A new mathematical programming approach in fuzzy linear regression models

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Abstract
        In this paper, we introduce a new mathematical programming approach to estimate the parameters of fuzzy linear regression with crisp/fuzzy input and fuzzy output. The advantage of this approach is, both of dependent and independent variables are influenced on the objective function. Therefore, this model rectifies some problems about outliers. Also, in this paper we use possibility function in constraints. To compare the performance of the proposed approach with previous methods, three examples are presented.

Keywords: Fuzzy data; Fuzzy linear regression; Fuzzy numbers; Mathematical programming; Possibility; Necessity.

Introduction

        Fuzzy linear regression was proposed by Tanaka et al[1] in 1982.Many different fuzzy regression approaches have been proposed by different researchers since then, and also this subject has drawn much attention from more and more people concerned. In general, there are two approaches in fuzzy regression analysis: linear programming-based method [1 – 6] and fuzzy least squares method [2, 7- 15]. The first method is based on minimizing fuzziness as an optimal criterion. The second method used least-square of errors as a fitting criterion. The advantage of first approach is its simplicity in programming and computation, while that the degree of fuzziness between the observed and predicted values is minimized by using fuzzy least squares method. Tanaka et al[1] regarded fuzzy data as a possibility distribution, the deviations between the observed values and the estimated values were supposed to be due to fuzziness of system structure. This structure was represented as a fuzzy linear function whose parameters were given by fuzzy sets. They resorted to linear programming to develop their regression model. Sakawa and Yano [16] concentrated on a fuzzy linear regression model that assumes the residuals are caused by the vagueness, both the parameters the model and of the input data simultaneously.

        Heshmaty and Kandel [17] applied fuzzy regression to forecast the computer sales in the United State in a uncertain environment. In addition, other researchers have devoted their efforts to improve the application capability of this fuzzy regression methodology. In this direction, Moskowitz and Kim [18] studied the relation ship among the H value, membership function shape, and the spreads of fuzzy parameters in fuzzy linear regression models, also, developing a systematic linear approach to assess the proper H parameter values. Kim et al [19]

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and Kim and Chen [11] made a comparison of fuzzy and nonparametric linear regression and concluded that when size of a data is small, error terms have small variability, and the relationships among variable are not well specified, fuzzy linear regression outperforms nonparametric linear regression with respect to descriptive capability which is concerned with how close the estimated model parameters are to the true parameters value.

Redden and Woodal [20] showed the proposed models by Sakawa and Yano [16] are extremely sensitive to outlier. The existence of outlier in a set of experimental data can cause incorrect interpretation of the fuzzy linear regression results, Peters [21] consider this problem for the non-fuzzy input and non-fuzzy output data type. Chen [22] focuses on non-fuzzy input and fuzzy output data type and proposes approaches to handle the outlier problem.

In this paper, we propose a new mathematical programming approach when the independent variables are crisp/fuzzy and the dependent variable is fuzzy to rectify above problem.

The paper is organized as follows. In section 2, some elementary properties of fuzzy numbers and fuzzy linear regression are described. The propose method is presented in section 3. Three numerical examples are illustrated to compare the proposed method with previous ones, in section 4.conclusions are drawn in section 5.

Preliminaries

In this section, we describe fuzzy regression methods based on the linear fuzzy model with symmetric triangular fuzzy coefficient [23, 6]. The aim of fuzzy regression is to minimize the fuzziness of the linear fuzzy model that includes all the given data. So we need some definitions to describe fuzzy regression.

Recall that a fuzzy number $\tilde{A}$ is a convex normalized fuzzy subset of the real line **R** with an upper semi-continuous membership function of bounded support [2].

Definition 2.1. A symmetric fuzzy number $\tilde{A}$, denoted by $\tilde{A} = (\alpha, c)_L$ is defined as $\tilde{A}(x) = L((x - \alpha)/c)$, $c > 0$, Where α and c are the center and spread of $\tilde{A}$, respectively, and $L(x)$ is a shape function of fuzzy numbers such that :

i) $L(x) = L(-x)$,
ii) $L(0) = 1, L(1) = 0$,
iii) $L(x)$ is strictly decreasing function for $x \geq 0$,
iv) L is invertible on $[0,1]$.

The set of all symmetric fuzzy numbers is denoted by $F_L(R)$. If $L(x) = 1 - |x|$, then the fuzzy number is a symmetric triangular fuzzy number.

