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MARCINKIEWICZ-TYPE STRONG LAW OF LARGE NUMBERS FOR DOUBLE ARRAYS OF NEGATIVELY DEPENDENT RANDOM VARIABLES

H.A. Azarnoosh*

Department of Statistics, School of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Islamic Republic of Iran

Abstract

In the following work we present a proof for the strong law of large numbers for pairwise negatively dependent random variables which relaxes the usual assumption of pairwise independence. Let $\{X_{ij}\}$ be a double sequence of pairwise negatively dependent random variables. If $P\{|X_{ij}| \geq t\} \leq P\{|X| \geq t\}$ for all non-negative real numbers t and $E|X|^p \log^+|X| < \infty$, for 1 < p < 2, then we prove that $\frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - EX_{ij})}{(mn)^{1/p}} \to 0$ *a.e. as* $m \vee n \to \infty$ (1). In addition, it also converges to 0 in L^1. The results can be generalized to an r-dimensional array of random variables under condition $E|X|^p (\log^+|X|)^{r-1} < \infty$, thus, extending Choi and Sung's result [7] of one dimensional case for negatively dependent random variables.

Keywords: Strong law of large numbers; Negatively dependent

Introduction

The history and literature on laws of large numbers is vast and rich as this concept is crucial in probability and statistical theory. The literature on concepts of negative dependence is much more limited but still very interesting. Lehmann (1966) provides an extensive introductory overview of various concepts of positive and negative dependence in the bivariate case [12]. Negative dependence has been particularly useful in obtaining laws of large numbers [1,3-5,13]. We have extended, a novel argument of Etemadi (1981) for pairwise negatively dependent random variables [2], *i.e.*, if $\{X_n\}$ is a sequence of pairwise negatively dependent identically distributed random variables with $E|X| < \infty$, then

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* *E-mail: azarnush@math.um.ac.ir*

$$\frac{\sum_{i=1}^{n}(X_i - EX_i)}{n} \to 0 \quad a.e. \tag{2}$$

Also for a double sequence $\{X_{ij}\}$ of pairwise negatively dependent random variables we have proved that if $E|X_{11}|\log^+|X_{11}| < \infty$, then

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - EX_{ij})}{mn} \to 0 \;\; a.e. \;\; as \;\; m \vee n \to \infty . \tag{3}$$

In 1985 Choi and Sung [7] have shown that if $\{X_n\}$ are pairwise independent and dominated in distribution by a random variable X with $E|X^p|(\log^+|X|)^2 < \infty$, for $1 < p < 2$, then $\frac{\sum_{i=1}^{n}(X_i - EX_i)}{n^{1/p}} \to 0$ *a.e.* In addition, if $E|X|^p < \infty$, then it converges to 0 in L^1.

In 1999 Hong and Hwang [11] extended the Choi and Sung's result of the one-dimensional case to a multi-dimensional array of pairwise independent random variables. They have proved (1) under the strong condition $E|X^p|(\log^+|X|)^3 < \infty$.

Now, we are interested to extend the Hong and Hwang's result under the weaker condition $E|X^p|\log^+|X| < \infty$ for pairwise dependent random variables.

In the following we present some background materials on negative dependence which will be used in obtaining the SLLN in the next section.

Definition. Random variables X and Y are negatively dependent (ND) if

$$P\{X \le x; Y \le y\} \le P\{X \le x\}P\{Y \le y\} \tag{4}$$

for all x, y ∈ R. A collection of random variables is said to be pairwise ND (PND) if every pair of random variables in the collection satisfies (4).

The following properties are listed for reference in obtaining the main result in the next section. Detailed proofs can be founded in [9].

Lemma 1. If $\{X_n; n \ge 1\}$ is a sequence of ND random variables, then

(a) $E(X_i Y_j) \le E(X_i)E(Y_j)$, $i \ne j$

(b) $Cov(X_i, Y_j) \le 0$, $i \ne j$.

Lemma 2. If $\{X_n; n \ge 1\}$ is a sequence of ND random variables, and $\{f_n\}$ is a sequence of monotone increasing, (or monotone decreasing) Borel functions, then $\{f_n(X_n)\}$ is a sequence of ND random variables.

Corollary 1. Let $\{X_n; n \ge 1\}$ be a sequence of ND random variables then $\{X_n^+; n \ge 1\}$ and $\{X_n^-; n \ge 1\}$ are.

