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# COVARIANCE MATRIX OF MULTIVARIATE REWARD PROCESSES WITH NONLINEAR REWARD FUNCTIONS

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## Abstract

Multivariate reward processes with reward functions of constant rates, defined on a semi-Markov process, first were studied by Masuda and Sumita, 1991. Reward processes with nonlinear reward functions were introduced in Soltani, 1996. In this work we study a multivariate process $\underline{Z}(t) = \left(Z_1(t), \cdots, Z_p(t)\right),\ t \geq 0$, where $Z_1(t), \cdots, Z_p(t)$ are reward processes with nonlinear reward functions $\rho_1, \cdots, \rho_p$ respectively. The Laplace transform of the covariance matrix, $\sum(t)$, is specified for given $\rho_1, \cdots, \rho_p$, and if they are real analytic functions, then the covariance matrix is fully specified. This result in particular provides an explicit formula for the variances of univariate reward processes. We also view $\sum(t)$ as a solution of a renewal equation.

**Keywords:** Semi-Markov processes; Reward processes; Laplace transform



## 1. Introduction

Let $\{J(t), t \geq 0\}$ be a semi-Markov process with a Markov renewal process $\{(J_n, T_n), n = 0, 1, 2, ...\}$. The state space of $\{J_n\}$ is assumed to be $N = \{0, 1, 2, ...\}$, see [1], [2] and [13] for more details. Based on $\{J(t), t \geq 0\}$, a multivariate reward process may be defined as $\underline{Z}(t) = (Z_1(t), Z_2(t), ..., Z_p(t))$, where

$$Z_i(t) = \sum_{n: T_{n+1} \prec t} \rho_i(J_n, T_{n+1} - T_n) + \rho_i(J(t), X(t)), \quad (1.1)$$

where $X(t)$ is the age process. Each function $\rho_i$ in (1.1) is called a reward function, and is a real function of two variables; $\rho_i : N \times R \to R$ where $\rho_i(j, \tau)$ measures the excess reward when time $\tau$ is spent in the state j. If $\rho_i(j, \tau) = j\tau, \quad i = 1, ..., p$ then the reward process $\underline{Z}(t)$ becomes the multivariate reward process treated by Masuda and Sumita 1991. In the case that $\underline{\rho}$ is of the form

$$\rho_i(k, x) = \sum_{n=1}^{m_i} g_{in}(k) x^n, \quad (1.2)$$

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where $g_{in}$, $i = 1, \cdots, m_i$ are given functions, $m_i \prec \infty$ an explicit formula for the mean vector $E\underline{Z}(t),\ t \geq 0$ is given in [10]. Khorshidian and Soltani have treated the asymptotic behavior of mean, variance and covariances of univariate and multivariate reward processes as $t \to \infty$, see [5], [6], [10] and [11]. They have shown that $E\,Z(t) \to C_0 + C_1(t)$, and $\sum(t) \to B_0 t + B_1 t^2$, as $t \to \infty$, and obtained the related coefficients explicitly, in terms of moments of the semi-Markov matrix.

In this work we explicitly determine $\sum(t)$ the covariance matrix of $\underline{Z}(t)$ for each $t \geq 0$ as a solution of a Laplace transform equation when $\rho_i$ are as in (1.2), $m_i \leq \infty$ and also view $\sum(t)$ as a solution of a renewal equation for general $\rho_i$. We also determine the variances of univariate reward processes with nonlinear reward functions.

For more details on semi-Markov processes see [1], [2] and [13]. Concerning the asymptotic behavior of a semi-Markov process see [3] and [4].

## 2. Notations and Preliminaries

Corresponding to a semi-Markov process $\{J(t), t \geq 0\}$, let $A_{ij}(x)$, measures the transition probability from state $i$ to the state $j$ within the time interval $(0, x]$, *i.e.*,

$$A_{ij}(x) = P\{J_{n+1} = j, T_{n+1} - T_n \leq x \mid J_n = i\}.$$

Let $a_{ij}(x)$ denote the density of $A_{ij}(x)$ and let

$$A_i(x) = \sum_{j \in N} A_{ij}(x), \quad \overline{A}_i(x) = 1 - A_i(x),$$

$$P_{ij}(t) = P\{J(t) = j \mid J(0) = i\}.$$

The Markov renewal function is denoted by $R(t) = \sum_{k=0}^{\infty} A^{k*}(t)$ where $A^{k*}(t)$ is the k-fold convolution of $A(t)$ by itself. The initial probability vector is denoted by $p(0)$ and the unit vector by $\underline{e}$. The joint distributions corresponding to the processes $\{J(t), X(t), t \geq 0\}$ and $\{J(t), X(t), \underline{Z}(t), t \geq 0\}$, respectively, are given by

$$G_{ij}(x,t) = p\{J(t) = j,\ X(t) \leq x | J(0) = i\},$$

$$F_{ij}(x, \underline{z}, t) = p\{J(t) = j,\ X(t) \leq x,\ \underline{Z}(t) \leq \underline{z} | J(0) = i\},$$

where by $\underline{Z}(t) \leq \underline{z}$ we mean $(Z_1(t) \leq z_1, ..., Z_p(t) \leq z_p)$. A vector $(\omega_1, ..., \omega_p)$ in $R^p$ is denoted by $\underline{\omega}$. The following Laplace transforms are of frequent use in subsequent sections,

$$\alpha_{ij}(s) = \int_0^\infty e^{-sx} dA_{ij}(x),$$

$$\alpha_i(s) = \int_0^\infty e^{-sx} dA_i(x),$$

$$\Phi_{ij}(\upsilon, \underline{\omega}, s) = \int_0^\infty \int_{R^p} \int_0^\infty e^{-\upsilon x - \underline{\omega}' \underline{z} - st} f_{ij}(x, \underline{z}, t) dx\, d\underline{z}\, dt,$$

$$\sigma_{ij}(\underline{\omega}, s) = \int_0^\infty \int_{R^p} e^{-\underline{\omega}' \underline{z} - st} f_{ij}(0^+, \underline{z}, t)\, d\underline{z}\, dt,$$

$$C_{kj}(\underline{\omega}, s) = \int_0^\infty e^{-\underline{\omega}' \underline{\rho}(k,x) - sx} dA_{kj}(x),$$

$$E_j(\underline{\omega}, s) = \int_0^\infty e^{-\underline{\omega}' \underline{\rho}(j,x) - sx}\, \overline{A}_j(x) dx.$$

Throughout this paper a matrix with entries $y_{ij}$ $i, j \in N$ is denoted by $y = [y_{ij}]$ and a diagonal matrix with entries $y_i$ $i \in N$ is denoted by $y_D = [\delta_{ij} y_j]$.