Definition 2.2. Suppose $\tilde{A} = (\alpha, c)_L$ is a symmetric fuzzy number and $\lambda \in R$, then $\lambda \tilde{A} = (\lambda \alpha, |\lambda| c)_L$.

Definition 2.3. [24], For $\tilde{A} \in F_L(R)$ and $0 \leq r \leq 1$, the r-level cuts of $\tilde{A}$ is as follows:

$$[\tilde{A}]_r = \begin{cases} \{x \mid \tilde{A}(x) \geq r\}, & 0 < r \leq 1 \\ \{x \mid \tilde{A}(x) > 0\}, & r = 0 \end{cases} \qquad (1).$$

Dubois and Prade, [2], proposed three indices for the equalities between two fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Definition 2.4. $\tilde{A}$, $\tilde{B}$ are two fuzzy numbers, then

i) $Pos(\tilde{A} = \tilde{B}) = \sup\limits_{x \in R} \{\min \{\tilde{A}(x), \tilde{B}(x)\}\}$,

ii) $Nes(\tilde{A} \subset \tilde{B}) = \inf\limits_{x \in R} \{\max \{1 - \tilde{A}(x), \tilde{B}(x)\}\}$,

iii) $Nes(\tilde{A} \supset \tilde{B}) = \inf\limits_{x \in R} \{\max \{\tilde{A}(x), 1 - \tilde{B}(x)\}\}$,

where *Pos* and *Nes* are short for Possibility and Necessity, see[2].

The fuzzy linear regression model is as

$$\tilde{Y}_i = \tilde{A}_0 \otimes \tilde{X}_{i0} \oplus \tilde{A}_1 \otimes \tilde{X}_{i1} \oplus \ldots \oplus \tilde{A}_p \otimes \tilde{X}_{ip}, \quad i = 1,2,\ldots,n \qquad (2)$$

where $\tilde{X}_{ij} = (x_{ij}, r_{ij})_L$ are symmetric fuzzy numbers of the j-th independent variable in the i-th observation and $\tilde{A}_j = (\alpha_j, c_j)_L$, $j = 0,1,\ldots,p$ are fuzzy parameters. Also, the dependent variable is symmetric fuzzy number, i.e., $\tilde{\tilde{Y}}_i = (\bar{y}_i, e_i)_L$.

If the independent variables are crisp, the fuzzy linear regression model (2) reduce to the following model by using definition 2.2:

$$\tilde{Y}_i = \tilde{A}_0 X_{i0} \oplus \tilde{A}_1 X_{i1} \oplus \ldots \oplus \tilde{A}_p X_{ip}, \quad i = 1,2,\ldots,n. \qquad (3)$$

Tanaka et al. [23,6] formulated a linear programming problem for linear regression with crisp input and fuzzy output, called MIN problem, as follows:

$$\textit{Minimize} \quad \sum_{i=1}^{n} \sum_{j=0}^{p} c_j |x_{ij}|$$

Subject to:
$$\sum_{j=0}^{p} \alpha_j x_{ij} + |L^{-1}(h)| \sum_{j=0}^{p} c_j |x_{ij}| \geq \bar{y}_i + |L^{-1}(h)| e_i \qquad i = 1,2,\ldots,n$$

$$\sum_{j=0}^{p} \alpha_j x_{ij} - |L^{-1}(h)| \sum_{j=0}^{p} c_j |x_{ij}| \leq \bar{y}_i - |L^{-1}(h)| e_i \qquad i = 1,2,\ldots,n$$

$$\alpha_j \in R \quad c_j \geq 0 \qquad j = 0,1,\ldots,p.$$

(4)

In this model, the constraints guarantee the support of the estimated values from the regression model includes the support of the observed values in h − level $(0 \leq h \leq 1)$.

There have been a few criticisms of Tanaka et al.'s approach. One of shortcoming is, if we replace x_i by $(x_i - \bar{x}_i)$, then the estimated function will be very different, [25]. Some articles have proposed major changes to Tanaka et al.'s approach. Savic & Pedrycz [14] and Tanaka et al. [26] suggested, first find the centers, α_j, then solve LP with these α_j's. Hojati et al. [27] and Sakawa & Yano [16] and Razzaghnia et al. [4,5] changed the objective function according to

improve mentioned problems. In the next section, we propose a new mathematical programming approach to overcome the above shortcoming.

The proposed approach

In this section, we propose a new mathematical programming approach to compute the distance between the observed and the estimated values and change the constraints of Tanaka et al.'s model to determine the fuzzy parameters $\tilde{A}_j$ for $j = 0,1,\ldots,p$ of model (2), where $\tilde{A}_j$ is assumed to be a symmetric fuzzy number as defined by definition 2.1.

