Moment Inequalities and Applications for Negative Dependence Random Variables

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Abstract

In this paper, we extend some moment inequalities for partial sums of negative dependence (ND) random variables. Based on these inequalities, some useful inequalities are obtained.

Key Words: Negative Dependence, Moment Inequalities.

1.Introduction and preliminaries

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, p)$. The moment inequalities and their applications for independent random variables have been studied by Petrov (1995). Shao (2000) and Su (1997) studied moment inequalities for NA random variables. In this paper, we extend these inequalities for ND random variables. To prove our main results we need the following definition, lemmas and theorems.

Definition 1. The random variables $X_1, \cdots, X_n$ are said to be ND if we have

$$P[\bigcap_{j=1}^{n}(X_j \leq x_j)] \leq \prod_{j=1}^{n} P[X_j \leq x_j], \quad (1)$$

and

$$P[\bigcap_{j=1}^{n}(X_j > x_j)] \leq \prod_{j=1}^{n} P[X_j > x_j], \quad (2)$$

for all $x_1, \cdots, x_n \in R$. An infinite sequence $\{X_n,\ n \geq 1\}$ is said to be ND if every finite subset $X_1, \cdots, X_n$ is ND. The conditions (1) and (2) are equivalent for $n = 2$, but these do not agree for $n \geq 3$ (see [2])

Lemma 1.([1]) Let $X_1, \cdots, X_n$ be ND random variables and $f_1 \cdots, f_n$ be a sequence of Borel functions which all are monotone increasing (or all are monotone decreasing) , then $f_1(X_1), \cdots, f_n(X_n)$ are ND random variables.

Lemmas 2.([1]) Let $X_1, \cdots, X_n$ be ND nonnegative random variables. Then

$$E[\prod_{j=1}^{n} X_j] \leq \prod_{j=1}^{n} E[X_j].$$

Corollary 1. Let $X_1, X_2, \cdots, X_n$ be ND random variables, then for every real t we have

$$Ee^{tS_n} \leq \prod_{i=1}^{n} Ee^{tX_i}.$$

Lemmas 3.([3]) If X be an arbitrary random variable and $p > 0$.Then

$$E|X|^p = p\int_0^\infty x^{p-1}P[|X| \geq x]dx.$$

Lemmas 4.([3]) Let $\{X_n, n \geq 1\}$ be a sequence of random variables with $E|X|^p \leq \infty$ for some $p \geq 1$ and $S_n = \sum_{i=1}^{n} X_i$. Then

$$E|S_n|^p \leq n^{p-1}M_{p,n},$$

where $M_{p,n} = \sum_{j=1}^{n} E|X_j|^p$.

Theorem 1.([3]) Let X be a random variable and $\beta_r = E|X|^r$ for any $0 < r < s$. Then

$$\beta_r^{\frac{1}{r}} \leq \gamma^{\frac{1}{r}-\frac{1}{s}}.\beta_s^{\frac{1}{s}},$$

where $\gamma = P[X \neq 0]$.

Lemmas 5.([3])If Y is a random variable with d.f. $\frac{1}{n}\sum_{k=1}^{n} P[X_k \leq x]$. Then for any $r > 0$,

i)

$$E|Y|^r = \frac{1}{n}\sum_{k=1}^{n} E|X_k|^r$$

ii)

$$P[Y \neq 0] = \frac{1}{n}\sum_{k=1}^{n} P[X_k \neq 0].$$

Corollary 2. Applying Theorem 1 to Y with $r = 2$ and $s = p \geq 2$, we conclude that

$$B_n^{\frac{1}{2}} \leq [\sum_{k=1}^{n} P[X_k \neq 0]]^{\frac{1}{2}-\frac{1}{p}}.M_{p,n}^{\frac{1}{p}}.$$

where $B_n = \sum_{j=1}^{n} EX_j^2$.

2. The Main Results

In this section, first we extend an inequality for ND r.v.,s that Petrov [3] proved for independent r.v.'s and Su (1997) proved for NA r.v.'s. Then by using this inequality, we obtain some moment inequalities for ND random variables.