The following formula is an important and informative relation between the given Laplace transforms, see [7], [10].

$$\begin{aligned}\Phi(\upsilon, \underline{\omega}, s) &= \sigma(\underline{\omega}, s) \quad E_D(\underline{\omega}, s + \upsilon) \\ &= \left(I - C(\underline{\omega}, (s))^{-1} \quad E_D(\underline{\omega}, s + \upsilon)\right)\end{aligned} \tag{2.1}$$

Recall from [10] that in the univariate case the Laplace transform of $EZ_\rho(t)$ when $\rho(k,x) = kx^n$ is given by

$$\begin{aligned}L(EZ_\rho(t)) &= -p'(0) \frac{\partial^2 \Phi(0, \underline{\omega}, s)}{\partial \underline{\omega}} \Big|_{\underline{\omega}=0} \underline{e} \\ &= -p'(0)(I - \alpha(s)) \left( \sum_{i=1}^{n-1} \frac{n!(-1)^i \alpha_D^{(i)}(s)}{i! s^{n+1-i}} - n! \frac{(I - \alpha_D(s))}{s^{n+1}} \right) \rho_D \underline{e}\end{aligned}$$

where $\rho_D = [\delta_{kj}\ k]$; also when reward function is of the form (1.2), it has been shown that

$$E(Z_\rho(t)) = \int_0^t p'(0)\sum_{i=1}^m nE^{n-1}(\tau)\rho_{D:n}\underline{e}d\tau , \qquad (2.2)$$

where $g_n(k)$, $k \in N$, are the entries of the matrix $\rho_{D:n}$, $n = 1,...,m$; and

$$E^n(\tau) = \int_0^\infty x^n G(dx,\tau). \qquad (2.3)$$

The Formula (2.2) enables one to compute the mean of the cumulative reward up to time $t$, whenever the reward function is a polynomial. Indeed if $\rho$ satisfying

$$\rho(i,t) = \int_0^t \rho'(i,x)\, dx ,$$

$\rho(i,0) = 0$ and $(d/dx)\rho(i,x)$ exists, then $Z(t)$ in (1.1) can be written as

$$Z(t) = \int_0^t \rho'\ (\ J(s),\ X(s))\, ds ,$$

which implies that

$$E_i Z(t) = \int_0^t ds \sum_{j\in N}\int_0^\infty G_{ij}(dx,s)\rho'(j,x), \qquad (2.4)$$

where $E_i$ denotes the conditional expectation given $J_0=i$. Note that the Formula (2.2), will follow from (2.4).

The next section is devoted to the evaluation of $\sum(t)$ in the case that $\rho$ is given by (1.2).

## 3. Covariance Matrix under Polynomial Reward Functions

In this section we assume that

$$\rho_i(k,x) = g_i(k)x^{n_i} \quad ,i = 1,2,..., p ,$$

and by using (2.1) we obtain an explicit formula for $EZ_i(t)Z_j(t)$. First note that,

$$\int_0^\infty e^{-st} E\{e^{-\underline{\omega}'\underline{z}(t)}\}dt = p'(0)\Phi(0,\underline{\omega},s)\underline{e} ,$$

$$\int_0^\infty e^{-st} E\{Z_i(t)Z_j(t)\}dt = \underline{p}'(0)\frac{\partial^2\Phi(0,\underline{\omega},s)}{\partial\omega_i\partial\omega_j}\Big|_{\underline{\omega}=\underline{0}}\ \underline{e}. \qquad (3.1)$$

Also it follows from (2.1) that

$$\Phi(0,\underline{\omega},s) = \sigma(\underline{\omega},s)E_D(\underline{\omega},s).$$

**Theorem 3.2.** Suppose that $\rho_i(k,x) = g_i(k)x^{n_i}$, $i = 1,2,..., p$, then

$$\begin{aligned} EZ_i(t)Z_j(t) &= \\ &= p'(0)R(t) * \left(n_j\rho_{D:i}t^{n_i}a(t) * E^{n_j-1}(t)\rho_{D:j} \right. \\ &\left. + n_i\rho_{D:j}t^{n_j}a(t) * E^{n_i-1}(t)\rho_{D:i}\right)\underline{e} \\ &+ (n_i+n_j)p'(0)\int_0^t E^{n_i+n_j-1}(\tau)d\tau\, \rho_{D:i}\rho_{D:j}\underline{e}, \end{aligned} \qquad (3.3)$$

where $E^n(\tau)$ is given by (2.3) and $\rho_{D:i} = [\delta_{kl}g_i(k)]$.

**Proof.** Without loss of generality, we evaluate $EZ_1(t)Z_2(t)$. Differentiating of (2.1) gives that

$$\frac{\partial\Phi(0,\underline{\omega},s)}{\partial\omega_1} = \frac{\partial\sigma(\underline{\omega},s)}{\partial\omega_1}E_D(\underline{\omega},s) + \sigma(\underline{\omega},s)\frac{\partial E_D(\underline{\omega},s)}{\partial\omega_1},$$

and

$$\begin{aligned}\frac{\partial^2\Phi(0,\underline{\omega},s)}{\partial\omega_1\partial\omega_2} &= \frac{\partial^2\sigma(\underline{\omega},s)}{\partial\omega_1\partial\omega_2}E_D(\underline{\omega},s) \\ &+ \frac{\partial\sigma(\underline{\omega},s)}{\partial\omega_1}\frac{\partial E_D(\underline{\omega},s)}{\partial\omega_2} \\ &+ \frac{\partial\sigma(\underline{\omega},s)}{\partial\omega_2}\frac{\partial E_D(\underline{\omega},s)}{\partial\omega_1} \\ &+ \sigma(\underline{\omega},s)\frac{\partial^2 E_D(\underline{\omega},s)}{\partial\omega_1\partial\omega_2}\end{aligned} \qquad (3.4)$$

