SADDLE POINT VARIATIONAL METHOD FOR DIRAC CONFINEMENT

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Abstract

A saddle point variational (SPV) method was applied to the Dirac equation as an example of a fully relativistic equation with both negative and positive energy solutions. The effect of the negative energy states was mitigated by maximizing the energy with respect to a relevant parameter while at the same time minimizing it with respect to another parameter in the wave function. The Cornell potential and a power-law scalar and vector potentials were used in our calculations for the quark confinement. Cares were taken to avoid the Klein paradox by the dominance of the scalar component over the vector part. Two parameters variational method gives excellent and stable results. Our findings for the total energy per unit mass $(\frac{E}{m})$, relativistic magnetic moment $(\vec{r}\times\vec{a})$, electromagnetic energy for a unit charge $(\left\langle\frac{1}{r}\right\rangle)$ and magnetic moment of quarks were in good agreement with the exact solutions.

Introduction

During the last decades efforts have been focused on the problem of quark confinement. A crucial point is to assume that the quarks are confined as taken in bag models [1], where the Lorentz invariance is assumed. In first approximation the quarks may be considered in a central potential, behaving as independent particles [2]. In the various versions of the *QCD* motivated models, the confining potentials are taken as a mixture of Lorentz scalar and vector parts. This is because a Lorentz vector potential in the Dirac equation cannot confine the particles, and the confining component which is the consequence of multigluon exchange must be a scalar. Due to the relativistic motion of at least light quarks, a relativistic treatment is necessary. However, the Dirac equation with both negative and positive energy solutions may lead to the Klein paradox. This difficulty may be resolved by the dominance of the scalar component over the vector component in the Dirac equation [3].

In most cases the Dirac equation must be solved by a numerical integration. In this direction the approximate analytical methods are important. Due to existence of a definite lower bound to the energy spectrum, the variational methods are often applied in nonrelativistic bound state problems. However a simple version of this technique cannot be immediately applied to the

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relativistic Dirac equation. In this work we apply a saddle point variational method [4-6] to the fundamental state of Dirac equation.

Mathematical Scheme

The single particle Dirac equation is

$$H\Psi = [\vec{\alpha}.\vec{p} + (m + V_s)\beta + V_v]\Psi = E\Psi \tag{1}$$

where the particle is considered to be in the static Lorentz scalar (V_S) and vector (V_V) potentials. For the radial parts of the potentials we consider the general form

$$V_s = ar^{\mu}\ ,\ V_v = br^{\nu} \tag{2}$$

denoting the variational parameters by R and γ, the radial wave function for the stationary state of the confined quark is considered to be

$$\Psi = \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = \begin{pmatrix} \Psi_A \\ \gamma R(\vec{\sigma}.\vec{p})\Psi_A \end{pmatrix} \tag{3}$$

where

$$\Psi_A = N\exp\left[-\frac{1}{2}\left(\frac{r}{R}\right)^{\lambda}\right]\chi \tag{4}$$

and χ is the Pauli spinor. The free parameter λ which allows deviation from the hydrogen-type and the Gaussian wave functions is taken as fixed during the variational calculation, though it enables us to optimize the total energy at the end. By the normalization condition one may obtain

$$N = \left[\frac{4}{3}\pi R^3\Gamma\left(\frac{3}{\lambda}+1\right)(1+y)\right]^{-\frac{1}{2}} \tag{5}$$

where

$$y = \left(\frac{\gamma\lambda}{2}\right)^2 \frac{\Gamma\left(2+\frac{1}{\lambda}\right)}{\Gamma\left(\frac{3}{\lambda}\right)} \tag{6}$$

The expectation value of the Hamiltonian may be cast into the form

$$\langle H\rangle = \langle T\rangle + \langle m\beta\rangle + \langle V_s\beta\rangle + \langle V_v\rangle \tag{7}$$

where the kinetic term is given by

$$\langle T\rangle = \langle\vec{\alpha}.\vec{p}\rangle = \frac{2}{\gamma R}\frac{y}{(1+y)} \tag{8}$$

and for the potential terms we obtain

$$\langle V_s\beta\rangle = aR^{\mu}\frac{A-By}{1+y} \tag{9}$$

$$\langle V_v\rangle = bR^{\nu}\frac{A'+B'y}{1+y} \tag{10}$$

where

$$A = \frac{\Gamma\left(\frac{\mu+3}{\lambda}\right)}{\Gamma\left(\frac{3}{\lambda}\right)} \qquad B = \frac{\Gamma\left(2+\frac{1+\mu}{\lambda}\right)}{\Gamma\left(2+\frac{1}{\lambda}\right)}$$

$$A' = \frac{\Gamma\left(\frac{\nu+3}{\lambda}\right)}{\Gamma\left(\frac{3}{\lambda}\right)} \qquad B' = \frac{\Gamma\left(2+\frac{1+\nu}{\lambda}\right)}{\Gamma\left(2+\frac{1}{\lambda}\right)}.$$

