Fixed Point Method for Solving Fuzzy Nonlinear Equations

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Abstract

In this paper, we propose the numerical solution for a fuzzy nonlinear equation by fixed point method (simple iteration method). The fuzzy quantities are presented in parametric form. Some numerical illustrations are given to show the efficiency of method.

***Keywords*:** Fixed point method; Fuzzy parametric form; Fuzzy nonlinear equations

Introduction

The numerical solution of consistent system of algebraic nonlinear equations, i.e., F(x) = 0, arise quite often in engineering and the natural sciences. Many engineering design problems that must satisfy specified constraints can be expressed as a nonlinear equalities or inequalities. The concept of fuzzy numbers and arithmetic operation with these numbers were first introduced and investigated by [1]. One of the major applications of fuzzy number arithmetic is nonlinear equations whose parameters are all or partially represented by fuzzy numbers [2, 9, 11]. Standard analytical techniques like Buckley and Qu method, [3, 4, 5, 6], can not suitable for solving the equations such as

$$\text{(i)} \quad ax^4 + bx^3 + cx^3 + dx + x = f,$$

$$\text{(ii)} \quad x - \sin(x) = g,$$

Where x, a, b, c, d, e, f and g are fuzzy numbers. We therefore need to develop the numerical methods to find the roots of such equations. Abbasbandy et al. [12], investigated a model for solving a fuzzy nonlinear system as

$$F(x) = c,$$

Whose all parameters are fuzzy numbers. Here, we consider these equations, in general, as

$$F(x) = x.$$

In section 2, we recall some fundamental results of fuzzy numbers. In section 3, we propose fixed point method for solving fuzzy nonlinear equations. In section 4, we illustrate some examples and conclusions in the last section.

Preliminaries

Definition 1. A fuzzy number is a fuzzy set like $u : \Re \rightarrow I = [0.1]$ which satisfies, [10, 13],

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1. u is upper semi-continuous,
2. u(x) = 0 outside some interval [c, d],
3. There are real numbers a, b such that $c \le a \le b \le d$ and
3.1 u(x) is monotonic increasing on [c, a],
3.2 u(x) is monotonic decreasing on [b, d],
3.3 $u(x) = 1,\ a \le x \le b$.

The set of all these fuzzy numbers is denoted by E. An equivalent parametric is also given in [8] as follows.

Definition 2. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r)$, $0 \le r \le 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
2. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,
3. $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1$.

A popular fuzzy number is the trapezoidal fuzzy number $(x_0, y_0, \sigma, \beta)$ with interval defuzzifier [x_0, y_0] and left fuzziness σ and right fuzziness β where the membership function is

$$u(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma) & x_0 - \sigma \le x \le x_0, \\ 1 & x \in [x_0, y_0], \\ \frac{1}{\beta}(y_0 - x + \beta) & y_0 \le x \le y_0 + \beta, \\ 0 & \text{otherwise}. \end{cases}$$

Its parametric form is

$$\underline{u}(r) = x_0 - \sigma + \sigma r, \qquad \overline{u}(r) = y_0 + \beta + \beta r.$$

Let TF(R) is the set of all trapezoidal fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows.

For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and scalar k > 0 we define addition (u + v) and multiplication by k as

$$(\underline{u + v})(r) = \underline{u}(r) + \underline{v}(r), \qquad (\overline{u + v})(r) = \overline{u}(r) + \overline{v}(r). \tag{1}$$

$$(\underline{ku})(r) = k\,\underline{u}(r), \qquad (\overline{ku})(r) = k\,\overline{u}(r), \tag{2}$$

Definition 3. We call fuzzy number u is positive (resp. negative) if its membership function is such that $u(x) = 0,\ \forall x < 0$ (resp. $\forall x > 0$) this is denoted u > 0 (resp. u < 0).