Definition 3.1: [12] Let $\overline{C}[0,1]$ denote the class of all real-valued bounded functions over $[0,1]$ where is left continuous on $[0,1]$, has a right limit for all $r \in [0,1)$ and is right continuous at $r = 0$. The mapping j, embeds $F_L(R)$ into $\overline{C}[0,1] \times \overline{C}[0,1]$ as follows :

$$j(\tilde{A}) = (\underline{A}(r), \overline{A}(r)) = (\alpha - L^{-1}(r)c, \alpha + L^{-1}(r)c) \qquad (5)$$

where $\underline{A}(r)$ and $\overline{A}(r)$ denote the left and right endpoints of the closed interval $[\tilde{A}]_r$ ($\underline{A}(0)$ and $\overline{A}(0)$ are the endpoints of the closed interval $[\tilde{A}]_0$). As stated in [24]:

i) $\underline{A}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$;

ii) $\overline{A}(r)$ is a bounded left continuous non-increasing function over $[0,1]$;

iii) $\underline{A}(r) \leq \overline{A}(r), \quad 0 \leq r \leq 1$.

Definition 3.2: [28], Let $\tilde{A}, \tilde{B} \in F_L(R)$ where $j(\tilde{A}) = (\underline{A}(r), \overline{A}(r))$ and $j(\tilde{B}) = (\underline{B}(r), \overline{B}(r))$. The metric D_2 on $F_L(R)$ is defined as follows:

$$D_2^2(\tilde{A}, \tilde{B}) = \int_0^1 (\underline{A}(r) - \underline{B}(r))^2 \, dr + \int_0^1 (\overline{A}(r) - \overline{B}(r))^2 \, dr \qquad (6),$$

, i.e., the distance between $\tilde{A}$ and $\tilde{B}$ is in $\overline{C}[0,1] \times \overline{C}[0,1]$.

In this paper, we extended the constraint conditions (4) to possibility and necessity linear regression analysis with fuzzy input-output data, where $\tilde{X}_{ij} = (x_{ij}, r_{ij})_L$ and $\tilde{\overline{Y}}_i = (\overline{y}_i, e_i)_L$.

Using extension principle [29] based on T_M, the difficulty in treating model (2) of fuzzy input-output data is that $\tilde{A}_j \otimes \tilde{X}_{ij}$ may not be a symmetric fuzzy number. Although the product of two symmetric fuzzy numbers may not be a symmetric fuzzy number, Dubois and Prade [2] presented an approximation form. Here, the idea of approximation is used to determine fuzzy parameters of the model (2).

By applying extension principle, [29], and the approximation formula for $\tilde{A}_j \otimes \tilde{X}_{ij}$, the estimated value ($\tilde{Y}_i$) can be approximated by symmetric fuzzy number as $(y_i, d_i)_L$ where

$$y_i = \sum_{j=0}^p \alpha_j x_{ij}, \qquad (7)$$

$$d_i = \sum_{j=0}^p (c_j |x_{ij}| + r_{ij} |\alpha_j|).$$

The objective function of mathematical model: In fuzzy linear regression model (2), the objective function, is to minimize the total distance between observed and estimated values, i.e. $\sum\limits_{i=1}^{n} D_2^2(\tilde{\tilde{Y}}_i, \tilde{Y}_i)$.

The constraints of mathematical model:

(1) $Pos(\tilde{\tilde{Y}}_i = \tilde{Y}_i) \ge h$

(2) $Nes(\tilde{\tilde{Y}}_i \subset \tilde{Y}_i) \ge h$,

These constraint conditions guarantee the degree of fitness of the fuzzy linear regression model is greater than or equal to a threshold h , where $0 \le h \le 1$.

To compute the constraints of the proposed model, we need following theorem.