Finally it can be shown

$$\langle m\beta\rangle = m\frac{1-y}{1+y} \tag{11}$$

Now we apply the *SPV* procedure to the equation (1). The variation of $\langle H\rangle$ with respect to the parameter R gives

$$\frac{\partial\langle H\rangle}{\partial R} = -\frac{1}{R}[\langle T\rangle - \mu\langle V_s\beta\rangle - \nu\langle V_v\rangle] \tag{12}$$

For the critical point condition $\left(\frac{\partial\langle H\rangle}{\partial R}\right)_{R=R_0,\gamma=\gamma_0} = 0$, we get

$$\langle T\rangle_0 = \mu\langle V_s\beta\rangle_0 + \nu\langle V_v\rangle_0 \tag{13}$$

which is in fact nothing more than the relativistic Virial theorem with the general form of [7]

$$\langle T\rangle = \langle \vec{r}.\vec{\nabla} V\rangle \tag{14}$$

In (13), the null subscripts stand for $R = R_0$ and $\gamma = \gamma_0$, the critical point of $\langle H\rangle$. The variation of $\langle H\rangle$ with respect to the parameter γ yields to

$$\frac{\partial\langle H\rangle}{\partial\gamma} = \frac{1}{\gamma(1+y)}\Big[(1+y)\langle T\rangle - 2y\frac{1+D}{1-Dy}\langle V_s\beta\rangle - 2y\frac{1-D'}{1+D'y}\langle V_v\rangle - \frac{4my}{1+y}\Big] \tag{15}$$

where

$$D = \frac{B}{A} \quad D' = \frac{B'}{A'} \tag{16}$$

By a little algebra for the condition $\left(\frac{\partial\langle H\rangle}{\partial\gamma}\right)_{R=R_0,\gamma=\gamma_0} = 0$ we obtain

$$(1-y_0^2)\langle T\rangle_0 - 2y_0(1+y_0)\left[\frac{1-D'}{1+D'y_0}\langle V_v\rangle_0 + \frac{1+D}{1-Dy_0}\langle V_s\beta\rangle_0\right] - 4my_0 = 0 \tag{17}$$

Finally the critical point (R_0, γ_0) may be obtained by solving the coupled equations (13) and (17). The sufficient condition for the critical point (R_0, γ_0) to be a saddle point is given by [8]

$$\left(\frac{\partial^2\langle H\rangle}{\partial R^2}\frac{\partial^2\langle H\rangle}{\partial\gamma^2} - \left(\frac{\partial^2\langle H\rangle}{\partial R\partial\gamma}\right)^2\right)_0 \langle 0 \tag{18}$$

For the potentials of the type (2) we find

$$\left(\frac{\partial^2\langle H\rangle}{\partial R^2}\right)_0 = \frac{1}{R_0^2}\left[\langle T\rangle_0 - \mu^2\langle V_s\beta\rangle_0 - \nu^2\langle V_v\rangle_0\right] \tag{19}$$

$$\left(\frac{\partial^2\langle H\rangle}{\partial\gamma^2}\right)_0 = \frac{1}{\gamma_0^2(1+y_0)^2} \times\left[-(y_0^2 - 4y_0 + 1)\langle T\rangle_0 - \frac{4(1+D)y_0}{1-Dy_0}\langle V_s\beta\rangle_0 - \frac{4(1-D')y_0}{1+D'y_0}\langle V_v\rangle_0 - \frac{8my_0}{1+y_0}\right] \tag{20}$$

The simplest *QCD*-motivated potential has the form

$$V = V_s + V_v \tag{21}$$

where

$$V_s = K_s m^2 r \qquad V_v = -\frac{K_v}{r} \tag{22}$$

known as the Cornell potential. Compared with the equation (2), it can be observed that

$$a \equiv K_s m^2 \quad and \quad b \equiv -K_v \qquad \mu \equiv 1 \quad and \quad \nu \equiv -1. \tag{23}$$

Hence for this special case, (13) yields

$$\langle V_s\beta\rangle_0 = \langle T\rangle_0 + \langle V_v\rangle_0 \tag{24}$$

The application of (24) in (17) results in

$$R_0 = \frac{1}{2m}\left[\frac{1}{\gamma_0}\left(1 - y_0 - 2y_0\frac{1+D}{1-Dy_0}\right) + K_v(A' + B'y_0)\left(\frac{1-D'}{1+D'y_0} + \frac{1+D}{1-Dy_0}\right)\right] \tag{25}$$

