A SHORT PROOF FOR THE EXISTENCE OF HAAR MEASURE ON COMMUTATIVE HYPERGROUPS

R.A. Kamyabi-Gol*

Department of Mathematics, Faculity of Mathematical Science, University of Mashhad, Mashhad, Islamic Republic of Iran

Abstract

In this short note, we have given a short proof for the existence of the Haar measure on commutative locally compact hypergroups based on functional analysis methods by using Markov-Kakutani fixed point theorem.

Keywords: Hypergroups; Haar measure; Markov-Kakutani

Introduction

A fundamental open question about hypergroups is the existence of a Haar measure for any hypergroup (for a definition the reader can consult with Jewett [6]). If a hypergroup \$K\$ is compact or discrete, then \$K\$ possesses a Haar measure. All known examples have a Haar measure [6, s5]. Spector in [11] claims that any commutative hypergroup possesses a Haar measure but as Ross in [9] mentioned there are several technical problems in his proof. Ross in [9] has given a lengthy proof for the existence of the Haar measure on commutative hypergroups. Recently Izzo in [5] has given a short proof of the existence of Haar measure on a commutative locally compact group by using the Markov-Kakutani fixed-point theorem [1, pp. 151-152]. Based on his idea, we give a short proof of the existence of the Haar measure on commutative hypergroups. For definitions and notations we follow Jewett [6].

For the reader's convenience, we include the Markov-Kakutani fixed point theorem. Let \$\cal S\$ be a compact convex subset of a Hausdorff topological vector space and \$\cal F\$ be a commutative family of continuous affine mappings of \$\cal S\$ into \$\cal S\$

that is abelian. Then there exists $p\ S$ such that $\ D = p$ for all $\Delta \in F$ (for a proof see [1]).

Note 1.1. For a vector space X, let X^{\parallel} be the space of all linear functionals on X with the weak topology induced by X. Then, if C is a closed subset of X^{\parallel} such that the set $\left| \text{Lambda x:} \right|$ Lambda in C\} is bounded, for any $x \in C$ is compact [3, pp. 423-424].

Lemma 1.2. Let K be a hypergroup and U a symmetric neighborhood of the identity $k \in K$. Then there exists a subset M of K such that for any finite subset $\{a_1, a_2, cdots a_n\}$ of K, the set $a_1*a_2 cdots*a_n*U*U$ contains at least one element of M and the set $a_1*a_2 cdots*a_n*U$ contains at most one element of M.

Proof. Let $\{ A = \{ T \in K:, Mbox \{ for any \} p \in q \in T, Mbox \{ there is a finite subset \} \{ a_1, a_2, cdots, a_n \} Mbox \{ of \} K Mbox \{ such that \} p notin q^{A} (A, Mbox \{ where \} br \{ A \} = U^{*} cdots * br \{ a_1 \} \}.$

^{*} *E-mail: kamyabi@math.umac.ir*

Then $\cal A$ is non-empty (all single subsets of K are in $\cal A$) and any chain $\left\{T_a\right\}_{al\in I}$ I}\$ in $\cal A$ has an upper bound $\cal A_a$ has a maximal element \$M\$. By using [6, 4.1A, 4.1B], we have \$M\cap a*U*U\neq \emptyset\$. Now for $\all a_1$, a_2,\cdots,a_n\}\$ an arbitrary finite subset of \$K\$, we have \$ M\cap a_1*a_2*\cdots*a_n*U*U=M\cap(\cup {x\ina_1*a_2*\cdots*a_n},x*U*U)=\cup_{x\ina_1*a_2*\cdots*a_n},x*U*U)=\cup_{x\ina_1*a_2*\cdots*a_n},(M\cap x*U*U)\neq\emptyset\$.

To show that \$M\$ intersects \$a_1, a_2,\cdots, a_n*U*U\$ at most at one point, let there are \$s_1\$ and \$s_2\$ in \$M\$ that \$s_1\neq s_2\$ and \$s_i\in a_1*a_2*\cdots*a_n*U\$ for \$i=1,2.\$ Then by using [6, 4.1A, 4.1B] we have \$s_1\in s_2*A*\br{A}\$, where \$A\$ is \$U*\br{a_n}*\cdots*\br{a_2}\$ and this contradicts \$M\in {\cal A}\$. So the proof of the Lemma is complete.