1 The fixed point method

Now our aim is to obtain a solution for fuzzy nonlinear equation F(x) = x. Without any loss of generality, assume that x is positive and then the parametric form is as follows:

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}, r) = \underline{x}(r), \\ \overline{F}(\underline{x}, \overline{x}, r) = \overline{x}(r), \end{cases} \qquad \forall r \in [0.1].$$

For all, $r \in [0.1]$ the above system is equivalent to

$$F(X(r)) = X(r),$$

Where

$$X(r) = (\underline{x}(r), \overline{x}(r)),$$

and

$$F(\underline{x}(r), \overline{x}(r)) = (\underline{F}(\underline{x}, \overline{x}, r), \overline{F}(\underline{x}, \overline{x}, r)),$$

Definition 4. A function G from $D \subseteq \Re^n$ into $\Re^n$ has a fixed point at $p \in D$ if G (P) = P.

Theorem 1. [7] Let $D = \{(x_1, x_2, ..., x_n)^T | a_i \le x_i \le b_i, i = 1,2,...,n\}$ for some collection of constants $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$. Suppose G is a continues function from $D \subseteq \Re^n$ into $\Re^n$ with the property that

$$G(x) = (g_1(x), g_2(x), ... g_n(x)) \in D,$$

Whenever $x \in$ D. Then G has a fixed point in D.

Suppose, in addition, that G has continuous partial derivatives and a constant K < 1 exists with

$$\left| \frac{\partial g_i(x)}{\partial x_j} \right| \le \frac{K}{n}, \qquad \text{whenever } x \in D,$$

for each j = 1, 2, ..., n and each component function g_i. Then the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ defined by an arbitrarily selected $x^{(0)}$ in D and generated by

$$x^{(k)} = G(x^{(k-1)}), \qquad \text{for each } k \ge 0,$$

Converges to the unique fixed point $P \in D$ and

$$\|x^{(k)} - P\|_\infty \le \frac{K^k}{1-K} \|x^{(1)} - x^{(0)}\|_\infty.$$

In fact this theorem will be used to show that G has unique fixed point in D for x on its domain. Note that G having a unique fixed point in D dose not implies that the solution to the original system is unique on this domain. To approximate fixed point P, we choose $x^{(0)}$, and the sequence of vectors generated by

$$x^{(k)} = G(x^{(k-1)}), \qquad (3)$$

Converges to the unique solution of the system. For initial guess, one can use the trapezoidal fuzzy number

$$x_0 = (\underline{x}(1), \overline{x}(1), \underline{x}(1) - \underline{x}(0), \overline{x}(1) - \overline{x}(0)),$$

and in parametric form

$$\underline{x}_0(r) = \underline{x}(1) + (\underline{x}(1) - \underline{x}(0))(r-1), \quad \overline{x}_0(r) = \overline{x}(1) + (\overline{x}(0) - \overline{x}(1))(1-r).$$

Numerical application

Here we present two examples to illustrating the fixed point method for fuzzy nonlinear equations.

Definition 5. Let $(\underline{x}(r), \overline{x}(r))$ denotes the obtained solution by iterating of (3), the fuzzy number $U(r) = (\underline{u}(r), \overline{u}(r))$ defined by

$$\underline{u}(r) = \min\{\underline{x}(r), \overline{x}(r), \underline{x}(1), \overline{x}(1)\},$$
$$\overline{u}(r) = \max\{\underline{x}(r), \overline{x}(r), \underline{x}(1), \overline{x}(1)\},$$

is called the obtained solution of (3). If $(\underline{x}(r), \overline{x}(r))$ is a fuzzy number then $\underline{u}(r) = \underline{x}(r), \overline{u}(r) = \overline{x}(r)$, and then U is called a strong fuzzy obtained solution. Otherwise, U is a weak fuzzy obtained solution.