Meanwhile, taking into account the results of the expectation values obtained so far, (24) directly gives

$$\frac{2y_0}{\gamma_0} - K_v(A' + B'y_0) - K_s m^2 R_0^2 (A - By_0) = 0. \tag{26}$$

From the relations (25) and (26) we derive

$$\frac{K_s}{4}(A - By_0)\left[\frac{1}{\gamma_0}\left(1 - y_0 - 2y_0\frac{1+D}{1-Dy_0}\right)\right.$$
$$\left. + K_v(A' + B'y_0)\left(\frac{1-D'}{1+D'y_0} + \frac{1+D}{1-Dy_0}\right)\right]^2 \qquad (27)$$
$$+ K_v(A' + B'y_0) - \frac{2y_0}{\gamma_0} = 0.$$

We solve this equation for γ_0 and hence R_0 may be obtained from (25). It is remarkable that by the form of the constant a introduced in (23), the quantity $\frac{\langle H\rangle_0}{m}$ is independent of m. This point may be easily verified by application of (24) in (7) and m-dependency form of R_0 in (25). Putting everything together we get

$$\langle H\rangle = \frac{2}{R_0(1+y_0)}\left[\frac{2y_0}{\gamma_0} - K_v(A' + B'y_0)\right] + m\frac{1-y_0}{1+y_0} \qquad (28)$$

An alternative potential which is commonly used in quark confinement is power-law potential. Allowing $\mu = \nu$, and noting that in this case $A' = A$ and $B' = B$, the relations (9) and (10) lead to

$$\langle V\rangle = \langle V_s\beta\rangle + \langle V_v\rangle = R^{\mu}\frac{(a+b)A - (a-b)By}{1+y} \qquad (29)$$

The application of the *SPV* method leads to

$$\langle T\rangle_0 = \mu\langle V\rangle_0 \qquad (30)$$

and

$$\left(1 - y_0^2\right)\langle T\rangle_0$$
$$- 2y_0(1+y_0)\frac{1+D''}{1-D''y_0}\langle V\rangle_0 - 4my_0 = 0 \qquad (31)$$

where

$$D'' = -\frac{a-b}{a+b}\frac{B}{A} \qquad (32)$$

The coupled equations (30) and (31) may be reduced to

$$R_0 = \frac{1}{2m\gamma_0}\left(1 - y_0 - \frac{2y_0}{\mu}\frac{1+D''}{1-D''y_0}\right) \qquad (33)$$

$$\left(\frac{1-y_0}{2m\gamma_0} - \frac{y_0}{m\mu\gamma_0}\frac{1+D''}{1-D''y_0}\right)^{\mu+1} = \frac{2y_0}{\mu\gamma_0}\frac{1}{(a+b)A - (a-b)By_0} \qquad (34)$$

These are sister equations for (25) and (27) to be solved for R_0 and γ_0 in the case of power-law potential.

Magnetic Moment of Quarks

In electrodynamics the magnetic moment $\vec{\mu}$ due to an electric current distribution $\vec{J}_e(\vec{r})$ is given by

$$\vec{\mu} = \frac{1}{2}\int\left[\vec{r} \times \vec{J}_e(\vec{r})\right]d^3r\ . \qquad (35)$$

The electric current carried by a quark of charge e_q in the state Ψ_{jm} with a vector current of $J^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi \equiv (J^0, \vec{J})$ is

$$\vec{J}_e(\vec{r}) = e_q\Psi^+_{jm}\vec{\alpha}\Psi_{jm} = e_q\vec{J}(\vec{r}) \qquad (36)$$

where

$$\Psi_{jm}(\vec{r}) = \begin{pmatrix} g_k(r)Y^m_{jl} \\ i f_k(r)Y^m_{jl'} \end{pmatrix} \qquad (37)$$

For the ground state we have [9]