Theorem 2. Let $X_1, X_2, \cdots, X_n$ be ND random variables with $E(X_i) = 0$, $EX_i^p < \infty$, $i = 1, 2, \cdots, n$ and $p \geq 2$. Then for any $t > p/2$ and $x > 0$,

$$P[|S_n| \geq x] \leq \sum_{j=1}^{n} P[|X_j| \geq \frac{x}{t}] + 2e^t[1 + \frac{x^2}{tB_n}]^{-t} \quad (3)$$

Proof. Since proof is similar to Lemma 2.3 (Ref.[3]) and Theorem 1 (Ref.[5]), thus we omit details. We define $Y_j = \min(X_j, y)$, $y > 0$ and $T_n = \sum_{j=1}^{n} Y_j$. It is clear that for every real x that $x > y$,

$$[S_n \geq x] \subseteq [T_n \geq x] \cup [T_n \neq S_n].$$

Hence for every $h > 0$ and by Markov's inequality

$$P[S_n \geq x] \leq P[T_n \geq x] + P[T_n \neq S_n] \leq e^{-hx} Ee^{hT_n} + \sum_{j=1}^{n} P[X_j \geq y]. \quad (4)$$

Let $F_j(x) = P[X_j \leq x]$ for all real x and $j = 1, 2, ..., n$. Since $Y_j \leq X_j$ and $EY_j \leq EX_j = 0$. Thus for every $h > 0$,

$$Ee^{hY_j} \leq 1 + \int_{\infty}^{y} (e^{hx} - 1 - hx) dF_j(x) + (e^{hy} - 1 - hy) P[X_j \geq y]. \quad (5)$$

Since the function $g(x) = \frac{e^{hx}-1-hx}{x^2}$, for every $h > 0$ is non-decreasing, hence by (5) obtain

$$Ee^{hY_j} \leq 1 + \frac{e^{hy} - 1 - hy}{y^2} EX_j^2 \leq \exp[\frac{e^{hy-1-hy}}{y^2} EX_j^2].$$

By Lemma 1, the random variables $Y_1, Y_2, ..., Y_n$ are ND because the random variables $X_1, X_2, ...X_n$ are ND. Therefore,

$$e^{-hx} Ee^{hT_n} \leq e^{-hx} \prod_{j=1}^{n} Ee^{hY_j} \leq \exp[-hx + \frac{e^{hy} - 1 - hy}{y^2} B_n]. \quad (6)$$

We put here $h = \frac{1}{y} \ln(\frac{xy}{B_n} + 1)$, then

$$e^{-hx} Ee^{hT_n} \leq \exp[-\frac{x}{y} \ln(\frac{xy}{B_n} + 1) + \frac{x}{y} - \frac{B_n}{y^2} \ln(\frac{xy}{B_n} + 1)] \leq \exp[\frac{x}{y} - \frac{x}{y} \ln(\frac{xy}{B_n} + 1)].$$

Note that an analogous estimate holds for $P[-S_n \geq x]$, with replacement of $Z_j = \min(-X_j, y)$ and $T_n' = \sum_{j=1}^{n} Z_j$ by $Y_j = \min(X_j, y)$ and $T_n = \sum_{j=1}^{n} Y_j$. Since by Lemma 1 $Z_1, Z_2, ...Z_n$ are ND, hence for every $x > 0$ and $h > 0$,

$$P[|S_n| \geq x] \leq P[S_n \geq x]$$

$$+ P[-S_n \geq x] \leq P[T_n \geq x] + P[T_n \neq S_n] + P[T'_n \geq x] + P[T'_n \neq -S_n]$$

$$\leq \sum_{j=1}^{n} P[|X_j| \geq y] + e^{-hx}(Ee^{hT_n} + Ee^{hT'_n})$$

$$\leq \sum_{j=1}^{n} P[|X_j| \geq y] + 2\exp[\frac{x}{y} - \frac{x}{y}\ln(\frac{xy}{B_n} + 1)],$$

Let $t = \frac{x}{y} > p/2$, we have

$$P[|S_n| \geq x] \leq \sum_{j=1}^{n} P[|X_j| \geq \frac{x}{t}] + 2e^t[1 + \frac{x^2}{tB_n}]^{-t}$$

Hence complete the proof.

Corollary 3. Under the assumptions of Theorem 2 we have

i)

$$E|S_n|^p \leq A_p(M_{p,n} + B_n^{p/2}), \quad (7)$$

ii)

$$E|S_n|^p \leq 2A_p n^{p/2-1} M_{p,n}. \quad (8)$$

Where $A_p > 0$ depends only on p.