Also from (2.1),

$$\sigma(\underline{\omega},s) = \sigma(\underline{\omega},s)C(\underline{\omega},s) + I ,$$

which implies that,

$$\frac{\partial\sigma(\underline{\omega},s)}{\partial\omega_1} = \frac{\partial\sigma(\underline{\omega},s)}{\partial\omega_2}C(\underline{\omega},s) + \sigma(\underline{\omega},s)\frac{\partial C(\underline{\omega},s)}{\partial\omega_1},$$

or

$$\frac{\partial\sigma(\underline{\omega},s)}{\partial\omega_1} = (I - C(\underline{\omega},s))^{-1}\frac{\partial C(\underline{\omega},s)}{\partial\omega_1}(I - C(\underline{\omega},s))^{-1}, \qquad (3.5)$$

similarly

$$\frac{\partial\sigma(\underline{\omega},s)}{\partial\omega_2} = (I - C(\underline{\omega},(s))^{-1}\frac{\partial C(\underline{\omega},s)}{\partial\omega_2}(I - C(\underline{\omega},(s))^{-1}.$$

By using a similar method and formula (3.5) we obtain that

$$\frac{\partial^2\sigma(\underline{\omega},s)}{\partial\omega_1\partial\omega_2} = (I - C(\underline{\omega},s))^{-1} \times \left\{ \frac{\partial C(\underline{\omega},s)}{\partial\omega_1}(I - C(\underline{\omega},s)^{-1}\frac{\partial C(\underline{\omega},s)}{\partial\omega_2} + \frac{\partial C(\underline{\omega},s)}{\partial\omega_2}(I - C(\underline{\omega},s))^{-1}\frac{\partial C(\underline{\omega},s)}{\partial\omega_1} + \frac{\partial^2 C(\underline{\omega},s)}{\partial\omega_1\partial\omega_2} \right\} \times (I - C(\underline{\omega},s))^{-1}. \quad (3.6)$$

For $\rho_i(k,x) = g_i(k)x^{n_i}, i = 1,..., p,$ we have

$$C_{kj}(\underline{\omega},s) = \int_0^\infty e^{-\sum_{i=1}^p \omega_i g_i(k)x^{n_i} - sx} dA_{kj}(x),$$

which implies

$$C_{kj}(\underline{0},s) = \int_0^\infty e^{-sx} dA_{kj}(x).$$

Therefore $C(\underline{0},s) = \alpha(s)$ and

$$\frac{\partial C(\underline{\omega},s)}{\partial\omega_1}\Big|_{\underline{\omega}=\underline{0}} = (-1)^{n_1+1}\rho_{D:1}\alpha^{(n_1)}(s),$$

$$\frac{\partial C(\underline{\omega},s)}{\partial\omega_2}\Big|_{\underline{\omega}=\underline{0}} = (-1)^{n_2+1}\rho_{D:2}\alpha^{(n_2)}(s), \quad (3.7)$$

where $\rho_{D:i} = [\delta_{kl} g_i(k)]$. Also we obtain that

$$\frac{\partial^2 C_{kj}(\underline{\omega},s)}{\partial\omega_1\partial\omega_2}\Big|_{\underline{\omega}=\underline{0}} = (-1)^{n_1+n_2} g_1(k)g_2(k)\alpha_{kj}{}^{(n_1+n_2)}(s),$$

therefore

$$\frac{\partial^2 C(\underline{\omega},s)}{\partial\omega_1\partial\omega_2}\Big|_{\underline{\omega}=\underline{0}} = (-1)^{n_1+n_2}\rho_{D:1}\rho_{D:2}\alpha^{(n_1+n_2)}(s). \quad (3.8)$$

On the other hand

$$E_j(\underline{\omega},s) = \int_0^\infty e^{-\sum_{i=1}^p \omega_i g_i(k)x^{n_i} - sx}\overline{A}_j(x)dx,$$

giving that $E_j(\underline{0},s) = \frac{1-\alpha_j(s)}{s}$, or

$$E_D(\underline{0},s) = \frac{I - \alpha_D(s)}{s}. \quad (3.9)$$

Also

$$\frac{\partial E_j(\underline{\omega},s)}{\partial\omega_1}\Big|_{\underline{\omega}=\underline{0}} = (-1)^{n_1+1} g_1(j)\frac{d^{n_1}}{ds^{n_1}}\frac{1-\alpha_j(s)}{s},$$

and therefore

$$\frac{\partial E_j(\underline{\omega},s)}{\partial\omega_1}\Big|_{\underline{\omega}=\underline{0}} = \rho_{D:1}\left(\sum_{i=1}^{n_1}\frac{n_1!(-1)^i\alpha_D{}^{(i)}(s)}{i!s^{n_1+1-i}} + (-1)^{2n_1+1}n_1!\frac{(I-\alpha_D(s))}{s^{n_1+1}}\right), \quad (3.10)$$

and

$$\frac{\partial^2 E_j(\underline{\omega},s)}{\partial\omega_1\partial\omega_2} = \int_0^\infty g_1(j)g_2(j)x^{n_1+n_2}e^{-\sum_{i=1}^p \omega_i g_i(j)x^{n_i} - sx} \times \overline{A}_j(x)dx,$$

which implies that

$$\frac{\partial^2 E_j(\underline{\omega},s)}{\partial\omega_1\partial\omega_2}\Big|_{\underline{\omega}=\underline{0}} = (-1)^{n_1+n_2} g_1(j)g_2(j)\frac{d^{n_1+n_2}}{ds^{n_1+n_2}}\frac{1-\alpha_j(s)}{s}.$$

In the matrix form

$$\frac{\partial^2 E_D(\underline{\omega},s)}{\partial\omega_1\partial\omega_2}\Big|_{\underline{\omega}=\underline{0}} = \rho_{D:1}\rho_{D:2}\left(\sum_{i=1}^{n_1+n_2}\frac{(n_1+n_2)!(-1)^{i-1}\alpha_D{}^{(i)}(s)}{i!s^{n_1+n_2+1-i}} + (n_1+n_2)!\frac{(I-\alpha_D(s))}{s^{n_1+n_2+1}}\right). \quad (3.11)$$

It follows from (3.5) and (3.7) that

$$\frac{\partial\sigma(\underline{\omega},s)}{\partial\omega_1}\Big|_{\underline{\omega}=\underline{0}} =$$

$$(I-\alpha(s))^{-1}(-1)^{n_1+1}\rho_{D:1}\alpha^{(n_1)}(s)(I-\alpha(s))^{-1}, \tag{3.12}$$

similarly

$$\frac{\partial\sigma(\underline{\omega},s)}{\partial\omega_2}\Big|_{\underline{\omega}=\underline{0}} = (I-\alpha(s))^{-1}(-1)^{n_2+1}\rho_{D:2}\alpha^{(n_2)}(s)(I-\alpha(s))^{-1}.$$