$$\mu = e_q\int_0^{\infty}\mu(r)dr \qquad (38)$$

where $\mu(r)$, the scalar magnetization density is given by

Table 1. *SPV* and the exact [5] results for Cornell potential

						$\frac{E}{m}$		$\frac{\langle \frac{1}{r} \rangle}{m}$		$\langle \vec{r}\times\vec{\alpha} \rangle$	
Group	λ	a	b	γ_0	$\frac{R_0}{m}$	SPV	exact	SPV	exact	SPV	exact
1	1.00			0.63	0.60	1.111		0.833		0.692	
	1.20	0.1	0.4	0.46	0.85	1.108	1.108	0.793	0.79	0.706	0.713
	2.00			0.24	1.6	1.128		0.684		0.718	
2	1.00			1.11	0.27	0.705		1.849		0.458	
	1.06	0.2	0.8	0.98	0.32	0.720	0.720	1.735	1.82	0.478	0.543
	2.00			0.39	0.92	0.861		1.153		0.583	
3	1.00			0.76	0.38	1.436		1.301		0.511	
	1.30	0.4	0.4	0.49	0.62	1.423	1.423	1.202	1.20	0.525	0.525
	2.00			0.29	1.02	1.447		1.046		0.528	
4	1.00			1.19	0.19	0.902		2.678		0.328	
	1.09	0.8	0.8	1.00	0.23	0.934	0.934	2.448	2.57	0.349	0.384
	2.00			0.43	0.60	1.120		1.715		0.409	
5	1.00			0.87	0.22	2.155		2.155		0.328	
	1.35	1.6	0.4	0.52	0.38	2.116	2.12	2.021	2.01	0.338	0.336
	2.00			0.33	0.59	2.146		1.810		0.338	
6	1.00			1.27	0.11	1.352		4.423		0.204	
	1.10	3.2	0.8	1.04	0.14	1.405	1.41	4.013	4.15	0.218	0.236
	2.00			0.45	0.36	1.690		2.877		0.251	

$$\mu(r) = \frac{2}{3} r^3 f_{-1}(r) g_{-1}(r) \tag{39}$$

and for our trial wave function

$$g_{-1}(r) = \sqrt{4\pi}\, N_0 \exp\left[-\frac{1}{2}\left(\frac{r}{R_0}\right)^{\lambda} \right] \tag{40}$$

$$f_{-1}(r) = \sqrt{4\pi}\, N_0 \frac{\gamma_0 \lambda}{2}\left(\frac{r}{R_0}\right)^{\lambda-1} \times \exp\left[-\frac{1}{2}\left(\frac{r}{R_0}\right)^{\lambda} \right] \tag{41}$$

In this case we get

$$\mu_q = e_q \frac{\gamma_0 R_0}{1+y_0}. \tag{42}$$

Results and Conclusion

Our results for the Cornell potential are shown in Table 1. We have applied the *SPV* method [4,5] to the Hamiltonian (1) with the Gaussian and hydrogen-type wave functions. However the parameter λ enables us to consider deviations from these wave functions. The present results for $\lambda = 1$ and $\lambda = 2$ are generally in agreement with the SPV results of ref. [5]. The critical point of $\langle H \rangle$, i.e. (R_0, γ_0), has been calculated for different sets of potential parameters *a, b* and the free parameter of the wave function λ. We have tested the saddle point condition (18) for each set of these parameters. The *SPV* results for total energy per unit mass, the relativistic magnetic moment $\langle \vec{r}\times\vec{\alpha} \rangle$ in units of quark magneton $(\frac{q}{2m})$ and the electromagnetic energy for a unit charge $\langle \frac{1}{r} \rangle$ are compared with corresponding results based on exact calculations. These quantities have been obtained for six different sets of the potential parameters. In the second item of each group we have optimized the energy by variation of the free parameter λ. With this choice of λ, the other quantities are also generally in better agreement with the exact results, compared with the results of the Gaussian and hydrogen-type trial wave functions. While the ratio of the scalar potential coefficient to the vector potential coefficient in the first two groups is $\frac{1}{4}$, in the second group the strength of the potentials has been raised by a factor of 2. This causes a reduction in $\frac{\mathrm{E}}{\mathrm{m}}$ and $\langle \vec{r}\times\vec{\alpha} \rangle$ while the electromagnetic energy for a unit charge increases. In the last two groups, this ratio has been changed to 4, however the strengths of the coefficients are different. A comparison of the results obtained for the different groups shows that for a fixed ratio of the potential coefficients, simultaneous changes in the strengths of the potentials causes similar changes in physical quantities in these groups. For example the diminution of the quantity $\frac{E}{m}$ in the second group compared with the first group is about %35 and in the last group compared with the preceding group is about %34. However this situation is demolished when the ratio is altered.

The results for the energy and the magnetic moment of the quarks are displayed in Table 2. The calculations have been carried on for the power law potential with $a = b = 1.137\ fm^{-3}$, $\mu = \nu = 2$, $m_u = 10Mev$ and m_s = 252*Mev*. The results are remarkable.

The agreement between the results of two-parameter variational method and the exact [5,10] solutions is encouraging. We conclude that one may rely on the results of the *SPV* method in two and three-body relativistic bound state problems where the analytical solutions do not exist or where the reduced equations indicate ambiguity in results [11]. In particular if a suitable choice of the wave function is used, the *SPV* method will lead to remarkable results.

Table 2. *SPV* and exact [10] results for the energy and the magnetic moment ratios of the quarks for power-law potential ($\lambda = 2$)

			$E(Mev)$		$\mu(\mu_N)$	
Quark	γ_0	R_0	SPV	exact	SPV	exact
U	0.57	0.63	542	540	1.6	1.53
S	0.38	0.56	665	664	- 0.59	-0.56

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