differential equation is discussed in Section 3. The proposed algorithm is illustrated in Section 4 and the conclusions are in Section 5.

Preliminaries

Prior to introduce fuzzy differential inclusion we must denote fuzzy sets and fuzzy numbers as follows. We place a tilde over a symbol to denote a fuzzy set so $\tilde{X}, \tilde{A}, \ldots$, all represent fuzzy subsets in R. We write $\tilde{X}(t)$ for the membership function of $\tilde{X}$

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evaluated at $t \in R$. An α-cut of $\tilde{X}$ written $[\tilde{X}]_\alpha$, is defined as $\{t : \tilde{X}(t) \geq \alpha\}$, for $0 < \alpha < 1$ and

$$[\tilde{X}]_0 = \overline{\bigcup_{\alpha \in (0,1]} [\tilde{X}]_\alpha}.$$

A triangular fuzzy number $\tilde{N}$ is defined by three numbers $a_1 < a_2 < a_3$ where the graph of $\tilde{N}(t)$ is triangle with base on the interval $[a_1, a_3]$ and vertex at $t = a_2$ where $\tilde{N}(a_1) = \tilde{N}(a_3) = 0$, $\tilde{N}(a_2) = 1$ and we write $\tilde{N} = (a_1 / a_2 / a_3)$, [11,12].

For $x \in R^n$ and $A, B \subset R^n$ let

$$\rho(x, A) = \inf\{|x - a|, a \in A\},$$
$$\beta(A, B) = \sup\{\rho(a, B), a \in A\},$$
$$d_H(A, B) = \max\{\beta(A, B), \beta(B, A)\}.$$

The Hausdorff distance d_H defines a metric on the nonempty and compact subsets of R^n. For two fuzzy sets $\tilde{A}, \tilde{B}$ the Hausdorff metric is defined as

$$\tilde{d}_H(\tilde{A}, \tilde{B}) = \sup_{\alpha \in [0,1]} d_H([\tilde{A}]_\alpha, [\tilde{B}]_\alpha).$$

We can replace functions and initial values in the problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases} \qquad (1)$$

by set-valued functions which leads to the following differential inclusion (DI), [13]

$$\begin{cases} \dot{x}(t) \in F(t, x(t)), \\ x(0) \in X_0, \end{cases} \qquad (2)$$

where $F : [0, T] \times R^n \to 2^{R^n} \setminus \{\phi\}$ is a set-valued function and $X_0 \subset R^n$ is compact and convex. A function $x : [o, T] \to R^n$ is a solution of (2) if it is an absolutely continuous and satisfies (2) almost everywhere. Let χ denote the set of all solutions of (2), the *reachable* set $X(t)$ at time $t \in [0, T]$ is defined as, [13]

$$X(t) = \{x(t) \mid x \in \chi\}.$$

The set $X(t)$ is the set of all possible solutions of (1) at time t.

A reasonable generalization of this approach which takes vagueness into account is to replace sets by fuzzy sets, i.e. (2) becomes the fuzzy differential inclusion, [14]

$$\begin{cases} \dot{x}(t) \in \tilde{F}(t, x(t)), \\ x(0) \in \tilde{X}_0, \end{cases} \qquad (3)$$

on $[0,T]$ with a fuzzy function $\tilde{F}:[0,T]\times R^n \to E^n$, where fuzzy set $\tilde{X}_0 \in E^n$ and E^n is the set of normal, upper semi-continuous, fuzzy convex, and compactly supported fuzzy sets on $\mathbf{R}^n$. Also $\dot{x}(t)$ is the usual crisp derivative of the crisp differentiable function $x(t)$ with respect to t. In this paper we introduce a numerical method for finding reachable set $\tilde{X}(t)$ that are based on the theoretical consideration of the following theorem.

Theorem 1. [11] *Suppose the fuzzy function $\tilde{F}:[0,T]\times R^n \to E^n$ to be continuous in t and also satisfies Lipschitz condition*

$$\tilde{d}_H(\tilde{F}(t,x),\tilde{F}(t,y)) \le L|x-y|$$

on $\mathbf{R}^n$ with Lipschitz constant $L>0$. Consider the set $\tilde{\chi}$ of solutions to (3). The reachable set $\tilde{X}(t)$ associated with $\tilde{\chi}$ is a normal, upper semi-continuous, and compactly supported fuzzy set for all $t\in[0,T]$. If $\tilde{F}$ is also concave, i.e.,

$$\alpha\tilde{F}(t,x)+\beta\tilde{F}(t,y) \subset \tilde{F}(t,\alpha x+\beta y),$$

For all $\alpha,\beta>0$, $\alpha+\beta=1$, then $\tilde{X}(t)\in E^n$.