Theorem 3.1: [16], (1) $Pos(\tilde{\tilde{Y}}_i = \tilde{Y}_i) \ge h, i = 1,2,\ldots,n$ if and only if

$y_i - L^{-1}(h)d_i \le \bar{y}_i + L^{-1}(h)e_i$, and $y_i + L^{-1}(h)d_i \ge \bar{y}_i - L^{-1}(h)e_i$,

(2) $Nes(\tilde{\tilde{Y}}_i \subset \tilde{Y}_i) \ge h, i = 1,2,\ldots,n$ if and only if

$y_i - L^{-1}(h)d_i \le \bar{y}_i - L^{-1}(h)e_i$, and $y_i + L^{-1}(h)d_i \ge \bar{y}_i + L^{-1}(h)e_i$. (see Fig. 1 and Fig. 2.)
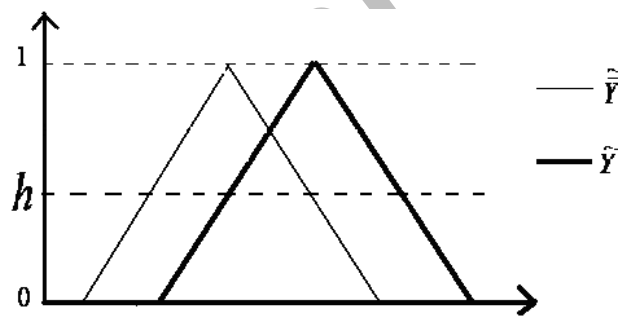

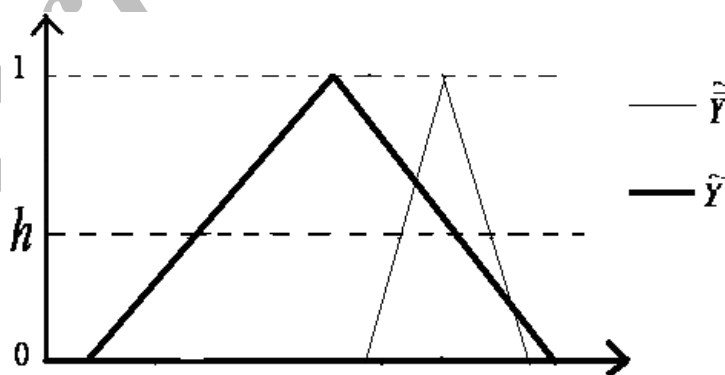**Fig. 1**: Possibility of two fuzzy numbers

Fig. 2: Necessity of two fuzzy numbers

The model: By definition 3.1. and 3.2. we have:

$$j(\tilde{\tilde{Y}}_i) = (\bar{y}_i - L^{-1}(r)e_i, \bar{y}_i + L^{-1}(r)e_i) ,$$
$$j(\tilde{Y}_i) = (y_i - L^{-1}(r)d_i, y_i + L^{-1}(r)d_i) .$$

The resulting mathematical model is as follows:

$$PM1 \qquad : Minimize \ Z(h) = \sum_{i=1}^{n} D_2^2(\widetilde{\overline{Y}}_i, \widetilde{Y}_i) =$$

$$\sum_{i=1}^{n} \int_0^1 [(\overline{y}_i - L^{-1}(r)e_i) - (y_i - L^{-1}(r)d_i)]^2 \, dr$$

$$+ \sum_{i=1}^{n} \int_0^1 [(\overline{y}_i + L^{-1}(r)e_i) - (y_i - L^{-1}(r)d_i)]^2 \, dr \qquad (8)$$

Subject to:
$$y_i - L^{-1}(h)d_i \le \overline{y}_i + L^{-1}(h)e_i, \ i = 1,2,\ldots,n,$$
$$y_i + L^{-1}(h)d_i \ge \overline{y}_i - L^{-1}(h)e_i, \ i = 1,2,\ldots,n. \qquad (9)$$
$$\alpha_j \in R \quad c_j \ge 0 \qquad j = 0,1,\ldots,p.$$

$$PM2 \qquad : Minimize \quad (8)$$

Subject to:
$$y_i - L^{-1}(h)d_i \le \overline{y}_i - L^{-1}(h)e_i, \ i = 1,2,\ldots,n,$$
$$y_i + L^{-1}(h)d_i \ge \overline{y}_i + L^{-1}(h)e_i, \ i = 1,2,\ldots,n. \qquad (10)$$
$$\alpha_j \in R \quad c_j \ge 0 \qquad j = 0,1,\ldots,p.$$

The constraint conditions in (9) means that the h-level sets of the estimated values and observed values should intersect with each other and the constraint conditions in (10) means that the h-level sets of the estimated values should include the h-level sets of the observed values.

The model is a mathematical programming model and can be solved by the existing soft wares.

Theorem 3.2: Given the data $((x_i, r_i)_L, (\overline{y}_i, e_i)_L), i = 1,2,\ldots,n,$ there is an optimal solution $\widetilde{A}_j = (\alpha_j, c_j)_L, j = 0,1,\ldots,p$ for $0 \le h < 1$ in PM1 and PM2.