Corollary 2. Let $\{X_n; n \ge 1\}$ be a sequence of ND random variables and $Y_i = X_i I_{\{X_i \le i\}} + i I_{\{X_i > i\}}$, where I is an indicator function. Then $\{Y_i\}$ is a sequence of ND random variables.

Main Results

Let $\{X_{ij}\}$ be a double sequence of pairwise negatively dependent random variables and $Y_{ij} = X_{ij} I_{\{X_{ij} \le (ij)^{1/p}\}} + (ij)^{1/p} I_{\{X_{ij} > (ij)^{1/p}\}}$ and $Z_{ij} = X_{ij} - Y_{ij}$, for $1 < p < 2$. Throughout this paper, c denotes an unimportant positive constant which is allowed to changes and d_k is the number of all divisors of integer k.

To prove the main theorems, we need the following lemma.

Lemma 3. Let $\{X_{ij}\}$ be a double sequence of pairwise negatively dependent random variables. If $P\{|X_{ij}| \ge t\} \le P\{|X| \ge t\}$ for all non-negative real numbers t, then

(a) $$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|Y_{ij}|^2}{(ij)^{2/p}} \le cE|X|^p \log^+|X|,$$

(b) $$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|Z_{ij}|}{(ij)^{1/p}} \le cE|X|^p \log^+|X| \quad for \quad 1 < p < 2 \tag{5}$$

Proof. By Corollary 1 $\{X_{ij}^+\}$ and $\{X_{ij}^-\}$ are sequences of PND random variables so they satisfy the assumption of the lemma and $X_{ij} = X_{ij}^+ - X_{ij}^-$. Thus without loss of generality we can assume that $X_{ij} \geq 0$. The estimation of EY_{ij}^2 is given by

$$EY_{ij}^2 \leq \int_0^{(ij)^{1/p}} x^2 dF(x) + \int_{(ij)^{1/p}}^{\infty} dF(x)$$

$$= \int_0^{(ij)^{1/p}} x^2 dF(x) + (ij)^{2/p} P\left[X > (ij)^{1/p}\right]. \qquad (6)$$

Where F(x) is the distribution of X. If we use the fact that $\sum_{k=i+1}^{\infty} \frac{d_k}{k^{2/p}} = O\left(\frac{\log i}{(i+1)^{2/p-1}}\right)$ [11], we obtain

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{(ij)^{2/p}} \int_0^{(ij)^{1/p}} x^2 dF(x)$$

$$\leq \sum_{k=1}^{\infty} \frac{d_k}{k^{2/p}} \int_0^{k^{1/p}} x^2 dF(x)$$

$$\leq c \sum_{i=0}^{\infty} \left(\sum_{k=i+1}^{\infty} \frac{d_k}{k^{2/p}} \right) \int_{i^{1/p}}^{(i+1)^{1/p}} x^2 dF(x)$$

$$\leq c \sum_{i=0}^{\infty} \frac{\log i}{(i+1)^{2/p-1}} \int_{i^{1/p}}^{(i+1)^{1/p}} x^2 dF(x)$$

$$\leq c \sum_{i=0}^{\infty} \int_{i^{1/p}}^{(i+1)^{1/p}} x^p \log x^p dF(x)$$

$$\leq cE|X|^p \log^+ |X| < \infty . \qquad (7)$$

and

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P\left[(ij)^{1/p} < X\right] = \sum_{k=1}^{\infty} d_k P\left[k < X^p\right]$$

$$= \sum_{k=1}^{\infty} d_k \sum_{i=k}^{\infty} P\left[i < X^p < i+1\right]$$

$$= \sum_{i=1}^{\infty} \left(\sum_{k=1}^{i} d_k \right) P\left[i < X^p < i+1\right]$$

$$\leq c \sum_{i=1}^{\infty} i \log i P\left[i < X^p < i+1\right]$$

$$= c \sum_{i=1}^{\infty} i \log i \int_{i^{1/p}}^{(i+1)^{1/p}} dF(x)$$

$$\leq c \sum_{i=1}^{\infty} \int_{i^{1/p}}^{(i+1)^{1/p}} \left(x^p \log x^p\right) dF(x)$$

$$\leq cE|X|^p \log^+ |X| < \infty , \qquad (8)$$

where we use the fact that $\sum_{k=1}^{n} d_k = O(n \log n)$. It follows that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\left|Y_{ij}\right|^2}{(ij)^{2/p}} \leq cE|X|^p \log^+ |X| < \infty , \qquad (9)$$

