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A SHORT PROOF FOR THE EXISTENCE OF HAAR MEASURE ON COMMUTATIVE HYPERGROUPS

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Abstract

In this short note, we have given a short proof for the existence of the Haar measure on commutative locally compact hypergroups based on functional analysis methods by using Markov-Kakutani fixed point theorem.

Keywords: Hypergroups; Haar measure; Markov-Kakutani

Introduction

A fundamental open question about hypergroups is the existence of a Haar measure for any hypergroup (for a definition the reader can consult with Jewett [6]). If a hypergroup K is compact or discrete, then K possesses a Haar measure. All known examples have a Haar measure [6, s5]. Spector in [11] claims that any commutative hypergroup possesses a Haar measure but as Ross in [9] mentioned there are several technical problems in his proof. Ross in [9] has given a lengthy proof for the existence of the Haar measure on commutative hypergroups. Recently Izzo in [5] has given a short proof of the existence of Haar measure on a commutative locally compact group by using the Markov-Kakutani fixed-point theorem [1, pp. 151-152]. Based on his idea, we give a short proof of the existence of the Haar measure on commutative hypergroups. For definitions and notations we follow Jewett [6].

For the reader's convenience, we include the Markov-Kakutani fixed point theorem. Let $\cal S$ be a compact convex subset of a Hausdorff topological vector space and $\cal F$ be a commutative family of continuous affine mappings of $\cal S$ into $\cal S$ that is abelian. Then there exists $p\in \cal S$ such that $\Lambda(p) = p$ for all $\Lambda\in \cal F$ (for a proof see [1]).

Note 1.1. For a vector space X, let $X^{\#}$ be the space of all linear functionals on X with the weak topology induced by X. Then, if C is a closed subset of $X^{\#}$ such that the set $\{\Lambda x:\, \Lambda\in C\}$ is bounded, for any $x\in X,$ then C is compact [3, pp. 423-424].

Lemma 1.2. Let K be a hypergroup and U a symmetric neighborhood of the identity $e\in K$. Then there exists a subset M of K such that for any finite subset $\{a_1, a_2,\cdots a_n\}$ of K, the set $a_1*a_2*\cdots*a_n*U*U$ contains at least one element of M and the set $a_1*a_2*\cdots*a_n*U$ contains at most one element of M.

Proof. Let ${\cal A}=\{T\subseteq K:\, \mbox{for any} p\neq q\in T, \mbox{there is a finite subset}\{a_1,a_2,\cdots,a_n\} \mbox{of} K \mbox{such that} p\notin q*A*\br{A},\mbox{where}\br{A}=U*\br{a_n}*\cdots *\br{a_1}\}$.

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Then $\cal A$ is non-empty (all single subsets of K are in $\cal A$) and any chain $\{T_\al\}_{\al\in I}$ in $\cal A$ has an upper bound $\cup_{\al\in I} T_\al$. So by Zorn's Lemma $\cal A$ has a maximal element M. By using [6, 4.1A, 4.1B], we have $M\cap a*U*U\neq \emptyset$. Now for $\{a_1, a_2,\cdots,a_n\}$ an arbitrary finite subset of K, we have $ M\cap a_1*a_2*\cdots*a_n*U*U=M\cap(\cup_{x\ina_1*a_2*\cdots*a_n}\,x*U*U)=\cup_{x\ina_1*a_2*\cdots*a_n}\,(M\cap x*U*U)\neq\emptyset$.

To show that M intersects $a_1, a_2,\cdots, a_n*U*U$ at most at one point, let there are s_1 and s_2 in M that $s_1\neq s_2$ and $s_i\in a_1*a_2*\cdots*a_n*U$ for $i=1,2.$ Then by using [6, 4.1A, 4.1B] we have $s_1\in s_2*A*\br{A}$, where A is $U*\br{a_n}*\cdots*\br{a_2}$ and this contradicts $M\in {\cal A}$. So the proof of the Lemma is complete.