Remark 3.1: If $h = 1$, then $L^{-1}(h) = 0$. Hence the following equation must hold

$$\overline{y}_i = \alpha_0 + \alpha_1 x_{i1} + \ldots + \alpha_p x_{ip}, i = 1,2,\ldots,n.$$

Thus in general, there is no solution because the given data do not usually satisfy the equation. To rectify this shortcoming, Razzaghnia et al. [30] extended the symmetric triangular fuzzy numbers to asymmetric trapezoidal fuzzy numbers.

Theorem 3.3: The value of objective function for h_2 is greater than or equal to the value of objective function for h_1 such that $h_1 < h_2$ in PM2.

Proof: Suppose that S_1 is all of the possible solution for PM2 at level h_1 and S_2 is all of the possible solution for PM2 at level h_2. It is sufficient to show $S_2 \subseteq S_1$. In PM2, $\alpha |x_i| \ge e_i \quad i = 1,2,\ldots,n$ and by definition 2.1, $L^{-1}(h_2) < L^{-1}(h_1)$ for $h_1 < h_2$, therefore $L^{-1}(h_2)\alpha|x_i| - L^{-1}(h_2)e_i \le L^{-1}(h_1)\alpha|x_i| - L^{-1}(h_1)e_i$ and $S_2 \subseteq S_1$.

To evaluate the performance of a fuzzy regression model, Kim & Bishu in [31] used the absolute difference between the membership values of the observed and estimated values as:

$$E_i = \int_{S(\tilde{Y}_i) \cup S(\tilde{\tilde{Y}}_i)} | \tilde{Y}_i(y) - \tilde{\tilde{Y}}_i(y) | \, dy \qquad (11)$$

Where $S(\tilde{Y}_i)$ and $s(\tilde{\tilde{Y}}_i)$ are the support of $\tilde{Y}_i$ and $\tilde{\tilde{Y}}_i$, respectively. In other words, E_i is the error in estimation. If E_i tends to zero, then the fitting is the best (see Fig. 3).
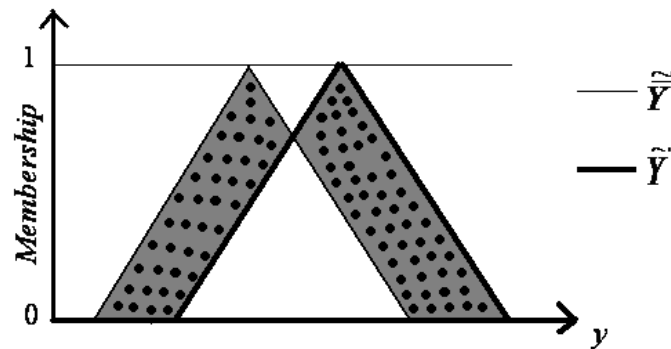

Fig. 3: difference of membership function between the observed and estimated fuzzy numbers

Numerical examples

In this section, we use three examples to compare our proposed method with previous models. In the first example, the independent variable is crisp and the dependent variable is symmetric triangular fuzzy numbers, i.e., $L(x) = 1 - |x|$. In the second example, both of independent and dependent variables are symmetric triangular fuzzy numbers. To illustrate that our model is not sensitive to outliers the third example is presented.

Example 4.1. Tanaka et al. in [23] designed an example to illustrate their regression model. In this example, there are five pairs of $(x_i, (\bar{y}_i, e_i))$ observations as shown in Table 1.

Table 1. crisp input and fuzzy output data for example 4.1(h=0)

input data	Fuzzy output data		Estimation intervals		
x_i	$\bar{y}_i$	e_i	Tanaka et al.	PM1	PM2
1	8.0	1.8	[2.1,9.8]	[4.6,8.6]	[2.3,9.8]
2	6.4	2.2	[4.2,11.9]	[6.2,10.5]	[4.1,11.9]
3	9.5	2.6	[6.3,14.0]	[7.7,12.4]	[6.0,14.0]
4	13.5	2.6	[8.4,16.1]	[9.3,14.2]	[7.9,16.0]
5	13.0	2.4	[10.5,18.2]	[10.8,16.1]	[9.8,18.1]

By using the proposed methods, the fuzzy linear regression models are constructed as:

$$PM1 : \tilde{Y}_i = (4.950,1.84)_L \oplus (1.71,0.16)_L x_i$$

$$PM2 : \tilde{Y}_i = (4.087,3.646)_L \oplus (1.985,0.106)_L x_i$$

To compare the performance of the models, the equation (11) is applied to calculate the errors in estimation the observed responses. The total error of the PM1 and PM2 (see Table 2), are better than the total error of Tanaka et al. model [23].

