INVESTIGATION OF BOUNDARY LAYERS IN A SINGULAR PERTURBATION PROBLEM INCLUDING A 4TH ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we investigate a singular perturbation problem including a fourth order O.D.E. with general linear boundary conditions. Firstly, we obtain the necessary conditions of solution of O.D.E. by making use of fundamental solution, then by compatibility of these conditions with boundary conditions, we determine that, for given perturbation problem, whether boundary layer is formed or not.

Introduction

One of the important subjects in applied mathematics is the theory of (regular or singular) perturbation problems. Normally, in a regular perturbation problem, the order of O.D.E. (Ordinary Differential Equations) remains unchanged as we let the perturbation parameters takes zero value. In a singular perturbation problem if we set the small parameter equal to zero, we will obtain an O.D.E of the lower order [6]. Consequently the solution to this reduced problem does not, in general, satisfy all the boundary conditions. In other words, boundary layers are formed at the points of the boundary where boundary conditions are not satisfied. From point of view of solving methods, the natural case of free boundary layer is more desirable than the other case [3].

We consider the following singular perturbation problem:

$$l_\varepsilon y_\varepsilon(x) \equiv \varepsilon y_\varepsilon^{(iv)}(x) + a y'_\varepsilon(x) + b y_\varepsilon(x) = 0, \quad x \in (0,1) \qquad (1)$$

$$l_i y_\varepsilon(x) \equiv \sum_{j=0}^{3} [\alpha_{ij}^{(0)} y_\varepsilon^{(j)}(0) + \alpha_{ij}^{(1)} y_\varepsilon^{(j)}(1)] = \alpha_i, i = 1,2,3,4 \quad (2)$$

where $\varepsilon > 0$ is the small parameter, a, b, $\alpha_{ij}^{(k)}$, k=1,2 and α_i are the known real constants, $y_\varepsilon(x)$ is the unknown function.

Basing on Lagrange's formula, it is easy to see that the adjoint equation for (1) is:

$$l_\varepsilon^* Z_\varepsilon(x) \equiv \varepsilon Z_\varepsilon^{(iv)}(x) - a Z'_\varepsilon(x) + b Z_\varepsilon(x) \qquad (3)$$

Assuming that

$$a, b \in R, \qquad a > 0 \qquad 1^0$$

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the following has been proven,

Theorem 1. Under condition 1^0, there is a fundamental solution, uniform in terms of parameter $\varepsilon \geq 0$, of the adjoint equation

$$l_\varepsilon^* Z_\varepsilon(x) = f(x) \tag{4}$$

having the following form (f(x) is an arbitrary function)

$$Z_\varepsilon(x-t) = \begin{cases} \dfrac{W_{42}(\varepsilon)}{\varepsilon W(\varepsilon)} e^{\theta_2(\varepsilon)(x-t)} + \dfrac{W_{43}(\varepsilon)}{\varepsilon W(\varepsilon)} e^{\theta_3(\varepsilon)(x-t)}; & 0 < t < x \\ -\dfrac{W_{41}(\varepsilon)}{\varepsilon W(\varepsilon)} e^{\theta_1(\varepsilon)(x-t)} - \dfrac{W_{44}(\varepsilon)}{\varepsilon W(\varepsilon)} e^{\theta_4(\varepsilon)(x-t)}; & x < t < 1 \end{cases} \tag{5}$$

where $\theta_j(\varepsilon), j = 1,\dots,4$ are the roots of the characteristic equation:

$$\varepsilon\theta^4 - \alpha\theta + b = 0 \tag{6}$$

$W(\varepsilon)$ is Vandermond's determinant and $W_{4k}(\varepsilon)$, $k = 1,\dots,4$ are the cofactors of elements of fourth row of main determinant $W(\varepsilon)$, that is

$$W(\varepsilon) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \theta_1(\varepsilon) & \theta_2(\varepsilon) & \theta_3(\varepsilon) & \theta_4(\varepsilon) \\ \theta_1^2(\varepsilon) & \theta_2^2(\varepsilon) & \theta_3^2(\varepsilon) & \theta_4^2(\varepsilon) \\ \theta_1^3(\varepsilon) & \theta_2^3(\varepsilon) & \theta_3^3(\varepsilon) & \theta_4^3(\varepsilon) \end{vmatrix}$$
$$= (\theta_2-\theta_1)(\theta_3-\theta_1)(\theta_4-\theta_1)\cdot(\theta_3-\theta_2)(\theta_4-\theta_2)(\theta_4-\theta_3)$$

Note that uniformity of the fundamental solution (5), that is, fundamentality and uniformity remain true results of the solution by Newton's diagram when $\varepsilon \to 0$. These asymptotes have the following form:

$$\theta_k(\varepsilon) = \alpha_{0k}\varepsilon^{-\frac{1}{3}} - \frac{b}{3a} - \frac{2b^2}{9a^2}\alpha_{0k}^2\varepsilon^{\frac{1}{3}} + O(\varepsilon^{\frac{2}{3}}), k = 1,2,3 \tag{7}$$

where $\alpha_{ok}^3 = a$ and

$$\theta_4(\varepsilon) = \frac{b}{a} + \frac{b^4}{a^5}\varepsilon + \frac{4b^7}{a^9}\varepsilon^2 + O(\varepsilon^3) \tag{8}$$

