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IMPROVED ESTIMATOR OF THE VARIANCE IN THE LINEAR MODEL

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Abstract

The improved estimator of the variance in the general linear model is presented under an asymmetric linex loss function.

Keywords: Equivariant estimator; Normal variance estimator; Improved estimator; Linex loss function

1. Introduction

Consider the canonical form of the general linear model and suppose $X \sim N_P(\mu,\tau I)$ and $U \sim N_n(O,\tau I)$ are to be independently observed. On the basis of these observations, τ is to be estimated, where the loss function is given by

$$L(\tau,\delta) = b\left\{e^{a\left(\frac{\delta}{\tau}-1\right)} - a\left(\frac{\delta}{\tau}-1\right)-1\right\}, \qquad (1.1)$$

where $a \neq 0$ is a shape parameter and $b>0$ is a scale parameter. This loss function which was introduced by Varian [1] and was extensively discussed by Zellner [2], is useful when overestimation is regarded as more serious than underestimation or *vice versa*. In this regard see Parsian and Sanjari Farsipour [3].

A sufficient statistic in this problem is (X,T), where if $\|.\|$ denotes the usual Euclidean norm, $T=\|U\|^2$.

2. MLE and Bayes Estimators

With U unobserved, we can write down the likelihood function, given our normality assumptions, and easily obtain the maximum likelihood estimator. The likelihood function is

$$L(\mu,\tau) =$$

$$(2\pi)^{-\frac{p+n}{2}}(\tau^{-1})\exp\left\{-\frac{1}{2\tau}(X-\mu)'(X-\mu)-\frac{1}{2\tau}U'U\right\}.$$

So we have **X** as an MLE of μ, and $\frac{1}{2}\sum_{i=1}^{n} U_i^2$ as an MLE of τ. Now, we calculate the risk function relative to the loss function in (1.1) of $T = \sum_{i=1}^{n} U_i^2$, we have

$$R(\tau,\hat{\tau}) = e^{-a}(1-a)^{-\frac{n}{2}} - \frac{an}{2} + a - 1 \qquad (2.1)$$

Now, let $\lambda=\tau^{-1}$, and introducing a diffuse prior, as the one cited in the article by Zellner [1], *i.e.*, $\pi(\lambda) = \frac{1}{\lambda}$ we can derive an optimal estimate that minimizes the posterior expected loss of our loss function in (1.1), as a solution of the following equation

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$$E_\lambda\left[\lambda e^{a\lambda\delta_B} \mid T=t\right]=e^a E_\lambda\left[\lambda \mid T=t\right]. \quad (2.2)$$

Hence, the Bayes estimator is $\delta_B=\frac{1}{2a}(1-e^{-\frac{2a}{3}})T$. Now we are able to obtain the risk function associated with this estimator as the following equation

$$R(\lambda,\delta_B)=\tfrac{1}{2}\left(1+e^{-\frac{2a}{3}}\right)^{-n}e^{-a}+\frac{n}{2}e^{-\frac{2a}{3}}-\frac{n}{2}+a-1, \quad (2.3)$$

and we can compare it with that we already derived under the assumption that U is observed. Obviously δ_B works better than T, since it is the best invariant estimator, and T is an invariant estimator.

For the loss function of the form $L(\delta,\lambda)=(\frac{\delta}{\lambda}-1)^2$ the problem was solved by some authors such as Brewster and Zidek [4] as well as Hodges and Lehmann [5].

3. Improved Estimators

The problem remains invariant under the transformation group A under which

$$(\mathbf{X},T)\to(\alpha\Gamma\mathbf{X}+\beta,\alpha^2T)$$
$$(\mu,\tau)\to(\alpha\Gamma\mu+\beta,\alpha^2\tau) \quad (3.1)$$
$$\delta\to\alpha^2\delta$$

where $\alpha>0, \beta\in\Re^P$ and Γ is a $p\times p$ orthogonal matrix. It follows that any nonrandomized $\mathcal{A}$-invariant estimator of τ is of the form cT, for some constant c>0. Since $\mathcal{A}$ acts transitively on the parameter space, the risk function of cT,

$$E_{\mu,\tau}\left[\rho\left(\frac{cT}{\tau}\right)\right]=E_{0,1}\left[\rho(cT)\right],$$

is independent of the unknown parameters, where ρ (.) is the scale invariant low function. Then the optimum choice for c is derived from the equation

$$E_{0,1}\left[\frac{\partial}{\partial c^*}\rho(c^*T)T\right]=0$$

and for the loss function (1.1), c^* is a multiplier of $\sum_{i=1}^n X_i^2$ [3].

Let $\mathcal{H}$ denote the subgroup of $\mathcal{A}$ obtained by requiring in (3.1) that $\beta=0$ and that Γ be a diagonal orthogonal matrix. Any $\mathcal{H}$-invariant estimator is of the form $\phi(\mathbf{Z})$T, where $\mathbf{Z}=(Z_1, Z_2, \dots, Z_p)'$ and $Z_i=|X_i|T^{-\frac{1}{2}}, i=1,\dots,p$. We can see that the risk of such an estimator is

$$R(\mu,\tau;\delta)=E_{\mu,\tau}\left[\rho\left(\frac{\phi(z)T}{\tau}\right)\right]$$
$$=E_{\xi,1}\left[\rho(\phi(z)T)\right]$$
$$=R(\xi;\delta),(say)$$

where $\xi=(\xi_1,\xi_2,\dots\xi_p)'$ and $\xi_i=|\mu_i|\tau^{-\frac{1}{2}}, i=1,\dots,p$. Since we deal only with $\mathcal{H}$–invariant estimators, we may assume without loss of generality that $\tau=1$.

