

UNIQUENESS OF SOLUTION FOR A CLASS OF STEFAN PROBLEMS

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Abstract

This paper deals with a theoretical mathematical analysis of one-dimensional solidification problem, in which kinetic undercooling is incorporated into the temperature condition at the interface. A model problem with nonlinear kinetic law is considered. We prove a local result intimate for the uniqueness of solution of the corresponding free boundary problem.

Keywords: Free boundary problem; Nonlinear integral equation

Introduction

It is well known that in many industrial areas, the solidification process plays a significant role. Mathematical models of solidification including interface kinetics effects have been considered for quite some time (see [1], and references therein). This class of free boundary problems, which arises in a number of physical situations, is that of on equilibrium problems, in which the phase - change temperature is dependent on the velocity of the front at which the phase-change occurs (for more physical problems, see [3-7]). Here, we study a model problem with nonlinear kinetic law at the interface in the one-dimensional case. Specifically, let the curve with $s(0)=b(0<b<1)$ be defined as the interface that separates the liquid and solid phases. With u denoting temperature (scaled so that it vanishes at equilibrium), we may write the system of equations as

$$u_t = K_l u_{xx} \text{ in } Q_1 = \{(x, t) | 0 < x < s(t), 0 < t \leq T\}, \quad (1.1)$$

$$u_t = K_s u_{xx} \text{ in } Q_1 = \{(x, t) | 0 < x < s(t), 0 < t \leq T\}, \quad (1.2)$$

and on the interface $x = s(t)$ as

$$u_1 = u_2 = g_1(V(f)), \quad (1.3)$$

$$Ku_x^+ - Ku_x^- = g_2(V(t)), \quad (1.4)$$

$$s(0) = b, \quad 0 < b < 1, \quad (1.5)$$

where K_l and K_s are thermal diffusivities of a liquid and a solid respectively, $L > 0$ is the latent heat and the superscripts $+$ and $-$ denote, respectively the right-hand and left-hand limits with respect to the special variable x . These equations are subject to the initial and boundary conditions

$$u(x,0) = \varphi_1(x) \quad 0 \leq x \leq b, \quad (1.6)$$

$$u(x,0) = \varphi_2(x) \quad b \leq x \leq 1, \quad (1.7)$$

$$u(i-1,t) = f_i(t) \quad t \geq 0, \quad (i=1,2) \quad (1.8)$$

where

$$V(t) = \frac{ds(t)}{dt} \quad (1.9)$$

is the propagation velocity of the free boundary. The free boundary problem considered here was formulated in [1], where reduction the problem to an integral equation was given. In the context of solid fuel combustion, $s(t)$ represents the boundary between the unburnt and burnt material, and u_1, u_2 , are the nondimensionalized temperature in the unburnt and burnt material respectively, (see [3-7] and references therein). The temperature at the free boundary controls its velocity $V(t) = g_1^{-1}(u_1(s(t), t))$. The heat exchange between the unburnt ($x < s(t)$) and burnt material is modeled by the boundary condition in (1.4) which, in principle, may be nonlinear.

Main Results

Theorem. Consider the problem (1.1)-(1.9). Suppose that the kinetic function and initial and boundary data satisfy the assumption $(H_1)-(H_3)$ in [1]. Then the problem (1.1)-(1.9) has not more than one solution.

To prove uniqueness for $t < \sigma$ suppose that $u_0 = (u_{01}, u_{02}), s_0$ is another solution of (1.1)-(1.9) for $t < \sigma$ and $v_0(t) = (v_{01}(t), v_{02}(t))^T$ is another solution of integral equations (26) and (27) in [1]. It suffices to prove uniqueness, for any $\bar{\sigma} < \sigma$.