Table 2. Error in estimation for example 4.1(h=0)

Number of observation	Error in estimation		
	Tanaka et al.	PM1	PM2
1	3.356	2.207	3.185
2	2.850	3.050	2.863
3	1.522	1.091	1.042
4	2.257	2.844	2.616
5	2.414	0.950	1.873
Total error	**12.399**	**10.142**	**11.579**

Example 4.2. This example is designed by Sakawa & Yano [16]. In this example both of dependent and independent variables are fuzzy. The fuzzy input-output data are shown in Table 3. In [16], a value $\in$ which represents the degree of conformity between the observed and estimated values must be specified beforehand. In this example the results for $\in = 0.5$ are adopted for comparison.

By using the PM1 and PM2, the fuzzy regression models are shown in Table 4.

Table 3. Fuzzy input-output data for Example 4.2

Fuzzy input data		Fuzzy output data	
x_i	r_i	$\overline{y}_i$	e_i
2.0	0.5	4.0	0.5
3.5	0.5	5.5	0.5
5.5	1.0	7.5	1.0
7.0	0.5	6.5	0.5
8.5	0.5	8.5	0.5
10.5	1.0	8.0	1.0
11.0	0.5	10.5	0.5
12.5	0.5	9.5	0.5

$$PM1:\widetilde{Y}_i = (3.53\,,0.28)_L \oplus (0.52,0.71)_L \otimes \widetilde{X}_i$$

$$PM2:\widetilde{Y}_i = (3.39,0.61)_L \oplus (0.56,1.66)_L \otimes \widetilde{X}_i$$

Table 4. Estimated parameters and objective function

	Sakawa & Yano	PM1	PM2
Z	18.26	12.02	15.58
α_0	3.20	3.53	3.39
c_0	0.17	0.28	0.61
α_1	0.58	0.52	0.56
c_1	0.08	0.71	1.66

Example 4.3. In this example, there are five pairs observations as shown in Table 5, that the 3rd observation is outlier.

Table 5. Numerical data and comparison of the estimated values for Example 4.3

Observations values		Estimated values			
x_i	$(\bar{y}_i , e_i)_L$	Sakawa & Yano	Tanaka et al.	PM1	PM2
1	(6.4,2.2)	(7.97,1.13)	(8.23,4.03)	(7.86,2.45)	(9.17,6.34)
2	(8.0,1.8)	(10.44,2.26)	(10.72,4.95)	(9.53,2.57)	(10.84,6.46)
3	$(16.5,2.6)^*$	(12.90,3.39)	(13.21,5.88)	(11.25,2.69)	(12.51,6.58)
4	(11.5,2.6)	(15.37,4.43)	(15.40,6.80)	(12.87,2.81)	(14.18,6.70)
5	(13.0,2.4)	(17.83,5.66)	(18.19,7.73)	(14.54,2.93)	(15.85,6.82)

* Indicates the outlier**.**

By using PM1 and PM2, the fuzzy linear regression models are

$$PM1: \tilde{Y}_i = (6.1955 , 2.3345)_L \oplus (1.670,0.120)_L x_i$$

$$PM2: \tilde{Y}_i = (7.5003 , 6.2297)_L \oplus (1.670,0.120)_L x_i .$$

The estimated fuzzy values by Sakawa and Yano $(\in = 0.4)$ [16] and Tanaka et al [23] and Proposed Method, were compared with the observed fuzzy values in the right half of Table 5. The outlier is not influence on the estimated values in proposed methods, but in Tanaka et al. and Sakawa and Yano, the outlier changed the third, forth and fifth estimated values. To compare the performance of these models, equation (11) is used again. The results are shown in Table 6.

Table 6. The estimation errors for Examples 4.3

Number of observations	Errors in estimations		
	Tanaka et al.	PM1	PM2
1	3.132	2.469	4.643
2	4.353	2.532	4.705
3	5.295	5.294	6.247
4	6.532	2.401	4.588
5	7.727	2.643	4.820
Total error	**27.039**	**15.339**	**25.003**

Conclusion

In this paper, we proposed a mathematical programming approach to evaluate fuzzy parameters in fuzzy linear regression with crisp/fuzzy input and fuzzy output data. In this approach, the variation of data didn't influence on estimations, because all of the unknown parameters used in objective functions, therefore it is useful for outliers and influence points in fuzzy regression.

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