IMPROVED ESTIMATOR OF THE VARIANCE IN THE LINEAR MODEL

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Abstract

The improved estimator of the variance in the general linear model is presented under an asymmetric linex loss function.

Keywords: Equivariant estimator; Normal variance estimator; Improved estimator; Linex loss function

1. Introduction

Consider the canonical form of the general linear model and suppose $X \sim N_P(\mu, \tau I)$ and $U \sim N_n(O, \tau I)$ are to be independently observed. On the basis of these observations, τ is to be estimated, where the loss function is given by

$$L(\tau,\delta) = b \left\{ e^{a\left(\frac{\delta}{\tau}-1\right)} - a\left(\frac{\delta}{\tau}-1\right) - 1 \right\},\tag{1.1}$$

where $a \neq 0$ is a shape parameter and b>0 is a scale parameter. This loss function which was introduced by Varian [1] and was extensively discussed by Zellner [2], is useful when overestimation is regarded as more serious than underestimation or *vice versa*. In this regard see Parsian and Sanjari Farsipour [3].

A sufficient statistic in this problem is (X,T), where if ||.|| denotes the usual Euclidean norm, $T=||U||^2$.

2. MLE and Bayes Estimators

With U unobserved, we can write down the likelihood function, given our normality assumptions,

and easily obtain the maximum likelihood estimator. The likelihood function is

$$L(\mu, \tau) =$$

$$(2\pi)^{-\frac{p+n}{2}}(\tau^{-1})\exp\left\{-\frac{1}{2\tau}(X-\mu)'(X-\mu)-\frac{1}{2\tau}U'U\right\}.$$

So we have **X** as an MLE of μ , and $\frac{1}{2} \sum_{i=1}^{n} U_i^2$ as an MLE of τ . Now, we calculate the risk function relative to the loss function in (1.1) of $T = \sum_{i=1}^{n} U_i^2$, we have

$$R(\tau, \hat{\tau}) = e^{-a} (1-a)^{-\frac{n}{2}} - \frac{an}{2} + a - 1$$
(2.1)

Now, let $\lambda = \tau^{-1}$, and introducing a diffuse prior, as the one cited in the article by Zellner [1], *i.e.*, $\pi(\lambda) = \frac{1}{\lambda}$ we can derive an optimal estimate that minimizes the posterior expected loss of our loss function in (1.1), as a solution of the following equation

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$$E_{\lambda} \Big[\lambda e^{a\lambda\delta_B} \mid T = t \Big] = e^a E_{\lambda} \Big[\lambda \mid T = t \Big].$$
(2.2)

Hence, the Bayes estimator is $\delta_B = \frac{1}{2a}(1-e^{-\frac{2a}{3}})T$. Now we are able to obtain the risk function associated with this estimator as the following equation

$$R(\lambda, \delta_B) = \frac{1}{2} \left(1 + e^{-\frac{2a}{3}} \right)^{-n} e^{-a} + \frac{n}{2} e^{-\frac{2a}{3}} - \frac{n}{2} + a - 1, \quad (2.3)$$

and we can compare it with that we already derived under the assumption that U is observed. Obviously δ_B works better than T, since it is the best invariant estimator, and T is an invariant estimator.

For the loss function of the form $L(\delta, \lambda) = (\frac{\delta}{\lambda} - 1)^2$

the problem was solved by some authors such as Brewster and Zidek [4] as well as Hodges and Lehmann [5].

3. Improved Estimators

The problem remains invariant under the transformation group *A* under which

$$(\mathbf{X}, T) \to (\alpha \Gamma \mathbf{X} + \beta, \alpha^2 T)$$

$$(\mu, \tau) \to (\alpha \Gamma \mu + \beta, \alpha^2 \tau)$$

$$\delta \to \alpha^2 \delta$$
(3.1)

where $\alpha > 0, \beta \in \Re^P$ and Γ is a $p \times p$ orthogonal matrix. It follows that any nonrandomized \mathcal{A} -invariant estimator of τ is of the form cT, for some constant c>0. Since \mathcal{A} acts transitively on the parameter space, the risk function of cT,

$$E_{\mu,\tau}\left[\rho\left(\frac{cT}{\tau}\right)\right] = E_{0,1}\left[\rho(cT)\right],$$

is independent of the unknown parameters, where ρ (.) is the scale invariant low function. Then the optimum choice for c is derived from the equation

$$E_{0,1}\left[\frac{\partial}{\partial c^*}\rho(c^*T)T\right] = 0$$

and for the loss function (1.1), c^* is a multiplier of $\sum_{i=1}^{n} X_i^2$ [3].

Let \mathcal{H} denote the subgroup of \mathcal{A} obtained by requiring in (3.1) that $\beta = 0$ and that Γ be a diagonal orthogonal matrix. Any \mathcal{H} -invariant estimator is of the form $\phi(\mathbf{Z})T$, where $\mathbf{Z}=(Z_1, Z_2, ..., Z_p)$ ' and $Z_i = |X_i| T^{-\frac{1}{2}}, i = 1,...,p$. We can see that the risk of such an estimator is

$$R(\mu,\tau;\delta) = E_{\mu,\tau} \left[\rho\left(\frac{\phi(z)T}{\tau}\right) \right]$$
$$= E_{\xi,1} \left[\rho(\phi(z)T) \right]$$
$$= R(\xi;\delta), (say)$$

where $\xi = (\xi_1, \xi_2, ..., \xi_p)'$ and $\xi_i = |\mu_i| \tau^{-\frac{1}{2}}, i = 1, ..., p$. Since we deal only with *H*-invariant estimators, we may assume without loss of generality that $\tau = 1$.