Equations (3.6)-(3.8) give that

$$\frac{\partial^2\sigma(\underline{\omega},s)}{\partial\omega_1\partial\omega_2}\Big|_{\underline{\omega}=\underline{0}} = (I-\alpha(s))^{-1} \times$$

$$\Big\{(-1)^{n_1+1}\rho_{D:1}\alpha^{(n_1)}(s)(I-\alpha(s))^{-1}(-1)^{n_2+1}\rho_{D:2}\alpha^{(n_2)}(s)$$

$$+(-1)^{n_2+1}\rho_{D:2}\alpha^{(n_2)}(s)(I-\alpha(s))^{-1}(-1)^{n_1+1}\rho_{D:1}\alpha^{(n_1)}(s)$$

$$+(-1)^{n_1+n_2+1}\rho_{D:1}\rho_{D:2}\alpha^{(n_1+n_2)}(s)\Big\}(I-\alpha(s))^{-1}. \tag{3.13}$$

Substituting (3.9)-(3.13) in (3.4) gives

$$\frac{\partial^2\Phi(\underline{0},\underline{\omega},s)}{\partial\omega_1\partial\omega_2}\Big|_{\underline{\omega}=\underline{0}} = \frac{(-1)^{n_1+n_2}}{s}(I-\alpha(s))^{-1} \times$$

$$\Big\{\rho_{D:1}\alpha^{(n_1)}(s)(I-\alpha(s))^{-1}\rho_{D:2} \times$$

$$\left[\alpha^{(n_2)}(s)(I-\alpha(s))^{-1}(I-\alpha_D(s))-\alpha_D{}^{(n_2)}(s)\right]$$

$$+\rho_{D:2}\alpha^{(n_2)}(s)(I-\alpha(s))^{-1}\rho_{D:1} \times$$

$$\left[\alpha^{(n_1)}(s)(I-\alpha(s))^{-1}(I-\alpha_D(s))-\alpha_D{}^{(n_1)}(s)\right]$$

$$+\rho_{D:1}\rho_{D:2}\Big[\alpha^{(n_1+n_2)}(s)(I-\alpha(s))^{-1}(I-\alpha_D(s))$$

$$-\alpha_D{}^{(n_1+n_2)}(s)\Big]\Big\}$$

$$+(-1)^{n_2+1}(I-\alpha(s))^{-1}\rho_{D:2}\alpha^{(n_2)}(s)(I-\alpha(s))^{-1}\rho_{D:1} \times$$

$$\left(\sum_{i=1}^{n_1-1}\frac{n_1!(-1)^i\alpha_D{}^{(i)}(s)}{i!s^{n_1+1-i}}+(-1)^{2n_1+1}n_1!\frac{(I-\alpha_D(s))}{s^{n_1}+1}\right)$$

$$+(-1)^{n_1+1}(I-\alpha(s))^{-1}\rho_{D:1}\alpha^{(n_1)}(s)(I-\alpha(s))^{-1}\rho_{D:2} \times$$

$$\left(\sum_{i=1}^{n_2-1}\frac{n_2!(-1)^i\alpha_D{}^{(i)}(s)}{i!s^{n_2+1-i}}+(-1)^{2n_2+1}n_2!\frac{(I-\alpha_D(s))}{s^{n_2}+1}\right)$$

$$-(I-\alpha(s))^{-1}\rho_{D:1}\rho_{D:2} \times$$

$$\left(\sum_{i=1}^{n_1+n_2}\frac{(n_1+n_2)!(-1)^i\alpha_D{}^{(i)}(s)}{i!s^{n_1+n_2+1-i}} -(n_1+n_2)!\frac{(I-\alpha_D(s))}{s^{n_1+n_2+1}}\right). \tag{3.14}$$

Now note that for $m=n_1, n_2, n_1+n_2$,

$$\left[\alpha^{(m)}(s)(I-\alpha(s))^{-1}(I-\alpha_D(s))-\alpha_D{}^{(m)}(s)\right]\underline{e}=\underline{0}$$

Therefore multiplying (3.14) in unit vector $\underline{e}$ provides

$$\frac{\partial^2\Phi(\underline{0},\underline{\omega},s)}{\partial\omega_1\partial\omega_2}\Big|_{\underline{\omega}=\underline{0}}\,\underline{e} =$$

$$(-1)^{n_2+1}(I-\alpha(s))^{-1}\rho_{D:2}\alpha^{(n_2)}(s)(I-\alpha(s))^{-1}\rho_{D:1} \times$$

$$\left(\sum_{i=1}^{n_1-1}\frac{n_1!(-1)^i\alpha_D{}^{(i)}(s)}{i!s^{n_1+1-i}}+(-1)^{2n_1+1}n_1!\frac{(I-\alpha_D(s))}{s^{n_1}+1}\right)\underline{e}$$

$$+(-1)^{n_1+1}(I-\alpha(s))^{-1}\rho_{D:1}\alpha^{(n_1)}(s)(I-\alpha(s))^{-1}\rho_{D:2} \times$$

$$\left(\sum_{i=1}^{n_2-1}\frac{n_2!(-1)^i\alpha_D{}^{(i)}(s)}{i!s^{n_2+1-i}}+(-1)^{2n_2+1}n_2!\frac{(I-\alpha_D(s))}{s^{n_2}+1}\right)\underline{e}$$

$$-(I-\alpha(s))^{-1}\rho_{D:1}\rho_{D:2} \times$$

$$\left(\sum_{i=1}^{n_1+n_2}\frac{(n_1+n_2)!(-1)^i\alpha_D{}^{(i)}(s)}{i!s^{n_1+n_2+1-i}}-(n_1+n_2)!\frac{(I-\alpha_D(s))}{s^{n_1+n_2+1}}\right)\underline{e} \tag{3.15}$$

Using the equation

$$\frac{(-1)^m m}{s}\frac{\partial^{m-1}\Phi(v,\underline{0},s)}{\partial v^{m-1}}\Big|_{v=0} =$$

$$(I-\alpha_D(s))^{-1}\left(\sum_{i=1}^{m-1}\frac{m!(-1)^i\alpha_D{}^{(i)}(s)}{i!s^{m+1-i}}-m!\frac{(I-\alpha_D(s))}{s^{m+1}}\right)$$

in (3.15) we obtain that

$$\frac{\partial^2\Phi(0,\underline{\omega},s)}{\partial\omega_1\partial\omega_2}\Big|_{\underline{\omega}=\underline{0}}\,\underline{e} = (-1)^{n_1+n_2+1}\frac{n_1}{s}(I-\alpha(s))^{-1}\rho_{D:2} \times$$

$$\alpha^{(n_2)}(s)\ \frac{\partial^{n_1-1}\Phi(v,\underline{0},s)}{\partial v^{n_1-1}}\Big|_{v=0}\rho_{D:1}\underline{e}$$

$$+(-1)^{n_1+n_2+1}\frac{n_2}{s}(I-\alpha(s))^{-1}\rho_{D:1} \times$$

$$\alpha^{(n_1)}(s)\frac{\partial^{n_2-1}\Phi(v,\underline{0},s)}{\partial v^{n_2-1}}|_{v=0}\,\rho_{D:2}\,\underline{e}$$

$$+(-1)^{n_1+n_2+1}\frac{n_1+n_2}{s}\frac{\partial^{n_1+n_2-1}\Phi(v,\underline{0},s)}{\partial v^{n_1+n_2-1}}|_{v=0}\,\rho_{D:1}\rho_{D:2}\,\underline{e}. \qquad (3.16)$$