Further, basing upon the fundamental solution (5) and making use of Lagrange's formula [1,2], and of its analogue [4,5], we obtain 8 necessary conditions, which include boundary values of the unknown function and its derivatives. We will write three out of them, those to be useful for us later.

$$\begin{aligned} &[aZ_\varepsilon(-1) - \varepsilon Z_\varepsilon'''(-1)]y_\varepsilon(0) + [\varepsilon Z_\varepsilon'''(0) - aZ_\varepsilon(0) \\ &-\frac{1}{2}]y_\varepsilon(1) + \varepsilon Z_\varepsilon''(-1)y_\varepsilon'(0) - \varepsilon Z_\varepsilon''(0)y_\varepsilon'(1) \\ &-\varepsilon Z_\varepsilon'(-1)y_\varepsilon''(0) + \varepsilon Z_\varepsilon'(0)y_\varepsilon''(1) + \varepsilon Z_\varepsilon(-1)y_\varepsilon'''(0) \\ &-\varepsilon Z_\varepsilon(0)y_\varepsilon'''(1) = 0 \end{aligned} \tag{9}$$

$$\begin{aligned} &-bZ_\varepsilon(0)y_\varepsilon(0) + bZ_\varepsilon(1)y_\varepsilon(1) - [\varepsilon Z_\varepsilon'''(0) + \frac{1}{2}]y_\varepsilon'(0) \\ &+\varepsilon Z_\varepsilon'''(1)y_\varepsilon'(1) + \varepsilon Z_\varepsilon''(0)y_\varepsilon''(0) - \varepsilon Z_\varepsilon''(0)y_\varepsilon''(1) \\ &-\varepsilon Z_\varepsilon''(1)y_\varepsilon''(1) - \varepsilon Z_\varepsilon'(0)y_\varepsilon'''(0) + \varepsilon Z_\varepsilon'(1)y_\varepsilon'''(1) = 0 \end{aligned} \tag{10}$$

$$\begin{aligned} &-bZ_\varepsilon(-1)y_\varepsilon(0) + bZ_\varepsilon(0)y_\varepsilon(1) - \varepsilon Z_\varepsilon'''(-1)y_\varepsilon'(0) \\ &+[\varepsilon Z_\varepsilon'''(0) - \frac{1}{2}]y_\varepsilon'(1) + \varepsilon Z_\varepsilon''(-1)y_\varepsilon''(0) \\ &-\varepsilon Z_\varepsilon''(0)y_\varepsilon''(1) - \varepsilon Z_\varepsilon'(-1)y_\varepsilon'''(0) + \varepsilon Z_\varepsilon'(0)y_\varepsilon'''(1) = 0 \end{aligned} \tag{11}$$

So, the following is true.

Theorem 2. Under condition 1^0, every solution of the equation (1), defined on (0,1), satisfies (9)-(11), where $Z_\varepsilon(x\text{-}t)$ is the fundamental solution of the equation (4), having the form (5).

Remark 1. It is easy to see that the three out of necessary conditions, given in the form (9)-(11), are linear independent conditions even if $\varepsilon \to 0$.

1. Case of Boundary Layer

Now, considering the limit of (5), we obtain

$$Z_0(x-t) = \frac{e(t-x)}{a} e^{\frac{b}{a}(x-t)} \tag{12}$$

where $e(t-x)$ is the Heavyside's function.

We consider that $Z_0(x-t)$ is the fundamental solution of following equation

$$-aZ_0'(x)+bZ_0(x)=0 \tag{13}$$

which is obtained by letting $\varepsilon = 0$ in equation (3), that is,

$$\varepsilon Z_\varepsilon^{(iv)}(x)-aZ_\varepsilon'(x)+bZ_\varepsilon(x)=0$$

In fact, we see that

$$\begin{aligned}&-aZ_0'(x-t)+bZ_0(x-t)\\&=-a\frac{\delta(t-x)}{a}(-1)e^{\frac{b}{a}(x-t)}-a\frac{e(t-x)}{a}\cdot\frac{b}{a}e^{\frac{b}{a}(x-t)}\\&+b\cdot\frac{e(t-x)}{a}e^{\frac{b}{a}(x-t)}=\delta(x-t)e^{\frac{b}{a}(x-t)}=\delta(x-t)\end{aligned}$$

where $\delta(x-t)$ is Dirac Delta function.

On the other hand, we consider the classic solution of equation (13), having the following form:

$$y_0(x)=y_0(0)e^{-\frac{b}{a}x}$$

where $y_0(0)$ is unknown constant.