Theorem 1.3. Every commutative hypergroup K has a left Haar measure.

Proof. Let $C_{00}(K)^{\#}$ be the space of all linear functionals on $C_{00}(K)$. We consider on $C_{00}(K)^{\#}$ the weak topology generated by $C_{00}(K)$. It is clear that if there exists a $\Lambda\in C_{00}(K)^{\#}$ such that $f(\Lambda) =0$ for all $f\in C_{00}(K),$ then $\Lambda = 0$. So $C_{00}(K)^{\#}$ with this topology is a locally convex space [4, p. 50]. Let U be a fixed symmetric neighborhood of the identity $e\in K$ with compact closure. Let $\cal S$ be the set of all positive linear functionals Λ on $C_{00}(K)$ that satisfy the following two conditions:

(i) $\Lambda(f)\leq 1$ whenever $f\leq 1$ in $C_{00}^+(K)$ and $spt f\subseteq a_1*a_2*\cdots*a_r*U$ for some finite subset $\{a_1, a_2, \cdots,a_r\}$ in K,

(ii) $\Lambda(f)\geq 1$ whenever $f\leq 1$ in $C_{00}^+(K)$ and $f=1$ on $a_1*a_2*\cdots*a_r*U*U$ for some finite subset $\{a_1, a_2,\cdots,a_r\}$ in K.

Then one can easily check that $\cal S$ is closed and convex. Moreover, any $f\in C_{00}^+(K)$ can be written as a finite sum of non-negative continuous functions, each of which has support in $a*U$ for some $a\in K.$ To see this, let $spt f=C,$ (compact set). Then $C\subseteq \cup_{1\leq i\leq n}\, a_i*U$ for some $a_i\in K$, $1\leq i\leq n$. By the partition of unity on compact sets, there are $h_i\in C_{00}^+(K)$ such that $0<\frac{h_i}{f}\leq 1$ on C. That is for any $x\in C$, $0<h_i(x)\leq f(x)$ and $h_1(x)+h_2(x)+\cdots +h_n(x)= f(x)$. Now it follows from (i) that the set $\{ \Lambda(f):\, \Lambda\in \cal S \}$ is bounded. So by Note 1.1., $\cal S$ is compact.

To see $ \cal S$ is non-empty, let M be as in Lemma 1.2. Put $\Lambda(f) = \sum_{s\in M}\, f(s)$, then $\Lambda \in {\cal S}$. Indeed, if $f\in C_{00}^+(K)$ and $f\leq 1$ with $spt f\subseteq a_1*a_2*\cdots*a_n*U$ for some $a_i\in K, 1\leq i\leq n$, then by Lemma 1.2., M intersects $a_1*a_2*\cdots*a_n*U$ at most at one point. Hence $\Lambda(f) \leq 1$. If $f\in C_{00}^+(K)$ and $f = 1$ on $a_1*a_2*\cdots*a_n*U*U$ for some $a_i\in K, 1\leq i\leq n$, then again by Lemma 1.2., M intersects $a_1*a_2*\cdots*a_n*U*U$ at least at one point. So $\Lambda(f) \geq 1$.

For each $x\in K$ and $\Lambda\in \cal S$, let $T_x:\, C_{00}(K)^{\#}\to C_{00}(K)^{\#}$ is defined by $T_x\Lambda(f)= \Lambda(\,_xf)$ for $f\in C_{00}(K)$ where$\,_xf(y)=f(xy)$. Then it is easy to see that T_x is affine and $T_x(\cal S)\subseteq \cal S$. Indeed, let $\Lambda\in \cal S$. If $f\in C_{00}^+(K)$ and $f\leq 1$ with $spt f\subseteq a_1*a_2*\cdots*a_n*U$ for some $a_i\in K,\quad 1\leq i\leq n$, then $\,_xf\in C_{00}^+(K)$ [6, 4.2E] and $\,_xf \leq 1$ with $spt(_xf)\subseteq \br{x}*a_1*a_2*\cdots*a_n*U$. So by (i) $\Lambda (_xf)\leq 1$. If $f\in C_{00}^+(K)$ and $f= 1$ on $a_1*a_2*\cdots*a_n*U*U$ for some $a_i\in K, 1\leq i\leq n$, then $\,_xf\in C_{00}^+(K)$ and $\,_xf=1$ on $\br{x}*a_1*a_2*\cdots*a_n*U*U$. So by (ii), $\ Lambda(_xf)\geq 1$.