Example 1. Consider the fuzzy nonlinear equation

$$(0.222, 0.222, 0.111, 0.111)x^2 + (0.200, 0.200, 0.100, 0.100) = x.$$

Without any loss of generality, assume that x is positive, then the parametric form of this equation is as follows

$$\begin{cases} (0.111 + 0.111r)\underline{x}^2(r) + (0.100 + 0.100r) = \underline{x}(r), \\ (0.333 - 0.111r)\overline{x}^2(r) + (0.300 - 0.100r) = \overline{x}(r). \end{cases}$$

To obtain initial guess we use above system for r = 0 and r = 1, therefore

$$\begin{cases} 0.222\,\underline{x}^2(1) + 0.200 = \underline{x}(1), \\ 0.222\,\overline{x}^2(1) + 0.200 = \overline{x}(1), \end{cases} \qquad \begin{cases} 0.111\,\underline{x}^2(0) + 0.100 = \underline{x}(0), \\ 0.333\,\overline{x}^2(0) + 0.300 = \overline{x}(0), \end{cases}$$

consequently $\underline{x}(0) = 0.101, \overline{x}(0) = 0.338$ and $\overline{x}(1) = \underline{x}(1) = 0.210$. Therefore initial guess is x0 = (0.210, 0.210, 0.109, 0.128). The component equations then become

$$\begin{cases} \underline{x}^{(k)}(r) = (0.111 + 0.111r)(\underline{x}^{(k-1)}(r))^2 + (0.100 + 0.100r), \\ \overline{x}^{(k)}(r) = (0.333 - 0.111r)(\overline{x}^{(k-1)}(r))^2 + (0.300 - 0.100r). \end{cases}$$

Because the parametric functions are bounded and continuous for r ∈ [0, 1] we obtain the solution after 10 iterations; which the maximum error would be less than 10−5. For more details see Figure 1.

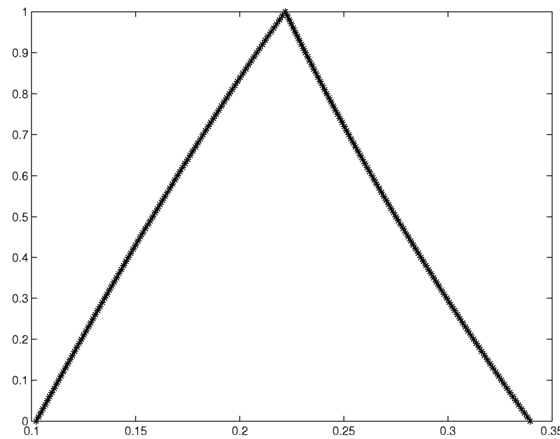

Figure 1 : First Solution of Fixed Point Method with x0 = (0.210, 0.210, 0.109, 0.128).

Let $\underline{x}(0) = 8.908$, $\overline{x}(0) = 2.665$ and $\overline{x}(1) = \underline{x}(1) = 4.295$. Hence x0 = (4.295, 4.295, 1.630, 4.613) be a weak initial guess. The component equations then become

$$\begin{cases} \underline{x}^{(k)}(r) = [(\underline{x}^{(k-1)}(r) - (0.100 + 0.100r))/(0.111 + 0.111r)]^{\frac{1}{2}}, \\ \overline{x}^{(k)}(r) = [\overline{x}^{(k-1)}(r) - (0.300 - 0.100r))/(0.333 - 0.111r)]^{\frac{1}{2}}. \end{cases}$$

Now parametric functions are bounded and continuous for r ∈ [0, 1] we obtain the weak solution after 15 iterations; which the maximum error would be less than 10−5. For more details see Figure 2.

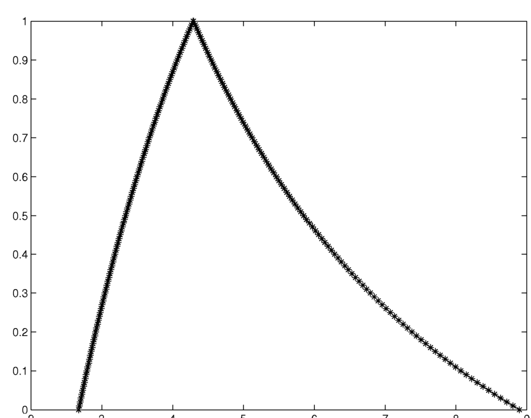

Figure 2: Weak Solution of Fixed Point Method with x0 = (4.295, 4.295, 1.630, 4.613).