On the other hand, X_i^2 has a chi-squared distribution with $1+2K_i$ degrees of freedom, where K_i denotes a Poisson random variable with mean $\lambda_i=\frac{1}{2}\xi_i^2$, and the $K_i s$, i=1,…,p, are independent of each other and of T. Let $\mathbf{K}=(K_1,K_2,\dots,K_p)$, the joint density of T and $\mathbf{Z}$ conditional on $\mathbf{K=k}=(k_1,k_2,\dots,k_p)$ is

$$f_{T,Z}(t,z\mid k)\propto t^{\frac{1}{2}(n+p)+k_\bullet-1}e^{-\frac{1}{2}t(1+\|z\|^2)}\Pi_{i=1}^p z_i^{2k_i},$$

Independent of ξ, where $k_\bullet=\sum_{i=1}^p k_i$.

Now since the loss (1.1) is strictly convex, it uniquely minimized at $\phi_k(z)$ satisfying

$$E\left\{\rho'(\phi_k(\mathbf{Z})T)T\mid\mathbf{Z}=z,\mathbf{K}=k\right\}=0$$

which is equivalent to

$$E\{Te^{a\phi}k^{(\mathbf{Z})T}\mid\mathbf{Z}=z,\mathbf{K}=k]\}=e^aE[T\mid\mathbf{Z}=z,\mathbf{K}=k].$$

Now, for any estimator $\phi(\mathbf{Z})$T define $\phi^*(z)$ = min$\{\phi(z),\phi_o(z)\}$, then let

$$R(\xi;\phi)=E_\xi\left\{E[\rho(\phi(\mathbf{Z})T)\mid\mathbf{Z},\mathbf{K}]\right\}$$
$$=E_\xi\left\{R(\phi(\mathbf{Z})\mid\mathbf{Z},\mathbf{K})\right\}.$$

Now, either $\phi^*(\mathbf{z})=\phi(\mathbf{z})$, then $R(\phi^*(\mathbf{z})\mid\mathbf{z},\mathbf{k})=R(\phi(\mathbf{z})\mid\mathbf{z},\mathbf{k})$ or $\phi^*(\mathbf{z})=\phi_o(\mathbf{z})<\phi(\mathbf{z})$, then since $R(\phi\mid\mathbf{z},\mathbf{k})$ is strictly convex, and $\phi_{\mathbf{k}}(\mathbf{z})\le\phi_o(\mathbf{z})$ for all

k, it follows that $R(\phi^*(\mathbf{z})\,|\,\mathbf{z},\mathbf{k}) < R(\phi(\mathbf{z})\,|\,\mathbf{z},\mathbf{k})$, see Figure 2.1, which is also cited in Maatta and Casella [6] in the univariate set up. Therefore, for any $\xi, R(\xi,\phi^*) \le R(\xi,\phi)$ with inequality if $P_\xi(\phi^*(\mathbf{Z}) \ne \phi(\mathbf{z})) > 0$. Now, let $\phi(\mathbf{z}) = c^* = \frac{1}{2a}(1-e^{-\frac{2a}{n+2}})$, then to find $\phi_o(\mathbf{z})$ in this case, note that

$$R(c\,|\,\mathbf{z},\mathbf{O}) \propto \int \rho(ct)t^{\frac{1}{2}(n+p)-1}e^{-\frac{1}{2}t(1+\|z\|^2)}dt.$$

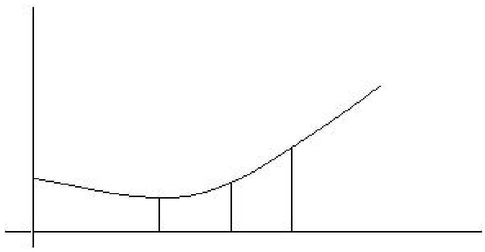

Figure 3.1.

So, using the transformation $t \to t(1+\|\mathbf{z}\|^2)$, we can see that

$$R(c\,|\,\mathbf{z},\mathbf{O}) \propto \int \rho(\tilde{c}t)t^{\frac{p}{2}}t^{\frac{n}{2}-1}e^{-\frac{1}{2}t}dt$$
$$\propto E\left[\rho(\tilde{c}T)T^{\frac{p}{2}}\right] \qquad (3.2)$$

where $\tilde{c} = c/(1+\|\mathbf{z}\|^2)$, so the minimum is attained at $\tilde{c} = \phi_o(\mathbf{z})/(1+\|\mathbf{z}\|^2)$. For finding the value of $\tilde{c}$, using (2.2), $\tilde{c}$ must satisfy the following relation

$$E\left[T^{\frac{p}{2}+1}e^{a\tilde{c}T}\right] = e^a E\left[T^{\frac{p}{2}+1}\right]$$

which is obtained by

$$\tilde{c} = \frac{1}{2a}\left(1-e^{-\frac{2a}{n+p+2}}\right).$$

Hence, $\phi_o(\mathbf{z}) = \frac{1}{2a}(1-e^{-\frac{2a}{n+p+2}})(1+\|\mathbf{z}\|^2)$, and so by the above discussion c^*T is dominated by

$$\delta^* = \min\left\{c^*, \tilde{c}\left(1+\|z\|^2\right)\right\}T. \qquad (3.3)$$

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