which proves (a), since $\left|X_{ij}\right| = X_{ij}^+ + X_{ij}^-$

Now, the estimation of EZ_{ij} is given by

$$EZ_{ij} = E\left(X_{ij} - Y_{ij}\right)$$

$$\leq \int_{(ij)^{1/p}}^{\infty} x dF(x) - \int_{(ij)^{1/p}}^{\infty} dF(x)$$

$$\leq \int_{(ij)^{1/p}}^{\infty} x dF(x) \qquad (10)$$

By the fact that $\sum_{i=1}^{n} d_k / k^{1/p} = O\left(n^{1-(1/p)} \log n\right)$, we can obtain (b) as follows

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(ij)^{1/p}} \int_{(ij)^{1/p}}^{\infty} x dF(x)$$

$$= \sum_{k=1}^{\infty} \frac{d_k}{k^{1/p}} \sum_{i=k}^{\infty} \int_{i^{1/p}}^{(i+1)^{1/p}} x dF(x)$$

$$= \sum_{i=1}^{\infty} \left(\sum_{k=1}^{i} \frac{d_k}{k^{1/p}} \right) \int_{i^{1/p}}^{(i+1)^{1/p}} x dF(x)$$

$$= c\sum_{i=1}^{\infty}\left(i^{1-(1/p)}\log i\right)\int_{i^{1/p}}^{(i+1)^{1/p}} x dF(x)$$

$$\le c\sum_{i=1}^{\infty}\int_{i^{1/p}}^{(i+1)^{1/p}} (x^p \log x^p) dF(x)$$

$$\le cE|X|^p \log^+|X| < \infty . \quad (11)$$

Theorem 1. Let $\{X_{ij}\}$ be a double sequence of pairwise negatively dependent random variables. If $P\{|X_{ij}| \ge t\} \le P\{|X| \ge t\}$ for all non-negative real numbers t and $E|X|^p \log^+|X| < \infty$, for $1 < p < 2$, and $S_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$, then

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{ij} - EX_{ij}\right)}{(mn)^{1/p}} \to 0 \ \textit{a.e. as } m \vee n \to \infty . \quad (12)$$

Proof. We shall follow the proof of Lemma 3, then without loss of generality we assume that $X_{ij} \ge 0$. Let $Y_{ij} = X_{ij} I_{\{X_{ij} \le (ij)^{1/p}\}} + (ij)^{1/p} I_{\{X_{ij} > (ij)^{1/p}\}}$ and $S_{mn}^* = \sum_{i=1}^{m}\sum_{j=1}^{n} Y_{ij}$.
If we let d_k to be the number of divisors of k *i.e.* the cardinality of $\{(i, j) : ij = k\}$ and F(x) be the distribution of X, then we obtain

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\{Y_{ij} \ne X_{ij}\} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\{X_{ij} > (ij)^{1/p}\}$$

$$\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\{X > (ij)^{1/p}\}$$

$$\le \sum_{k=1}^{\infty} d_k P\{X > k^{1/p}\}$$

$$= \sum_{k=1}^{\infty} d_k \sum_{i=k}^{\infty}\int_{i^{1/p}}^{(i+1)^{1/p}} dF(x)$$

$$= \sum_{i=1}^{\infty}\left(\sum_{k=1}^{i} d_k\right)\int_{i^{1/p}}^{(i+1)^{1/p}} dF(x)$$

$$\le \sum_{i=1}^{\infty} i \log i \int_{i^{1/p}}^{(i+1)^{1/p}} dF(x)$$

$$\le c\sum_{i=1}^{\infty}\int_{i^{1/p}}^{(i+1)^{1/p}} x^p \log x^p dF(x)$$

$$\le cE|X|^p \log^+|X| < \infty . \quad (13)$$

So $\{X_{ij}\}$ and $\{Y_{ij}\}$ are equivalent, then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(X_{ij} - Y_{ij}) \quad \textit{converges a.e.}$$

Furthermore,

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{ij} - Y_{ij}\right)}{(mn)^{1/p}} \to 0 \quad \textit{a.e.} \quad (14)$$

(Which is a two parameter analog of Theorem 5.2.1 of Chung [8]).