Now note that

$$\frac{1}{s}(I-\alpha(s))^{-1}=L[R(t)],$$

$$\rho_{D:i}\alpha^{(n_i)}(s)=L\left[(-1)^{n_i}\rho_{D:i}t^{n_i}\alpha(t)\right],\quad i=1,2$$

$$(-1)^{m-1}\frac{\partial^{m-1}\Phi(\upsilon,\underline{\omega},s)}{\partial\upsilon^{m-1}}|_{\upsilon=0}=L\left[E^{m-1}(t)\right].$$

Therefore by (3.1) and (3.16),

$$L[EZ_1(t)Z_2(t)]=p'(0)\frac{\partial^2\Phi(0,\underline{\omega},s)}{\partial\omega_1\partial\omega_2}|_{\underline{\omega}=0}\;\underline{e}$$

$$=n_2p'(0)L[R(t)]\;L\left[\rho_{D:1}t^{n_1}a(t)\right]L\left[E^{n_2-1}(t)\right]\rho_{D:2}\underline{e}$$

$$+n_1p'(0)L[R(t)]\;L\left[\rho_{D:2}t^{n_2}a(t)\right]L\left[E^{n_1-1}(t)\right]\rho_{D:1}\underline{e}$$

$$+(n_1+n_2)p'(0)L\left[\int_0^t E^{n_1+n_2-1}(\tau)d\tau\right]\rho_{D:1}\rho_{D:2}\underline{e}$$

giving the result.

**Corollary 3.17.** Let

$$\underline{\rho}=(\rho_1,\ldots,\rho_p),\qquad \rho_i(k,x)=g_i(k)x^{n_i},$$

then the covariance matrix of $\underline{Z}(t)=(Z_1(t),\ldots,Zp(t))$ is given by $\sum(t)=\left[\sum_{ij}(t)\right]_{i,j=1,\ldots,p}$ where

$$\textstyle\sum_{ij}(t)=p'(0)R(t)*\Big(n_j\rho_{D:i}t^{n_i}a(t)*E^{n_j-1}(t)\rho_{D:j}$$

$$+n_i\rho_{D:j}t^{n_j}a(t)*E^{n_i-1}(t)\rho_{D:i}\Big)\underline{e}$$

$$+(n_i+n_j)p'(0)\int_0^t E^{n_i+n_j-1}(\tau)d\tau\rho_{D:i}\rho_{D:j}\underline{e}$$

$$-n_in_j\left(p'(0)\int_0^t E^{n_i-1}(\tau)\rho_{D:n_i}d\tau\underline{e}\right)\times$$

$$\left(p'(0)\int_0^t E^{n_j-1}(\tau)\rho_{D:n_j}d\tau\underline{e}\right)$$

**Corollary 3.18.** Let $Z_\rho(t)$ be a one-dimensional reward process corresponding to $\rho(k,x)=kx^n$, then

$$Var(Z_p(t))=2n\underline{p'}(0)\Big(R(t)*\rho_Dt^na(t)*E^{n-1}(t)\rho_D$$

$$+\int_0^t E^{2n-1}(\tau)d\tau\rho^2D\Big)\underline{e}$$

$$-\left(np'(0)\int_0^t E^{n-1}(\tau)d\tau\rho_{D:n}\;\underline{e}\right)^2.$$

**Theorem 3.19.** Let $\rho_r(k,x)=\sum_{n=1}^{m_r}g_{rn}(k)x^n, r=1,2,\ldots,p,$ then

$$EZ_r(t)Z_s(t)=p'(0)R(t)*$$

$$\sum_{i=1}^{m_r}\sum_{j=1}^{m_s}\Big(\rho_{D:ri}t^ia(t)*jE^{j-1}(t)\rho_{D:sj}$$

$$+(\rho_{D:sj}t^ja(t)*iE^{i-1}(t)\rho_{D:ri}\Big)\underline{e}$$

$$+p'(0)\sum_{i=1}^{m_r}\sum_{j=1}^{m_s}(i+j)\int_0^t E^{i+j-1}(\tau)d\tau\rho_{D:ri}\rho_{D:sj}\underline{e}. \qquad (3.20)$$

where $\rho_{D:rn}=[\delta_{lk}g_{rn}(k)],\quad n=1,\ldots,m_r,\quad r=1,2,\ldots,p.$

**Proof.** Let $\rho_{rn}(k,x)=g_{rn}(k)x^n$, then $\rho_r(k,x)=\sum_{n=1}^{m_r}\rho_{rn}(k,x)$. If $Z_{rn}(t)$ and $Z_r(t)$, are the reward processes associated with $\rho_{rn}(.,.)$ and $\rho_r(.,.)$ respectively, then by the linearity of $Z_\rho$ in $\rho$; Lemma 4.1 in section 4, it follows that $Z_r(t)=\sum_{n=1}^{m_r}Z_{rn}(t)$ and therefore

$$EZ_r(t)Z_s(t)=\sum_{i=1}^{m_r}\sum_{j=1}^{m_s}EZ_{ri}(t)Z_{sj}(t).$$

Now apply Formula (3.3) in Theorem 3.2 to conclude the result.