Theorem 1.3. Every commutative hypergroup \$K\$ has a left Haar measure.

Proof. Let $C_{00}(K)^{\#}$ be the space of all linear functionals on $C_{00}(K)$. We consider on $C_{00}(K)^{\#}$ the weak topology generated by $C_{00}(K)^{\#}$. It is clear that if there exists a $\Lambda = 00(K)^{\#}$ such that f(Lambda) = 0 for all $f(Lambda)^{+}$ such that f(Lambda) = 0 for all f(Lambda) = 0, so $C_{00}(K)^{\#}$ with this topology is a locally convex space [4, p. 50]. Let U be a fixed symmetric neighborhood of the identity e(Lambda) = 0 for all Lambda = 0. So $C_{00}(K)^{+}$ be the set of all positive linear functionals $Lambda^{+} = 0$ for all $C_{00}(K)^{+} = 0$.

(i) $\Lambda = 1$ whenever f = 1 in $C_{00}^{+}(K)$ and $spt f = 1^{a_2}(cdots^a_r^{US}$ for some finite subset $A_{a_1, a_2, cdots, a_r^{S}}$ in K,

(ii) $\Lambda = 0^{-1}$ whenever $f \le 1^{-1}$ in $C_{00}^{+}(K)$ and $f=1^{-1}$ on $a_1*a_2*\cdots*a_r*U*U$ for some finite subset $A_a_1, a_2,\cdots,a_r\$ in K.

Then one can easily check that cal S is closed and convex. Moreover, any $f\ln C_{00}^+(K)$ can be written as a finite sum of non-negative continuous functions, each of which has support in a^*U for some $a\ln K$. To see this, let spt f=C, (compact set). Then $C \sum (cup_{1} e i) = i e n_{1}$, $a_i = U$ for some $a_i \ln K$, $1 e n_{1} = i e n_{1}$, $a_i = U$ for some $a_i \ln K$, $1 e n_{1} = n_{1}$, $a_i = 0$, $1 e n_{1} + 1$, $a_i = 0$, $1 e n_{1} + 1$, $a_i = 1$, $1 e n_{1} + 1$, $1 e n_{1}$

 $Lambda(f):\, Lambda(in \ S \) is bounded. So by Note 1.1., \Cal S is compact.$

To see \$ \cal S\$ is non-empty, let \$M\$ be as in Lemma 1.2. Put $\lambda(f) = \sum_{s \in M} \{s \in M\}, f(s)$ then $\lambda \in \{ cal S \}$. Indeed, if $f \in S$ $C_{00}^{+(K)}$ and $f \leq 1$ with $spt f \leq 0$ $a_1*a_2*\cdots*a_n*U$ for some $a_i \in K$, $1 \leq i \leq j$ by Lemma 1.2., \$M\$ intersects n\$. then \$a_1*a_2*\cdots*a_n*U\$ at most at one point. Hence $\Lambda (f) \leq 1$, If $f \in C_{00}^{+}(K)$ and f = 1on \$a_1*a_2*\cdots*a_n*U*U\$ for some \$a_i\in K, 1\leq i\leq n\$, then again by Lemma 1.2., \$M\$ intersects \$a_1*a_2*\cdots*a_n*U*U\$ at least at one point. So $\Lambda(f) \ge 1$.