Example 2. Consider fuzzy nonlinear equation

$$(2, 2, 1, 1)x^3 + (3, 3, 1, 1)x^2 + (4, 4, 1, 1) = (8, 8, 3, 5).$$

Without any loss of generality, assume that x is positive, then parametric form of this equation is as follows

$$\begin{cases} (1+r)\underline{x}^3(r) + (2+r)\underline{x}^2(r) + (3+r) = (5+3r), \\ (3-r)\overline{x}^3(r) + (3-r)\overline{x}^2(r) + (5-r) = (13-5r). \end{cases}$$

By solving the above system for r = 0 and r = 1, we obtain the initial guess x0 = (0.91, 0.91, 0.15, 0.15). If we apply two iterations from fixed point method, the maximum error is less than 10−3, Figure 3.

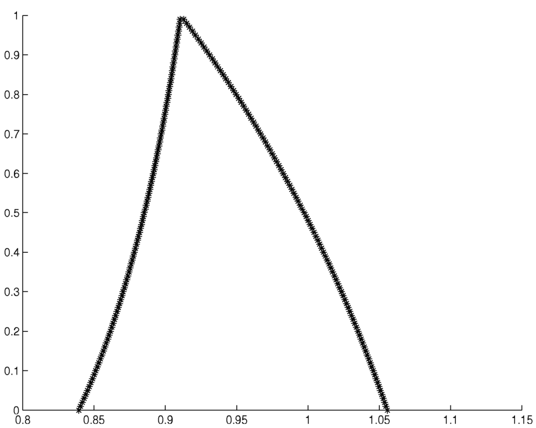

Figure 3: Positive Solution of Fixed Point Method with x0 = (0.91, 0.91, 0.15, 0.15).

Conclusions

In this paper, we have suggested numerical solving method for fuzzy nonlinear equations instead of standard analytical techniques which are not suitable everywhere. Initially we wrote fuzzy nonlinear equation in parametric form and then solve it by fixed point method. Finally, examples were presented to illustrate proposed method.

References

1. Zadeh, L. A., *J. Inform. Sci.*, **3**, 199 (1975).
2. Badard, R., *J. Inform. Sci*., **28**, 161 (1982).
3. Diamond, P., *J. Inform. Sci*., **46**, 144 (1988).
4. Peeva, K., *Fuzzy Sets and Systems*, **49**, 339 (1992).
5. Buckley, J.J., and Qu, Y., *J. Fuzzy Sets and Systems*, **38**, 43 (1990).
6. Buckley, J.J., and Qu, Y., *J. Fuzzy Sets and Systems*, **38**, 309 (1990).
7. J Buckley, J.J., and Qu, Y., *J. Fuzzy Sets and Systems*, **39**, 291 (1991).
8. Buckley, J.J., and Qu, Y., *J. Fuzzy Sets and Systems*, **43**, 33 (1991).
9. Abbasbandy, A. and Asady, B., *Appl. Math. Comput*., **156**, 381 (2004).
10. Dubois, D., and Prade, H., *Fuzzy Sets and Systems: Theory and Application*, Academic Press, New York (1980).
11. Zimmermann, H.J., *Fuzzy Sets Theory and its Application*, Kluwer AcademicPress, Dordrecht (1991).
12. Cong-Xin, W., and Ming, M., *J. Fuzzy Sets and Systems*, **44**, 33 (1991).
13. Burden, R. L., and Faires, J. D., *Numerical Analysis,* 7th , Boston, PWSKent (2001).