**Corollary 3.21.** Let

$$\rho_r(k,x)=\sum_{n=1}^{m_r}g_{rn}(k)x^n,\qquad r=1,\ldots,p,$$

then the covariance matrix of $\underline{Z}(t)=(Z_1(t),\ldots,Z_p(t))$ is given by $\sum(t)=[\Sigma_{rs}(t)]_{r,s=1,\ldots,p}$ where

$$\sum_{rs}(t) = p'(0)R(t)*\left(\sum_{i=1}^{m_r}\sum_{j=1}^{m_s}\rho_{D:ri}t^i a(t)*jE^{j-1}(t)\rho_{D:sj} + \rho_{D:sj}t^j a(t)*iE^{i-1}(t)\rho_{D:ri}\right)\underline{e}$$

$$+\underline{p}'(0)\sum_{i=1}^{m_r}\sum_{j=1}^{m_s}(i+j)\int_0^t E^{i+j-1}(\tau)\,d\tau\rho_{D:ri}\rho_{D:sj}\underline{e}$$

$$-\left(\int_0^t p'(0)\sum_{i=1}^{m_r} iE^{i-1}(\tau)\rho_{D:ri}\underline{e}d\tau\right)\times\left(\int_0^t p'(0)\sum_{i=1}^{m_s} jE^{j-1}(\tau)\rho_{D:sj}\underline{e}d\tau\right).$$

**Corollary 3.22.** Let $Z_\rho(t)$ be a univariate reward process corresponding to $\rho(k,x)=\sum_{n=1}^m g_n(k)x^n$, then

$$Var(Z_\rho(t)) = 2p'(0)R(t)*\sum_{i,j=1}^m \rho_{D:i}t^i a(t)*jE^{j-1}(t)\rho_{D:j}\underline{e}$$

$$+p'(0)\sum_{i=1}^m\sum_{j=1}^m (i+j)\int_0^t E^{i+j-1}(\tau)d\tau\rho_{D:i}\rho_{D:j}\underline{e}$$

$$-\left(\int_0^t p'(0)\sum_{i=1}^m iE^{i-1}(\tau)\rho_{D:i}\underline{e}d\tau\right)^2.$$

## 4. Covariance Matrix Under Real Analytic Reward Functions

**Lemma 4.1.** Let $(J_n, T_n)$ be a recurrent Markov-renewal process with state space $N$. Suppose $Z_\rho(t)$, $t \ge 0$, is a univariate reward process corresponding to a reward function $\rho$, given by (1.1). Then the following holds;

(i) $Z_\rho(t)$ is linear in $\rho$ for each $t$, *i.e.*,

$Z_{\rho_1+\rho_2}(t) = Z_{\rho_1}(t) + Z_{\rho_2}(t)$,

(ii) If $\rho_m \uparrow \rho$, as m $\to \infty$, then $EZ_{\rho_m}(t) \to EZ_\rho(t)$ for each $t$,

(iii) If the state space is a finite set then the conclusion of part (ii) is satisfied even if the sequence of reward functions only converges.

**Proof.** Part (i) is immediate from (1.1). For (ii) note that it follows from the assumption that $Z_{\rho_m}(t)_\omega \to Z_\rho(t)_\omega$ for every given $t$ and every ω for which $Sup_n\, T_n(\omega) \succ t$. Since the semi-Markov process is assumed to be recurrent the later event occurs with probability one. Therefore $Z_{\rho_m}(t) \uparrow Z_\rho(t)$ with probability one. The desisred result follows by applying the Monotone Convergence Theorem. For part (iii), let

$$Max\, C_\rho(t) = Max\{\rho(k,x),\quad 0 \prec x \le t,\quad k \in N\},$$

then for each given $t$, $C_{\rho_m}(t)$ converges to $C_\rho(t)$ and therefore is bounded in, $\rho_m$ *i.e.*, $C_{\rho_m}(t) \le M(t)$ for all $m$. Thus it follows from (1.1) that

$$Z_{\rho_m}(t) \le C_{\rho_m}(t)(N(t)+1) \le M(t)(N(t)+1), \qquad (4.2)$$

where $N(t)$ is the number of renewal epoches in $[0,t]$. The argument given in the proof of (ii) and (4.2) make it possible to apply the Dominated Convergence. Theorem to conclude the result. The proof is complete.

**Theorem 4.3.** Suppose a univariate reward function $\rho(k,x)$ assumes a series representation of the form

$$\rho(k,x) = \sum_{n=1}^\infty g_n(k)\,x^n, \qquad x \ge 0,$$

where $g_n(.)$, $n=1,2\ldots$ are nonnegative functions. Then

$$EZ_\rho(t) = \int_0^t p'(0)E_\rho(\tau)\,\underline{e}\,d\tau, \qquad (4.4)$$

where

$$E_\rho(\tau) = \int_0^t G(dx,\tau)(d/dx)\gamma_D(x) \qquad (4.5)$$

and $\rho(k,x)$, $k \in N$ are the entries of the diagonal matrix $\gamma_D(x)$.

**Proof.** Let $\rho_m(k,x) = \sum_{n=1}^m g_n(k)\,x^n$, $m \ge 1$, and let $Z_{\rho_m}(t)$ be the reward process corresponding to $\rho_m$ through (1.1). It follows that $EZ_{\rho_m}(t)$ can be expressed by (2.2). But

$$\sum_{n=1}^m nE^{n-1}(\tau)\rho_{D:n} = \int_0^\infty \sum_{n=1}^m nx^{n-1}G(dx,\tau)\rho_{D:n} = \int_0^\infty G(dx,\tau)\left\{(d/dx)\sum_{n=1}^m \rho_{D:n}\,x^n\right\} = \int_0^\infty G(dx,\tau)(d/dx)\,\gamma_{D:m}(x)$$

where $\gamma_{D:m}(x)$ is the same as $\gamma_D(x)$ with $\rho$ replaced by $\rho_m$. Now $(d/dx)\rho_m(k,x) \uparrow (d/dx)\rho(k,x)$, as

$m \to \infty$, which gives that

$$\int_0^\infty G(dx,\tau)(d/dx)\,\gamma_{D:m}\,(x) \uparrow \int_0^\infty G(dx,\tau)(d/dx)\,\gamma_D\,(x)$$

for each $\tau$, giving that

$$Lim_{m\to\infty}\; E\,Z_{\rho_m}(t) = \int_0^t p'(0)E_\rho(\tau)\,\underline{e}\,d\tau\,. \qquad (4.6)$$

The result follows by applying (4.6) and Lemma 4.1 (ii).

**Lemma 4.7.** Let $(J_n, T_n)$ be a recurrent Markov-renewal process with state space $\boldsymbol{N}$. Suppose $\underline{Z}(t)=(Z_1(t),\ldots,Z_p(t))$, $t \geq 0$, is a reward process corresponding to a multi-dimensional reward function $\underline{\rho}=(\rho_1,\ldots,\rho_p)$ given by (1.1). Then the followings hold;

(i) If $\rho_{rm} \uparrow \rho_r$, as $m \to \infty$, then

$$EZ_{rm}(t)Z_{sm}(t) \to EZ_r(t)Z_s(t) \qquad \forall\, t\,,$$

(ii) If the state space is a finite set and $EN^2(t) \prec \infty$ then the conclusion of (i) is satisfied even if the sequence of reward functions only converges.