Now, consider the initial value problem (3) with $n=1$, i.e.

$$\begin{cases} \dot{x}(t)\in \tilde{F}(t,x(t)), \\ x(0)\in \tilde{X}_0, \end{cases} \qquad (4)$$

on $J=[0,T]$ with a fuzzy concave function $\tilde{F}:J\times R\to E$ where fuzzy set $\tilde{X}\in E$ and the hypotheses of Theorem 1 are satisfied. We call a function $x_\alpha:J\to R$ an α-solution to (4), if it is absolutely continuous and satisfies

$$\begin{cases} \dot{x}_\alpha(t)\in F_\alpha(t,x(t)), \\ x_\alpha(0)\in [\tilde{X}_0]_\alpha, \end{cases} \qquad (5)$$

almost everywhere on J , where $F_\alpha(t,x(t))$ is the α- cut of the fuzzy set $\tilde{F}(t,x(t))$. The set of all α-solutions to (5) is denoted by χ_α, and the α-reachable set $X_\alpha(t)$ is defined as $X_\alpha(t)=\{x(t):x\in\chi_\alpha\}$. In this paper, the α-reachable set $X_\alpha(t)$ is approximated by Euler method.

Euler method

One of the basic question in numerical approximation of the α-reachable set $X_\alpha(t)$ is the follow: Given an α-reachable set $X_\alpha(t_i)$ of possible state at time t_i, what does the interval $X_\alpha(t_{i+1})$ look like? Suppose $\Delta t=t_{i+1}-t_i$. Our first approximation step is to characterize upper bound of α-reachable set $X_\alpha(t_{i+1})$ and second is to characterize lower bound of it. The Euler scheme is set valued generalization as

$$X_\alpha(t_{i+1}) = \bigcup_{x_\alpha(t_i)\in X_\alpha(t_i)} x_\alpha(t_i) + \Delta t F_\alpha(t_i, x_\alpha(t_i)), \qquad (6)$$

then we take $y_i \in X_\alpha(t_i)$, hence there exists a trajectory as $x_\alpha(t) \in \chi_\alpha$ where $x_\alpha(t_i) = y_i$.

Also, $x_\alpha(t)$ is absolutely continuous on $[0,T]$, then $x_\alpha(t)$ is bounded variable on $[0,T]$, i.e., for each partition $0 = t_0 < t_1 < ... < t_n = T$, where $\Delta t = t_{i+1} - t_i$, there exists $M > 0$ such that

$$\sum_{i=1}^{i=n} \left| x_\alpha(t_i) - x_\alpha(t_{i-1}) \right| \le M .$$

We take $\overline{m} = \max_{i=1,2,...,n} \left| x_\alpha(t_i) - x_\alpha(t_{i-1}) \right|$, then $\overline{m} \le \frac{M}{n}$, therefore $\overline{m} \to 0$ as $\Delta t \to 0$. On the other word let $X_a(t_i) = [x_{i,1}, x_{i,2}]$ then distance between $x_{i+1,1}$ and $x_{i+1,2}$ by $x_{i,1}$ and $x_{i,2}$ is respectively small. Therefore, it is better that y_i be selected as $x_{i,2}$, i.e., $y_i = x_{i,2}$. Refer to (5), if $x_\alpha(t)$ be an α-solution then

$$\dot{x}_\alpha(t_i) \cong \frac{x_\alpha(t_{i+1}) - x_\alpha(t_i)}{\Delta t},$$

and hence

$$\frac{x_\alpha(t_{i+1}) - x_\alpha(t_i)}{\Delta t} \in F_\alpha(t_i, x_\alpha(t_i)),$$

where $\Delta t = \frac{T}{n}$.