For each $x\in K$ and $\Delta \in S$, let $T_x:, C_{00}(K)^{\pm} \in C_{00}(K)^{\pm}$ is defined by $T_x\ (f)= \ for \ f$ $C_{00}(K)$ where $\lambda_x f(y) = f(xy)$. Then it is easy to see that \$T_x\$ is affine and \$T_x(\cal S)\subseteq \cal S\$. Indeed, let $\lambda = \frac{S}{S}$. If $f \in S$. $C_{00}^{+(K)}$ and $f \leq 1$ with $spt f \leq 0$ a_1*a_2*\cdots*a_n*U\$ for some \$a_i\in K,\quad $1 \le i \le n$, then $\lambda_x = 00^{+(K)} = 6, 4.2E$ 1\$ with \$spt(xf)\subseteq and $\lambda, xf \leq 0$ $br{x}*a_1*a_2*\cdots*a_n*U$. So by (i) Δa (xf) | 1, If $f \in C_{00}^{+}(K)$ and f = 1 on $a_1*a_2*\cdots*a_n*U*U$ for some $a_i\i K$, $l \in$ i\leq n\$, then $\ \pi C_{00}^+(K)\ and \ \pi f=1\ on$ $br{x}*a_1*a_2*\cdots*a_n*U*U$. So by (ii), $\$ Lambda(xf)\geq 1\$.

Also T_x is continuous, since if $||m_|a|$, $Lambda_a|=Lambda$ in $\{cal S\}$, then for any $fn C_{00}(K)$, $|lm_a|T_xLambda_a|(f)-T_xLambda (f)| = <math>|lm_a||Lambda_a|(xf)-Lambda (xf)| = 0$. Moreover for $x,y\in X_x = T_y(T_x)Lambda = T_y(T_x)Lambda$ for any $Lambda = T_y(T_x)Lambda$ for any $Lambda = T_y(T_x)Lambda$ for any $Lambda = T_x+Y$. This shows that the family $\{cal F\}=\{T_x:,x\in X\}$ and cal S (as above) have all properties in Markov-Kakutani fixed-point theorem. So there exists $Lambda_0 = Lambda_0$ for all $x\in X_x = Lambda_0$. Markov-Kakutani fixed-point theorem. So there exists $Lambda_0$ for all $x\in T_x = Lambda_0$.

Now since all elements of $\c S\ are non-zero$ positive linear functionals on $C_{00}(K)$, by [6, s5.2] the proof is complete.

Remark 1.4. Can the above proof be modified to show that every amenable hypergroup has a left Haar measure, using Day's generalization of Markov-Kakutani fixed-point theorem [2, Theorem 1] (see also [7, Theorem 4.2]). (For an extension to hypergroups see

[10, Theorem 3.3.1].)

It is attempted such modification, but there is a problem in the continuity of action of hypergroup K on cal S (metioned earlier) defined by $(x, Lambda) \log T_x Lambda (mbox{where} T_x Lambda (f)=Lambda(_xf) mbox{for} f c_{00}(K)$.

References

- 1. Conway J.B. A Course in Functional Analysis. Springer-Verlag, New-York (1985).
- 2. Day M.M. Fixed point theorems for compact convex sets, Ill. J. Math., 5: 585-590 (1961).
- 3. Dunford N. and Schwartz J. *Linear Operators*. Part I, Interscience Publ., New-York (1958).
- 4. Fell J.M.G. and Doran R.S. Representations of *algebras, Locally Compact Groups, and Banach *algebraic Bundeles. Vol. 1 (1988).

- 5. Izzo A.J. A functional analysis proof of the existence of Haar measure on locally compact abelian groups. *Proc. Amer. Math. Soc.*, **115**(2): 581-583 (1992).
- 6. Jewett R.I. Spaces with an abstract convolution of measures. *Advances in Math.*, **18**: 1-101 (1975).
- 7. Rickart C.E. Amenable groups and groups with the fixed point property. *Trans. Amer. Math. Soc.*, **127**: 221-232 (1967).
- 8. Ross K. Every commutative hypergroup possesses an invariant measure, (Preprint).
- 9. Skantharajah M. Amenable hypergroups. Doctoral Thesis, University of Alberta (1989).
- Spector R. Apercu de la theorie des hypergroupes, Analyse Harmonique sur les groupes de Lie. Lecture Notes in Math., 497, Springer-Verlag, Berlin-Heidelberg-New York (1975).
- Spector R. Apercu de la Theorie des Hypergroupes, Analyse Harmonique sur les Groupes de Lie. Lecture Notes in Math., 497, Springer-Verlag, Berlin-Heidelberg-New York (1975).