**Proof.** For (i) it follows from Lemma 4.1(ii) that $Z_{rm}(t)Z_{sm}(t) \uparrow Z_r(t)Z_s(t)$ with probability one. The desired result follows by applying the Monotone Convergence Theorem. For part (ii), it follows from the proof of Lemma 4.1 (iii) that

$$Z_{rm}(t)Z_{sm}(t) \leq M_r(t)M_s(t)(N(t)+1)^2, \qquad (4.8)$$

where $N(t)$ is the number of renewal epochs in $[0, t]$, and $M_r(t)$, $r=1,\cdots,p$, are some constants. The argument given in the proof of (i) and (4.8) together with the assumption that $EN^2(t) \prec \infty$ make it possible to apply the Dominated Convergence Theorem, which gives the result. The proof is complete.

**Theorem 4.9.** Suppose

$$\rho_r(k,x) = \sum_{n=1}^\infty g_{rn}(k)x^n\,, \quad r=1,\cdots,p,$$

where $g_{rn}(k)$, $k \in N$ are nonnegative, then

$$EZ_r(t)Z_s(t) = p'(0)R(t) * \big(\gamma_r(t)a(t) * E_s(t) + \gamma_s(t)a(t) * E_r(t)\big)\underline{e} + p'(0)\int_0^t E_{rs}(\tau)\,\underline{e}\,d\tau\,, \qquad (4.10)$$

where

$$E_r(\tau) = \int_0^\infty G(dx,\tau)\frac{d}{dx}\gamma_r(x),$$

$$E_{rs}(\tau) = \int_0^\infty G(dx,\tau)\frac{d}{dx}\{\gamma_r(x)\gamma_s(x)\},$$

and $\rho_r(k,x)$ are the entries of the diagonal matrices

$$\gamma_r(x), \quad r=1,\cdots,p.$$

**Proof.** Let

$$\rho_{rm}(k,x) = \sum_{n=1}^m g_{rn}(k)x^n\,, \quad m \geq 1\,, \quad r=1,\cdots,p,$$

and let $Z_{rm}(t)$ be the associated reward processes. It follows from Theorem 3.19 that $EZ_{rm}(t)\,Z_{sm}(t)$, can be expressed by (3.20), but

$$\sum_{i=1}^m \sum_{j=1}^m \rho_{D:ri}t^i * jE^{j-1}(t)\rho_{D:sj}\,\underline{e}$$

$$= \sum_{i=1}^m \rho_{D:ri}t^i * \sum_{j=1}^m jE^{j-1}(t)\rho_{D:sj}\,\underline{e}$$

$$= \gamma_{rm}(t)a(t) * \int_0^\infty \sum_{j=1}^m jx^{j-1}G(dx,t)\rho_{D:sj}\,\underline{e}$$

$$= \gamma_{rm}(t)a(t) * \int_0^\infty G(dx,t)\sum_{j=1}^m jx^{j-1}\rho_{D:sj}\,\underline{e}$$

$$= \gamma_{rm}(t)a(t) * \int_0^\infty G(dx,t)\frac{d}{dx}\gamma_{sm}(x)\underline{e}$$

where $\gamma_{rm}(x)$ is as $\gamma_r(x)$ with $\rho_{rm}(k,x)$ replaced by $\rho_r(k,x)$. Now $\frac{d}{dx}\rho_{rm}(k,x) \uparrow \frac{d}{dx}\rho_r(k,x)$ which implies that $\gamma_{rm}(x) \uparrow \gamma_r(x)$. Thus

$$\int_0^\infty G(dx,t)\frac{d}{dx}\gamma_{sm}(x) \uparrow \int_0^\infty G(dx,t)\frac{d}{dx}\gamma_s(x)\,.$$

Therefore as $m \to \infty$;

$$\sum_{i=1}^m \sum_{j=1}^m \rho_{D:ri}t^i a(t) * jE^{j-1}(t)\rho_{D:sj}\,\underline{e}$$

$$= \gamma_{rm}(t)a(t) * \int_0^\infty G(dx,t)\frac{d}{dx}\gamma_{sm}(x)\underline{e}$$

$$\uparrow \gamma_r(t)a(t) * E_s(t)\underline{e}\,. \qquad (4.11)$$

Similarly as $m \to \infty$;

$$\sum_{i=1}^m\sum_{j=1}^m \rho_{D:sj} t^j a(t) * iE^{i-1}(t)\rho_{D:ri}\underline{e} \uparrow \gamma_s(t)a(t) * E_r(t)\underline{e}$$

On the other hand

$$\sum_{i=1}^m\sum_{j=1}^m (i+j)p'(0)\int_0^t E^{i+j-1}(\tau)\rho_{D:ri}\rho_{D:sj}d\tau\underline{e}$$

$$= \int_0^t p'(0)\sum_{i=1}^m\sum_{j=1}^m (i+j)E^{i+j-1}(\tau)\rho_{D:ri}\rho_{D:sj}d\tau\underline{e}$$

$$= \int_0^t p'(0)\int_0^\infty G(dx,\tau)\sum_{i=1}^m\sum_{j=1}^m (i+j)x^{i+j-1}\rho_{D:ri}\rho_{D:sj}d\tau\underline{e}$$

$$= \int_0^t p'(0)\int_0^\infty G(dx,\tau)\sum_{i=1}^m ix^{i-1}\rho_{D:ri}\sum_{j=1}^m jx^j\rho_{D:sj}d\tau\underline{e}$$

$$+ \int_0^t p'(0)\int_0^\infty G(dx,\tau)\sum_{i=1}^m x^i\rho_{D:ri}\sum_{j=1}^m jx^{j-1}\rho_{D:sj}d\tau\underline{e}$$

$$= \int_0^t p'(0)\int_0^\infty G(dx,\tau)\gamma_{rm}(x)\frac{d}{dx}\{\gamma_{sm}(x)\}d\tau\underline{e}$$

$$+ \int_0^t p'(0)\int_0^\infty G(dx,\tau)\frac{d}{dx}\{\gamma_{rm}(x)\}\gamma_{sm}(x)d\tau\underline{e}$$

$$= \int_0^t p'(0)\int_0^\infty G(dx,\tau)\frac{d}{dx}\{\gamma_{rm}(x)\gamma_{sm}(x)\}d\tau\underline{e}$$