Now since $y_i = x_\alpha(t_i)$, we have

$$\frac{x_\alpha(t_{i+1}) - y_i}{\Delta t} \in F_\alpha(t_i, y_i) = [\underline{F}(t_i, y_i), \overline{F}(t_i, y_i)],$$

therefore $x_\alpha(t_{i+1}) - y_i \in \Delta t F_\alpha(t_i, y_i)$, or $x_\alpha(t_{i+1}) \in \Delta t F_\alpha(t_i, y_i)$, therefore in due to (6), $x_\alpha(t_{i+1}) \in X_\alpha(t_{i+1})$. We can select

$$y_{i+1} \in [y_i + \Delta t \underline{F}(t_i, y_i), y_i + \Delta t\, \overline{F}(t_i, y_i)],$$

randomly, but in good choice we take $y_{i+1} = y_i + \Delta t \overline{F}(x_i, y_i)$ and is denoted by $y_{i+1}^{(0)}$. Now we take $y_{i+1}^{(1)} = y_i^{(0)} + \varepsilon$, where $\varepsilon > 0$. If $y_{i+1}^{(1)} \in X_\alpha(t_{i+1})$ there exists a trajectory in χ_α such that $x_\alpha(t_{i+1}) = y_{i+1}^{(1)}$ and $\dot{x}(t_{i+1}) \in F_\alpha(t_{i+1}, y_{i+1}^{(1)})$ by using left derivative we have $\dot{x}_\alpha(t_{i+1}) \cong \frac{x_\alpha(t_i) - x_\alpha(t_{i+1})}{t_i - t_{i+1}}$, hence $\frac{x_\alpha(t_i) - y_{i+1}^{(1)}}{t_i - t_{i+1}} \in F_\alpha(t_{i+1}, y_{i+1}^{(1)})$, or

$$\frac{x_\alpha(t_i) - y_{i+1}^{(1)}}{t_i - t_{i+1}} \in [\underline{F}(t_{i+1}, y_{i+1}^{(1)}), \overline{F}(t_{i+1}, y_{i+1}^{(1)})].$$

Since $\Delta t > 0$ we conclude that

$$y_{i+1}^{(1)} - x_\alpha(t_i) \in [\Delta t \underline{F}(t_{i+1}, y_{i+1}^{(1)}), \Delta t \overline{F}(t_{i+1}, y_{i+1}^{(1)})].$$

In other hand we know $x_\alpha(t_i) \in [x_{i,1}, x_{i,2}]$ hence

$$y_{i+1}^{(1)} \in [x_{i,1} + \Delta t \underline{F}(t_{i+1}, y_{i+1}^{(1)}), x_{i,2} + \Delta t \overline{F}(t_{i+1}, y_{i+1}^{(1)})],$$

in general we iterate the sequence $y_{i+1}^{(k+1)} = y_{i+1}^{(k)} + \varepsilon$ until there exists a $m \in \mathrm{N}$ such that $y_{i+1}^{(m)}$ belong to $[x_{i,1} + \Delta t \underline{F}(t_{i,1}, y_{i+1}^{(m)}), x_{i,2} + \Delta t \overline{F}(t_{i+1}, y_{i+1}^{(m)})]$, but $y_{i+1}^{(m+1)}$ does not belong to

$$[x_{i,1} + \Delta t \underline{F}(t_{i+1}, y_{i+1}^{(m+1)}), x_{i,2} + \Delta t \overline{F}(t_{i+1}, y_{i+1}^{(m+1)})].$$

In this case upper bound of α-reachable set at $t = t_{i+1}$, i.e., $x_{i+1,2}$ be approximated by , $y_{i+1}^{(m)}$ or $x_{i+1,2} = y_{i+1}^{(m)}$, similarly for finding lower bound of $X_\alpha(t_{i+1})$ first we select z_i in $X_\alpha(t_i)$ which is a good selection, i.e., $z_i = x_{i,1}$.

By using Euler scheme we have $z_{i+1}^{(0)} \in [z_i + \Delta t \underline{F}(t_i, z_i), z_i + \Delta t \overline{F}(t_i, z_i)]$ therefore $z_{i+1}^{(0)}$ is in the $[x_{i+1,1}, x_{i+1,2}]$ it is means that there exists a trajectory as $x_\alpha(t) \in \chi_\alpha$ such that $x_\alpha(t_{i+1}) = z_{i+1}^{(0)}$ and $x_\alpha(t_i) \in X_\alpha(t_i)$. In general we iterate the sequence