Let

$$\bar{M} = \max\{M, l.u.b|_{0 \leq t \leq \bar{\sigma}} v_0(t)\}$$

where M introduced in section 4.2 in [1], and let be any positive number satisfying

$$\sqrt{\bar{\sigma}} <$$

$$\min\left\{\left(C_2\|\varphi_1'\| + C_3\|f_1'\| + C_4 + C_5\bar{M} + C_6 + C_7\right)^{-1} \frac{\bar{M}}{2}, \right. \\ \left. \left(D_2 + D_3 + D_4\bar{M} + D_5 + D_6\|\varphi_2'\| + D_7\|f_2'\|\right)^{-1} \frac{\bar{M}}{2}\right\}$$

where the constants C_i and D_i , $i=2,3,\dots,7$ are simple combination of $\pi, b, \frac{1}{b}, M, M', M_2, M_2', K$. Then by the

same calculations in [1] which were used to prove that T maps $B_{M,\sigma}$ into itself (where T and $B_{M,\sigma}$ introduced in subsection 4.2 in [1]) and is a contraction one shows that T maps $B_{\bar{M},\bar{\sigma}}$ into itself and is a contraction. Hence, there exists at most one fixed point of T in $B_{\bar{M},\bar{\sigma}}$. It follows that $v(t) = v_0(t)$ for $0 \leq t \leq \bar{\sigma}$, where $v(t)$ is solution of integral equations (26) and (27) in [1]. Hence also $s(t) = s_0(t)$, $u(x,t) = u_0(x,t)$ if $0 \leq t \leq \bar{\sigma}$, $0 \leq x \leq s(t)$ and $s(t) \leq x \leq 1$. We next consider the system (1.1)-(1.9) for $t > \bar{\sigma}$, i.e. (1.1)-(1.5), (1.8), (1.9) are considered for $t > \bar{\sigma}$ (instead of $t \geq 0$) where as (1.6), (1.7) are replaced by $u_1(x, \bar{\sigma}) = u_1(x, \bar{\sigma})$ for $0 \leq x \leq s(\bar{\sigma})$, $u_2(x, \bar{\sigma}) = u_2(x, \bar{\sigma})$ for $s(\bar{\sigma}) < x < 1$.

This problem can again be transformed into integral equations (26), (27) in [1] extend to the present integral equation provided M is replaced by M_0 where

$$M_0 = \frac{l.u.b}{\sigma < t < \bar{\sigma}} |V(t)g_1(V(t))|$$

Similarly to section 4 in [1], we reduce the problem (1.1)-(1.9) for u_0, s_0 in the interval $\bar{\sigma} \leq t < \sigma$ to an integral equation. Since $u_1(x, \bar{\sigma}) = u_1(x, \bar{\sigma})$, $u_2(x, \bar{\sigma}) = u_2(x, \bar{\sigma})$, the integral equation for $v(t)$ and $v_0(t)$ coincide. Repeating now the same argument as before we conclude that for $v(t) = v_0(t)$ for any $\bar{\sigma}$ satisfying

$$\sqrt{(\bar{\sigma} - \bar{\sigma})} <$$

$$\min\left\{\left(C_2\|\varphi_1'\| + C_3\|f_1'\| + C_4 + C_5\bar{M}_0 + C_6 + C_7\right)^{-1} \frac{\bar{M}_0}{2}, \right. \\ \left. \left(D_2 + D_3 + D_4\bar{M}_0 + D_5 + D_6\|\varphi_2'\| + D_7\|f_2'\|\right)^{-1} \frac{\bar{M}_0}{2}\right\}$$

$$\overline{M}_0 = \max\{M_0, \text{l.u.b}_{\overline{\sigma} \leq t \leq \sigma} |v_0(t)|\}$$

We can now proceed in the same manner as before in [2] step by step, nothing that in each step the time interval can be taken to be $\geq \varepsilon$ where satisfies

$$\sqrt{\varepsilon} <$$

$$\min\left\{\left(C_2\|\phi_1\| + C_3\|f_1'\| + C_4 + C_5\overline{M}_1 + C_6 + C_7\right)^{-1} \frac{\overline{M}_1}{2},\right.$$

$$\left.\left(D_2 + D_3 + D_4\overline{M}_1 + D_5 + D_6\|\phi_2'\| + D_7\|f_2'\|\right)^{-1} \frac{\overline{M}_1}{2}\right\}$$

where

$$\overline{M}_1 = \max\{\text{l.u.b}_{\overline{\sigma} < t < \sigma} |V(t)g_1(V(t))|, \text{l.u.b}_{\overline{\sigma} < t < \sigma} |v_0(t)|\}$$

Having proved existence and uniqueness for all $t < \sigma$ where σ is any positive number satisfying (36) in [1]. Let us stress that the previous proof (see (38), (39) in [1]) shows also the following:

If instead of (1.1)-(1.9) for $t > 0$ we consider (1.1)-(1.9) for $t > \lambda$, i.e., (1.1)-(1.5), (1.8), (1.9) hold for $t > \lambda$ and (1.6), (1.7) replaced by $u_1(x, \lambda) = u_1(x, \lambda)$ for $0 < x < s(\lambda)$ and $u_2(x, \lambda) = u_2(x, \lambda)$ for $s(\lambda) < x \leq 1$ respectively, and if

$$|V(\lambda)g_1(V(\lambda))|, s(\lambda), \frac{1}{s(\lambda)}$$

are bounded independently of λ , then there exists a unique solution for the problem in an interval $\lambda \leq t \leq \lambda + \varepsilon$, where ε is some positive number independent of λ .

Since for any solution of (1.1)-(1.9) the function $s(t)$ is monotone non-decreasing, $\frac{1}{s(\lambda)} \leq \frac{1}{b}$. To complete

the proof of theorem it suffices to prove the following statement:

For every $t_0 > 0$ there exists an $\varepsilon > 0$ such that if the system (1.1)-(1.9) has a unique solution for all $t < t_0$, then it also has a unique solution for all $t < t_0 + \varepsilon$ in view of the previous remarks it suffices to show: If

$u(x, t), s(t)$ is a solution of (1.1)-(1.9) for all $t < t_0$, then for all $\eta > 0$ sufficiently small, the functions

$$\text{l.u.b}|V(t_0 - \eta)g_1(V(t_0 - \eta))|, s(t_0 - \eta) \quad (2.1)$$

are bounded independently of η . If we prove that

$$\text{l.u.b}|v(t)| < \infty,$$

then from (28) in [1] follows the boundedness of $s(t)$ for $t < t_0$. Consequently, if we prove (2.2) then the proof of theorem is completed.

Proof of (2.2). We use for $v(t)$ the integral equation which corresponds to the system (1.1)-(1.9) in the interval $t_0 - \mu < t < \mu$ (μ sufficiently small) in [1]. Since

$$u_1(0, t_0 - \mu) = f_1(t_0 - \mu), u_2(0, t_0 - \mu) = f_2(t_0 - \mu),$$

the equations are

$$v_1(t) = -g_1(V(t))V(t) +$$

$$2 \int_0^{s(t_0 - \mu)} u_{1\xi}(\xi, t_0 - \mu) N(s(t), t; \xi, t_0 - \mu) d\xi$$

$$+ \int_{t_0 - \mu}^t v_1(\tau) G_x(s(t), t; s(\tau), \tau) d\tau$$

$$- 2 \int_{t_0}^t f_1'(\tau) N(s(t), t; 0, \tau) d\tau +$$

$$2g_1(V(t_0 - \mu))N(s(t), t; s(t_0 - \mu), t_0 - \mu)$$

$$+ 2 \int_{t_0 - \mu}^t g_1'(V(\tau))N(s(t), t; s(\tau), \tau) d\tau$$

$$- 2 \int_{t_0 - \mu}^t g_1(V(\tau))V(\tau)G_x(s(t) - 1, t; s(\tau), \tau) d\tau$$

$$\begin{aligned}
v_2(t) = & -g_1(V(t))V(t) \\
& -2g_1(V(t_0 - \mu))N(s(t) - 1, t; s(\tau) - 1, t_0 - \mu) \\
& + \int_{t_0 - \mu}^t g_1'(V(\tau))N(s(t) - 1, t; s(\tau) - 1, \tau) d\tau \\
& - 2 \int_{t_0 - \mu}^t v_2(\tau) G_x(s(t) - 1, t; s(\tau) - 1, \tau) d\tau \\
& + 2 \int_{s(t_0 - \mu)}^1 u_{2\xi}(\xi, t_0 - \mu) N(s(t) - 1, t; \xi, t_0 - \mu) d\xi \\
& + \int_{t_0}^t f_2'(\tau) N(s(t) - 1, t; 0, \tau) d\tau
\end{aligned}$$

In section 4 in [1] we proved $v_1(t)$ and $v_2(t)$ are bounded functions, we obtain that

$$\lim_{0 < t < t_0} |v(t)| < \infty$$

therefore we established theorem.

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