Now, for every subsequences $\{k_m\}$ and $\{l_n\}$ of positive integer such that $\liminf_{n\to\infty} \frac{k_m}{k_{m-1}} > 1$ and $\liminf_{n\to\infty} \frac{l_m}{l_{m-1}} > 1$ and for any $\varepsilon > 0$, we use Chebyshev's inequality and Lemmas 1 and 3 to obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P\left\{\left|\frac{S_{k_m l_n}^* - ES_{k_m l_n}^*}{(k_m l_n)^{1/p}}\right| > \varepsilon\right\} \le c\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{VarS_{k_m l_n}^*}{(k_m l_n)^{2/p}}$$

$$\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(k_m l_n)^{2/p}}\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} VarY_{ij}$$

$$\le c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{1}{(k_m l_n)^{2/p}}\right) EY_{ij}^2$$

$$\le c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{EY_{ij}^2}{(ij)^{2/p}}$$

$$\leq cE|X|^p \log^+|X| < \infty . \quad (15)$$

Where we use $\sum_{k=i+1}^{\infty} \frac{d_k}{k^{2/p}} = O\left(\frac{\log i}{(i+1)^{2/p-1}}\right)$, for more detail about $\sum_{n=i}^{\infty} \frac{1}{k_n^2}$ see Chandra and Goswami (1992) page 217 [6]. Therefore by the Borel Cantelli Lemma we have

$$P\left\{\left|\frac{S^*_{k_m l_n} - ES^*_{k_m l_n}}{(k_m l_n)^{1/p}}\right| > \varepsilon \quad i.o.\right\} = 0$$

and this is equivalent to

$$\frac{S^*_{k_m l_n} - ES^*_{k_m l_n}}{(k_m l_n)^{1/p}} \to 0 \quad a.e.$$

(See Chung Theorem 4.2.2, p. 73 [8]). Now for any positive number k and l such that, $k_m \leq k \leq k_{m+1}$ and $l_n \leq l \leq l_{n+1}$, we have

$$\frac{S^*_{kl} - ES^*_{kl}}{(kl)^{1/p}} \leq \frac{S^*_{k_{m+1}l_{n+1}} - ES^*_{k_{m+1}l_{n+1}}}{(k_{m+1}l_{n+1})^{1/p}} \cdot \left(\frac{k_{m+1}l_{n+!}}{k_m l_n}\right)^{1/p} + \frac{ES^*_{k_{m+1}l_{n+1}} - ES^*_{k_m l_n}}{(k_m l_n)^{1/p}} . \quad (16)$$

Similarly we can obtain a lower bound for left-hand side of (16) as follows

$$\frac{S^*_{k_m l_n} - ES^*_{k_m l_n}}{(k_m l_n)^{1/p}} \left(\frac{k_m l_n}{k_{m+1} l_{n+1}}\right)^{1/p} - \frac{ES^*_{k_{m+1}l_{n+1}} - ES^*_{k_m l_n}}{(k_{m+1}l_{n+1})^{1/p}} \leq \frac{S^*_{kl} - ES^*_{kl}}{(kl)^{1/p}} . \quad (17)$$

Then by using (16) and (17), it follows that

$$\left(1 - \frac{1}{ab}\right)^{1/p} EX \leq \liminf_{(k,l)\to\infty} \frac{S^*_{kl} - ES^*_{kl}}{(kl)^{1/p}} \leq \limsup_{(k,l)\to\infty} \frac{S^*_{kl} - ES^*_{kl}}{(kl)^{1/p}} \leq (ab-1)^{1/p} EX \quad (18)$$

then for any $a = \liminf_{m\to\infty} \frac{k_m}{k_{m-1}} > 1$ and $b = \liminf_{m\to\infty} \frac{l_n}{l_{n-1}} > 1$, it follows that

$$\frac{S^*_{mn} - ES^*_{mn}}{(mn)^{1/p}} \to 0 \quad a.e. \quad (19)$$

Combining (14) and (19), we get

$$\frac{S_{mn} - ES^*_{mn}}{(mn)^{1/p}} \to 0 \quad a.e. \quad (20)$$

Since

$$\frac{S_{mn} - ES_{mn}}{(mn)^{1/p}} = \frac{S_{mn} - ES^*_{mn}}{(mn)^{1/p}} - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{EZ_{ij}}{(mn)^{1/p}} \quad (21)$$

it remains to prove that the second term of the right-hand side converges to 0 a.e.