On the other hand, X_i^2 has a chi-squared distribution with 1+2K_i degrees of freedom, where K_i denotes a Poisson random variable with mean $\lambda_i = \frac{1}{2}\xi_i^2$, and the K_i^*s , i=1,...,p, are independent of each other and of T. Let **K**=($K_1, K_2, ..., K_p$), the joint density of T and **Z** conditional on **K**=**k**=($k_1, k_2, ..., k_p$) is

$$f_{T,Z}(t,z \mid k) \propto t^{\frac{1}{2}(n+p)+k_{\bullet}-1} e^{-\frac{1}{2}t(1+||z||^2)} \prod_{i=1}^{p} z_i^{2k_i},$$

Independent of ξ , where $k_{\bullet} = \sum_{i=1}^{p} k_i$.

Now since the loss (1.1) is strictly convex, it uniquely minimized at $\phi_k(z)$ satisfying

$$E\{\rho'(\phi_k(\mathbf{Z})T)T \mid \mathbf{Z} = z, \mathbf{K} = k\} = 0$$

which is equivalent to

$$E\{Te^{a\phi}k^{(\mathbf{Z})T} \mid \mathbf{Z} = z, \mathbf{K} = k]\} = e^{a}E[T \mid \mathbf{Z} = z, \mathbf{K} = k].$$

Now, for any estimator $\phi(\mathbf{Z})$ T define $\phi^*(z) = \min \{\phi(z), \phi_o(z)\}$, then let

$$\begin{aligned} R(\xi;\phi) &= E_{\xi} \left\{ E[\rho(\phi(\mathbf{Z})T) \mid \mathbf{Z}, \mathbf{K}] \right\} \\ &= E_{\xi} \left\{ R(\phi(\mathbf{Z}) \mid \mathbf{Z}, \mathbf{K}) \right\}. \end{aligned}$$

Now, either $\phi^*(\mathbf{z}) = \phi(\mathbf{z})$, then $R(\phi^*(\mathbf{z}) | \mathbf{z}, \mathbf{k}) = R(\phi(\mathbf{z}) | \mathbf{z}, \mathbf{k})$ or $\phi^*(\mathbf{z}) = \phi_o(\mathbf{z}) < \phi(\mathbf{z})$, then since $R(\phi | \mathbf{z}, \mathbf{k})$ is strictly convex, and $\phi_{\mathbf{k}}(\mathbf{z}) \le \phi_o(\mathbf{z})$ for all

k, it follows that $R(\phi^*(\mathbf{z}) | \mathbf{z}, \mathbf{k}) < R(\phi(\mathbf{z}) | \mathbf{z}, \mathbf{k})$, see Figure 2.1, which is also cited in Maatta and Casella [6] in the univariate set up. Therefore, for any $\xi, R(\xi, \phi^*) \le R(\xi, \phi)$ with inequality if $P_{\xi}(\phi^*(\mathbf{Z}) \neq \phi(\mathbf{z})) > 0$. Now, let $\phi(\mathbf{z}) = c^* = \frac{1}{2a}(1 - e^{-\frac{2a}{n+2}})$, then to find $\phi_o(\mathbf{z})$ in this case, note that

$$R(c \mid \mathbf{z}, \mathbf{O}) \propto \int \rho(ct) t^{\frac{1}{2}(n+p)-1} e^{-\frac{1}{2}t(1+||z||^2)} dt.$$



Figure 3.1.

So, using the transformation $t \rightarrow t(1+ || \mathbf{z} ||^2)$, we can see that

$$R(c \mid \mathbf{z}, \mathbf{O}) \propto \int \rho(\widetilde{c}t) t^{\frac{p}{2}} t^{\frac{n}{2} - 1} e^{-\frac{1}{2}t} dt$$

$$\propto E \left[\rho(\widetilde{c}T) T^{\frac{p}{2}} \right]$$
(3.2)

where $\tilde{c} = c/(1 + ||\mathbf{z}||^2)$, so the minimum is attained at $\tilde{c} = \phi_o(\mathbf{z})/(1 + ||\mathbf{z}||^2)$. For finding the value of \tilde{c} , using

(2.2), \tilde{c} must satisfy the following relation

$$E\left[T^{\frac{p}{2}+1}e^{a\widetilde{c}T}\right] = e^{a}E\left[T^{\frac{p}{2}+1}\right]$$

which is obtained by

$$\widetilde{c} = \frac{1}{2a} \left(1 - e^{-\frac{2a}{n+p+2}} \right).$$

Hence, $\phi_o(\mathbf{z}) = \frac{1}{2a} (1 - e^{-\frac{2a}{n+p+2}})(1 + ||\mathbf{z}||^2)$, and so by the above discussion c^*T is dominated by

$$\delta^* = \min\{c^*, \tilde{c}(1+||z||^2)\}T.$$
(3.3)

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