$z_{i+1}^{(k+1)} = z_{i+1}^{(k)} - \varepsilon$ until there exists a $m \in N$ such that $z_{i+1}^{(m)}$ belong to $[x_{i,1} + \Delta t \underline{F}(t_{i+1}, z_{i+1}^{(m)}), x_{i,2} + \Delta t \overline{F}(t_{i+1}, z_{i+1}^{(m)})]$ but $z_{i+1}^{(m+1)}$ does not belong to $[x_{i,1} + \Delta t \underline{F}(t_{i+1}, z_{i+1}^{(m+1)}), x_{i,2} + \Delta t \overline{F}(t_{i+1}, z_{i+1}^{(m+1)})]$. In this case lower bound of α-reachable set at $t = t_{i+1}$, i.e., $x_{i+1,1}$ can be approximated by $z_{i+1}^{(m)}$, in other word $x_{i+1,1} = z_{i+1}^{(m)}$.

Let the exact solution $[\tilde{X}(t_i)]_\alpha = [\underline{X}(t_i), \overline{X}(t_i)]$ in Euler method is approximated by $[\tilde{Y}(t_i)]_\alpha = [\underline{Y}(t_i), \overline{Y}(t_i)]_\alpha$, one can see $\left| x_{i,1} - \underline{Y}(t_i) \right| \le \varepsilon$ and $\left| x_{i,2} - \overline{Y}(t_i) \right| \le \varepsilon$ hence

$$(\left| x_{i,1} - \underline{Y}(t_i) \right|^2 + \left| x_{i,2} - \overline{Y}(t_i) \right|^2)^{\frac{1}{2}} \to 0,$$

as $i \to \infty$, therefore

$$\tilde{d}_H(\tilde{X}_i, \tilde{Y}(t_i)) \to 0$$

Numerical example

Consider the fuzzy differential inclusion on $\mathbf{R}^+$, (10)

$$\begin{cases} \dot{x}(t) \in -x(t) + \tilde{e}\cos t \\ x(0) \in \tilde{X}_0, \end{cases}$$

where $\tilde{e}$ and $\tilde{X}_0$ are symmetric triangular fuzzy numbers with level sets $[\tilde{e}]_\alpha = [0.05(\alpha - 1), 0.05(1 - \alpha)]$ and $[\tilde{X}_0]_\alpha = [0.5(\alpha - 1), 0.5(1 - \alpha)]$. The α-solution set is given for $t \ge 0$ by

$$x_\alpha(t) \in \frac{1}{2}(\sin t + \cos t)[\tilde{e}]_\alpha + ([\tilde{X}_0]_\alpha - \frac{1}{2}[\tilde{e}]_\alpha) e^{-t}.$$

Now, we obtain the approximation of 0-reachable set in Figure 1 and $\tilde{X}(5)$ by Euler method in Figure 2 with $\Delta t = 0.01$.

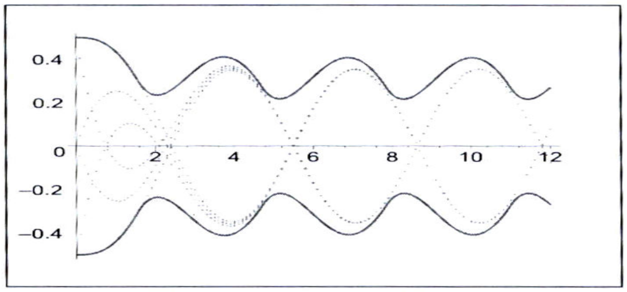

Figure 1. Estimation of 0-reachable set $[\tilde{X}(t)]_0$

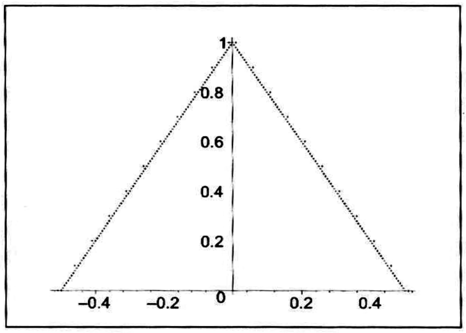

Figure 2. Estimation of $\tilde{X}(5)$

Conclusion

In this paper, we have outlined a Euler numerical method for solving one dimensional fuzzy differential equation based on fuzzy differential inclusion.

By Euler method we can approximate fuzzy α-reachable set, since $x_a(t)$ is an interval. Numerical example shows the efficiency of implemented numerical method.

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