Proof.

i) By Lemma 2 and (3) for every $x > 0$ and $p \geq 2$, we have

$$E|S_n|^p = p\int_0^\infty x^{p-1}P[|S_n| \geq x]dx$$

$$\leq p\sum_{j=1}^{n}\int_0^\infty x^{p-1}P[|X_j| \geq x/t]dx + 2pe^t\int_0^\infty x^{p-1}[1 + \frac{x^2}{tB_n}]^{-t}dx$$

$$= t^p M_{p,n} + pe^t t^{p/2}\beta(p/2, t-p/2)B_n^{p/2}$$

$$\leq A_p(M_{p,n} + B_n^{p/2}),$$

where,

$A_p = \max(p^p, p^{1+p/2}e^p\beta(p/2, p/2))$

for $t = p$.

ii) By Lemma 4 and Jensen's inequality for every $p \geq 2$, we have

$$E|S_n|^p \leq A_p(M_{p,n} + E(\sum_{j=1}^{n} X_j^2)^{p/2}))$$

$$\leq A_p(1 + n^{p/2-1})M_{p,n}$$

$$\leq 2A_p n^{p/2-1} M_{p,n}.$$

Corollary 4. Under the assumptions of Theorem 2,

$$E|S_n|^p \leq A_p[1 + (\sum_{j=1}^{n} P[X_j \neq 0])^{p/2-1}]M_{p,n}. \quad (9)$$

Proof. By Corollaries 2 and 3 we have

$$E|S_n|^p \leq A_p(M_{p,n} + B_n^{p/2})$$

$$\leq A_p[1 + (\sum_{j=1}^{n} P[X_j \neq 0])^{p/2-1}]M_{p,n}$$

This complete the proof.

Remark 2. If the sum $\sum_{j=1}^{n} P[X_j \neq 0]$ grows slower than n, particular, if $\sum_{j=1}^{n} P[X_j \neq 0] = O(n^\alpha)$, $0 < \alpha < 1$, then (6) gives a better estimate than (5) does.

Theorem 3. Let $X_1, X_2, \cdots, X_n$ are

ND random variables and $EX_k = 0$ for $k = 1, 2, \cdots, n$. Then for every $p > 1$,

$$E|S_n|^p \leq A_p(M_{p,n} + D_n^p), \qquad (10)$$

Where $D_n = \sum_{j=1}^n E|X_j|$ and A_p is a positive constant depending only on p.

Proof Since $g(x) = \frac{e^{hx}-hx-1}{x}$ for any $h > 0$ is a nondecreasing function; hence, by applying it in proof of Theorem 2 and $EX_j \leq E|X_j|$ for $h = \frac{1}{y}\ln(1 + \frac{y}{D_n}) > 0$, we obtain

$$\begin{aligned} P[|S_n| \geq x] &\leq \sum_{j=1}^n P[|X_j| \geq \frac{x}{t}] \\ &+ 2e[1 + \frac{x}{tD_n}]^{-t}, \qquad (11) \end{aligned}$$

where $t = \frac{x}{y} > p > 1$, and $x > 0$. Now by using (11) we have

$$\begin{aligned} E|S_n|^p &\leq t^p M_{p,n} \\ &+ 2pe\int_0^\infty x^{p-1}(1 + \frac{x}{tD_n})^{-t}dx \\ &= t^p M_{p,n} \\ &+ 2pt^p D_n^p e\int_0^\infty u^{p-1}(1+u)^{-t}du \\ &= t^p M_{p,n} + 2pt^p D_n^p e^t \beta(p, t-p), \end{aligned}$$

by putting $t = 2p$ and $A_p = \max((2p)^p, (2p)^{p+1}e^{2p}\beta(p,p))$, hence complete the proof.

Theorem 4. Let $X_1, X_2, \cdots, X_n$ are ND random variables with $EX_k = 0$ for $k = 1, 2, \cdots, n$. Then for every $p > 1$,

$$E|S_n|^p \leq A_p[1+(\sum_{j=1}^n P[X_j \neq 0])^{p-1}]M_{p,n}.$$

Proof By Theorem 1 for $r = 1$ and $s = p > 1$,we have

$$D_n \leq [\sum_{j=1}^n P[X_j \neq 0]]^{1-1/p} M_{p,n}^{1/p},$$

now Theorem 3 for every $p > 1$, complete the proof.

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5. References

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