Now $\gamma_{rm}(x)\gamma_{sm}(x) \uparrow \gamma_r(x)\gamma_s(x)$ which is also real analytic with positive coefficients, therefore

$$\frac{d}{dx}\{\gamma_{rm}(x)\gamma_{sm}(x)\} \uparrow \frac{d}{dx}\{\gamma_r(x)\gamma_s(x)\}$$

and hence by the Monotone Convergence Theorem

$$\int_0^\infty G(dx,\tau)\frac{d}{dx}\{\gamma_{rm}(x)\gamma_{sm}(x)\} \uparrow \int_0^\infty G(dx,\tau)\frac{d}{dx}\{\gamma_r(x)\gamma_s(x)\}$$

Therefore

$$\sum_{i=1}^m\sum_{j=1}^m (i+j)p'(0)\int_0^t E^{i+j-1}(\tau)d\tau\rho_{D:ri}\rho_{D:sj}\underline{e}$$

$$= \int_0^t p'(0)\int_0^\infty G(dx,\tau)\frac{d}{dx}\{\gamma_{rm}(x)\gamma_{sm}(x)\}d\tau\underline{e}$$

$$\uparrow p'(0)\int_0^t E_{rs}(\tau)\,\underline{e}\,d\tau\,. \qquad (4.12)$$

By passing through the limit in (3.20) and then substituting (4.11) and (4.12), we arrive at (4.10). Thus the desired result is obtained.

**Corollary 4.13.** Suppose that the reward function of a $p$-variate reward procees admits the power series representation

$$\rho_r(k,x) = \sum_{n=1}^\infty g_{rn}(k)x^n\,, \quad r = 1,\cdots,p,$$

then the covariance matrix of $\underline{Z}(t)=(Z_1(t),\ldots,Z_p(t))$ is given by $\sum(t) = \sum_{rs}(t)$, $r,s = 1,\cdots,p$,

$$\sum\nolimits_{rs}(t) = p'(0)R(t) * \big(\gamma_r(t)a(t) * E_s(t)$$

$$+ \gamma_s(t)a(t) * E_r(t)\big)\underline{e}$$

$$+ P'(0)\int_0^t E_{rs}(\tau)\underline{e}d\tau$$

$$-\left(P'(0)\int_0^t E_{rs}(\tau)\underline{e}d\tau\right)\left(P'(0)\int_0^t E_{rs}(\tau)\underline{e}d\tau\right)$$

**Corollary 4.14.** Let $Z_\rho(t)$ be a one-dimensional reward process corresponding to $\rho(k,x) = \sum_{n=1}^\infty g_n(k)x^n$, then

$$Var\, Z_\rho(t) = 2p'(0)R(t) * \gamma(t)a(t) * E_\rho(t)$$

$$+ P'(0)\int_0^t E_{\rho^2}(\tau)\underline{e}\,d\tau$$

$$-\left(P'(0)\int_0^t E_\rho(\tau)\,\underline{e}\,d\tau\right)^2,$$

where

$$E_{\rho^2}(\tau)=\int_0^\infty G(dx,\tau)\frac{d}{dx}\gamma^2(x).$$

## 5. A Renewal Theory Approach

In this section we use the renewal theory to obtain $\sum(t)$ for a more general function $\rho_r(k,x)$, rather than analytic functions. By conditioning on the first renewal epoch, in the univariate case, we obtain that

$$E_iZ(t)=(1-A_i(t))\rho(i,t)+\sum_{j\in N}\int_0^t A_{ij}(dx)\Big(\rho(i,x)+E_jZ(t-x)\Big)$$

$$=(1-A_i(t))\rho(i,t)+\int_0^t A_i(dx)\rho(i,x)+\sum_{j\in N}\int_0^t A_{ij}(dx)E_jZ(t-x)$$

$$=g(i,t)+\sum_{j\in N}\int_0^t A_{ij}(dx)E_jZ(t-x)$$

where $E_i$ is the conditional expectation given $J(0)=i$. The above equation has the form

$$f=g+A*f,$$

$$g(i,t)=(1-A_i(t))\rho(i,t)+\int_0^t A_i(dx)\rho(i,x),$$

and has the solution

$$E_iZ(t)=\sum_{j\in N}\int_0^t R_{ij}(dx)g(j,t-x), \qquad (5.1)$$

which provides a formula for $E_iZ(t)$. The behavior of $E_iZ(t),\ \ t\to\infty,$ is completely specified in [11]. In the multivariate case, by conditioning on the first renewal epoch one obtains that

$$E_iZ_r(t)Z_s(t)=$$

$$=(1-A_i(t))\rho_r(i,t)\rho_s(i,t)+\sum_{j\in N}\int_0^t A_{ij}(dx)\times E_j\{\rho_r(i,x)+Z_r(t-x)\}\{\rho_s(i,x)+Z_s(t-x)\}$$

$$=(1-A_i(t))\rho_r(i,t)\rho_s(i,t)+\int_0^t A_i(dx)\rho_r(i,x)\rho_s(i,x)$$

$$+\sum_{j\in N}\int_0^t A_{ij}(dx)\{\rho_r(i,x)E_jZ_s(t-x)+\rho_s(i,x)E_jZ_r(t-x)\}$$

$$+\sum_{j\in N}\int_0^t A_{ij}(dx)E_j\{Z_r(t-x)Z_s(t-x)\}$$

$$=g_{rs}(i,t)+\sum_{j\in N}\int_0^t A_{ij}(dx)E_j\{Z_r(t-x)Z_s(t-x)\}.$$

The equation given above is a Markov renewal equation with

$$g_{rs}(i,t)=$$

$$=(1-A_i(t))\rho_r(i,t)\rho_s(i,t)+\int_0^t A_i(dx)\rho_r(i,x)\rho_s(i,x)$$

$$+\sum_{j\in N}\int_0^t A_{ij}(dx)\{\rho_r(i,x)E_jZ_s(t-x)+\rho_s(i,x)E_jZ_r(t-x)\}$$

and has the solution

$$E_iZ_r(t)Z_s(t)=\sum_{j\in N}\int_0^t R_{ij}(dx)g_{rs}(j,t-x). \qquad (5.2)$$

The $\sum(t)$ and its asymptotic behavior may be specified by using (5.1), (5.2) and the Markov Renewal Limit Theorems (due to Cinlar). We expect the exact analysis to be hard and interesting and can be the basis of a further study.

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