To sum up, we'll have the following theorem:

Theorem 3. Under condition 1^0, if boundary conditions (2) are linear-independent, then boundary value problem (1) and (2) will have boundary layer, if at least, one out of the following is not true:

$$y_0(0)\sum_{j=0}^{3}(-\frac{b}{a})^j[\alpha_{1j}^{(0)}+\alpha_{1j}^{(1)}e^{-\frac{b}{a}}]=\alpha_1$$

$$y_0(0)\sum_{j=0}^{3}(-\frac{b}{a})^j[\alpha_{2j}^{(0)}+\alpha_{2j}^{(1)}e^{-\frac{b}{a}}]=\alpha_2$$

$$y_0(0)\sum_{j=0}^{3}(-\frac{b}{a})^j[\alpha_{3j}^{(0)}+\alpha_{3j}^{(1)}e^{-\frac{b}{a}}]=\alpha_3$$

$$y_0(0)\sum_{j=0}^{3}(-\frac{b}{a})^j[\alpha_{4j}^{(0)}+\alpha_{4j}^{(1)}e^{-\frac{b}{a}}]=\alpha_4$$

because we have only one unknown constant $y_0(0)$. Whereas there are 4 relations above.

2. Case of no Boundary Layer

Now consider the following boundary value problem for the equation (1) with attached boundary conditions:

$$\begin{aligned}&\sum_{j=0}^{3}[\alpha_{1j}^{(0)}\cdot y_\varepsilon^{(j)}(0)+\alpha_{1j}^{(1)}\cdot y_\varepsilon^{(j)}(1)]=\alpha_1\\&y_\varepsilon(1)-e^{-\frac{b}{a}}y_\varepsilon(0)=0\\&ay_\varepsilon'(0)+by_\varepsilon(0)=0\\&ay_\varepsilon'(1)+by_\varepsilon(1)=0\end{aligned} \tag{14}$$

Notice that the last three of above boundary conditions have been written by necessary conditions for solution of equation (13). On the other hand, we can obtain these boundary conditions from boundary conditions (2), by choosing following values for coefficients $\alpha_{ij}^{(k)}$ of (2).

- for i=1, we have no changes on $\alpha_{ij}^{(k)}$.
- for i=2, we put $\alpha_{20}^{(0)}=-e^{-\frac{b}{a}}$, $\alpha_{20}^{(1)}=1$ and $\alpha_{21}^{(0)}=\alpha_{21}^{(1)}=\alpha_{22}^{(0)}=\alpha_{22}^{(1)}=\alpha_{23}^{(0)}=\alpha_{23}^{(1)}=\alpha_2=0$
- for i=3, we put $\alpha_{30}^{(0)}=b$, $\alpha_{31}^{(0)}=a$ and $\alpha_{30}^{(1)}=\alpha_{31}^{(1)}=\alpha_{32}^{(0)}=\alpha_{32}^{(1)}=\alpha_{33}^{(0)}=\alpha_{33}^{(1)}=\alpha_3=0$
- and for i=4, we put $\alpha_{40}^{(1)}=b$, $\alpha_{41}^{(1)}=a$ and $\alpha_{40}^{(0)}=\alpha_{41}^{(0)}=\alpha_{42}^{(0)}=\alpha_{42}^{(1)}=\alpha_{43}^{(0)}=\alpha_{43}^{(1)}=\alpha_4=0$

Now we return to the boundary value problem (1)-(14). It is easy to see, that to provide linear-independent of boundary conditions (14), the following is sufficient:

$$\begin{vmatrix}\alpha_{10}^{(0)}&\alpha_{10}^{(1)}&\alpha_{11}^{(0)}&\alpha_{11}^{(1)}\\-e^{-\frac{b}{a}}&1&0&0\\b&0&a&0\\0&b&0&a\end{vmatrix}=\alpha_{10}^{(0)}-\frac{b}{a}\alpha_{11}^{(0)}+e^{-\frac{b}{a}}(\alpha_{01}^{(1)}-\frac{b}{a}\alpha_{11}^{(1)})\neq 0 \tag{15}$$

Then the following theorem is obtained.

Theorem 4. Under conditions 1^0, (15), and

$$\sum_{j=1}^{3}(-\frac{b}{a})^j[\alpha_{1j}^{(0)}+\alpha_{1j}^{(0)}e^{-\frac{b}{a}}]\neq 0$$

then for boundary value problem (1)-(14), we don't have any boundary layer, that is, the limit of the solution when $\varepsilon \to 0$, satisfies boundary conditions (14). Indeed under conditions of this theorem, the unknown constant $y_0(0)$ is being found from first boundary condition and rest boundary conditions of (14) are satisfied automatically, because they are resulted from necessary conditions of solution of equation (13). Therefore, in this problem, we will not have any boundary layer.

Remark 2. The case $a < 0$ is being investigated in the same way. In this case we'll have to obtain new fundamental solution of adjoint equation (1) and roots of characteristic equation by another asymptotic expansion with respect to parameter ε.

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