Also T_x is continuous, since if $\lim_\al\, \Lambda_\al=\Lambda$ in ${\cal S}$, then for any $f\in C_{00}(K)$, $\lim_\al|T_x\Lambda_\al(f)-T_x\Lambda (f)| = \ lim_\ al|\ Lambda_\ al(_xf)-\ Lambda (_xf)| = 0$. Moreover for $x,y\in K$, $T_x(T_y\Lambda) = T_{x*y}\, \Lambda = T_{y*x}\, \Lambda = T_y (T_x\ Lambda)$ for any $\Lambda\in C_{00}(K)^{\#}$. This shows that the family ${\cal F}=\{T_x:\,x\in K\}$ and $\cal S$ (as above) have all properties in Markov-Kakutani fixed-point theorem. So there exists $\Lambda_0\in \cal S$ such that $T_x\,\Lambda_0= \Lambda_0$ for all $x\in K$. In other words $T_x(\Lambda_0f) =\Lambda_0(_xf) = \Lambda_0(f) \mbox{for all} a\in K \mbox{and} f\in C_{00}(K)$.

Now since all elements of $\cal S$ are non-zero positive linear functionals on $C_{00}(K)$, by [6, s5.2] the proof is complete.

Remark 1.4. Can the above proof be modified to show that every amenable hypergroup has a left Haar measure, using Day's generalization of Markov-Kakutani fixed-point theorem [2, Theorem 1] (see also [7, Theorem 4.2]). (For an extension to hypergroups see

[10, Theorem 3.3.1].)

It is attempted such modification, but there is a problem in the continuity of action of hypergroup K on $\cal S$ (metioned earlier) defined by $(x, \Lambda) \longmapsto T_x\Lambda \mbox{where} T_x\Lambda (f)=\Lambda(_xf) \mbox{for} f\in C_{00}(K)$.

References

1. Conway J.B. *A Course in Functional Analysis*. Springer-Verlag, New-York (1985).
2. Day M.M. Fixed point theorems for compact convex sets, Ill. *J. Math.*, **5**: 585-590 (1961).
3. Dunford N. and Schwartz J. *Linear Operators*. Part I, Interscience Publ., New-York (1958).
4. Fell J.M.G. and Doran R.S. *Representations of *-algebras, Locally Compact Groups, and Banach *-algebraic Bundeles*. Vol. 1 (1988).
5. Izzo A.J. A functional analysis proof of the existence of Haar measure on locally compact abelian groups. *Proc. Amer. Math. Soc.*, **115**(2): 581-583 (1992).
6. Jewett R.I. Spaces with an abstract convolution of measures. *Advances in Math.*, **18**: 1-101 (1975).
7. Rickart C.E. Amenable groups and groups with the fixed point property. *Trans. Amer. Math. Soc.*, **127**: 221-232 (1967).
8. Ross K. Every commutative hypergroup possesses an invariant measure, (Preprint).
9. Skantharajah M. Amenable hypergroups. Doctoral Thesis, University of Alberta (1989).
10. Spector R. Apercu de la theorie des hypergroupes, Analyse Harmonique sur les groupes de Lie. Lecture Notes in Math., 497, Springer-Verlag, Berlin-Heidelberg-New York (1975).
11. Spector R. *Apercu de la Theorie des Hypergroupes, Analyse Harmonique sur les Groupes de Lie*. Lecture Notes in Math., **497**, Springer-Verlag, Berlin-Heidelberg-New York (1975).