By Lemma 3 (b), we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sum_{i=1}^{k_m}\sum_{j=1}^{l_n} EZ_{ij}}{(k_m l_n)^{1/p}} \leq c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{EZ_{ij}}{(ij)^{1/p}} \leq cE|X|^p \log^+|X| < \infty , \quad (22)$$

from which, it follows that

$$\lim_{(m,n)\to\infty} \sum_{i=1}^{k_m}\sum_{j=1}^{l_n} \frac{EZ_{ij}}{(k_m l_n)^{1/p}} = 0 . \quad (23)$$

But for every k and l such that $k_m \leq k < k_{m+1}$ and $l_n \leq l < l_{n+1}$ we have

$$R = \left|\sum_{i=1}^{k}\sum_{j=1}^{l} \frac{EZ_{ij}}{(kl)^{1/p}} - \sum_{i=1}^{k_m}\sum_{j=1}^{l_n} \frac{EZ_{ij}}{(k_m l_n)^{1/p}}\right|$$

$$\leq c\sum_{i=1}^{k_{m+1}}\sum_{j=1}^{l_{n+1}} \frac{EZ_{ij}}{(k_{m+1}l_{n+1,})^{1/p}} , \quad (24)$$

R converges to 0 which implies that, by (23)

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{EZ_{ij}}{(mn)^{1/p}}\to 0 . \quad (25)$$

This completes the proof.

Corollary 3. Let $\{X_{ij}\}$ be a double sequence of PND identically distributed random variables with $E|X_{11}|^p \log^+|X_{11}| < \infty$, and $1 \le p < 2$, then

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-EX_{ij})}{(mn)^{1/p}}\to 0 \ \ a.e. \ as \ m\vee n\to\infty \quad (26)$$

For p=1 see Theorem 2 of [2].

Remark. The generalization to r-dimensional arrays of random variables can be obtained easily under the condition $E|X|^p \log^+|X|^{r-1} < \infty$.

Theorem 2. Let $\{X_{ij}\}$ be a double sequence of pairwise negatively dependent random variables. If $P\{|X_{ij}| \ge t\} \le P\{|X| \ge t\}$ for all non-negative real numbers t and $EX^p \log^+|X| < \infty$, for $1 < p < 2$, then

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-EX_{ij})}{(mn)^{1/p}}\to 0 \ in \ L^1 \ as \ m\vee n\to\infty . \quad (27)$$

Proof. Since $\{X_{ij}\}$ are PND by Lemma 3 we have

$$\frac{E\left|\sum_{i=1}^{m}\sum_{j=1}^{n}(Y_{ij}-EY_{ij})\right|^2}{(mn)^{2/p}}\le\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}EY_{ij}^2}{(mn)^{2/p}} . \quad (28)$$

Since

$$\frac{E\left|\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-EX_{ij})\right|}{(mn)^{1/p}}$$

$$\le\frac{E\left|\sum_{i=1}^{m}\sum_{j=1}^{n}(Y_{ij}-EY_{ij})\right|}{(mn)^{1/p}}$$

$$+2\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}E|Z_{ij}|}{(mn)^{1/p}} \quad (29)$$

it suffices to show that $\sum_{i=1}^{m}\sum_{j=1}^{n}EY_{ij}^2/(mn)^{2/p}$ converges to 0 as $m\vee n\to 0$ By Lemma 3 (a), we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sum_{i=1}^{k_m}\sum_{j=1}^{l_n}Y_{ij}^2}{(k_m l_n)^{2/p}}\le\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}Y_{ij}^2}{(k_m l_n)^{2/p}}$$

$$\le c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{EY_{ij}^2}{(ij)^{2/p}}$$

$$\le cE|X|^p\log^+|X|<\infty , \quad (30)$$

from which, it follows that

$$\lim_{(m,n)\to\infty}\frac{\sum_{i=1}^{k_m}\sum_{j=1}^{l_n}Y_{ij}^2}{(k_m l_n)^{2/p}}=0 . \quad (31)$$

The rest of proof is similar to that used in the proof of (25) in Theorem 1.

Corollary 4. Let $\{X_{ij}\}$ be a double sequence of PND identically distributed random variables with $E|X_{11}|^p \log^+|X_{11}| < \infty$ and $1 \le p < 2$, then

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-EX_{ij})}{(mn)^{1/p}}\to 0 \ in \ L^1 \ as \ m\vee n\to\infty . \quad (32)$$

Remark. The generalization to r-dimensional arrays of random variables can be obtained easily under the condition $E|X|^p \log^+|X|^{r-1} < \infty$.

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