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ON THE LAWS OF LARGE NUMBERS FOR DEPENDENT RANDOM VARIABLES

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Abstract

In this paper, we extend and generalize some recent results on the strong laws of large numbers (SLLN) for pairwise independent random variables [3]. No assumption is made concerning the existence of independence among the random variables (henceforth r.v.'s). Also Chandra's result on Cesàro uniformly integrable r.v.'s is extended.

Keywords: Complete convergence; Strong law of large numbers; Pairwise negatively dependent r.v.'s; Negatively associated r.v.'s; Cesàro uniformly integrable r.v.'s

1. Introduction and Preliminaries

Let $\{X_n, n \geq 1\}$ be a sequence of integrable r.v.'s defined on the same probability space and put $S(n) = \sum_{i=1}^{n} X_i$, $\overline{X}_n = S(n)/n$. Landers and Rogge [8] proved a strong law of large numbers (SLLN) for pairwise independent and strongly uniformly integrable r.v.'s. Chandra and Goswami [3] proved a more general SLLN for pairwise independent and Cesàro uniformly integrable r.v.'s. Landers and Rogge [9] showed that Chandra's results hold for non-negative and uncorrelated instead of pairwise independent r.v.'s, but not without the assumption of non-negativity. Matula [10] has proved the SLLN for pairwise negatively dependent r.v.'s with the same distribution. Bozorgnia *et al.* [2] obtained the SLLN for weighted sums of an array of rowwise negatively dependent r.v.'s under certain moment conditions. Amini [1] has proved the SLLN for special negatively dependent r.v.'s and for weighted sums of uniformly bounded negatively dependent r.v.'s. In this paper, we modify and generalize some theorems of SLLN of Chandra and Goswami [3] for pairwise negatively dependent r.v.'s which are not necessarily identically distributed.

Definition 1. The random variables $X_1, \cdots, X_n\ (n \geq 2)$ are said to be pairwise negatively dependent (henceforth pairwise *ND*) if

(1) $P(X_i > x_i, X_j > x_j) \leq P(X_i > x_i)P(X_j > x_j)$,

for all x_i , $x_j \in \Re$, $i \neq j$. It can be shown that (1) is equivalent to

(2) $P(X_i \leq x_i, X_j \leq x_j) \leq P(X_i \leq x_i)P(X_j \leq x_j)$,

for all x_i , $x_j \in \Re$, $i \neq j$.

Definition 2 ([7]). The random variables $X_1, \cdots, X_n$ $(n \geq 2)$ are said to be negatively associated (*NA* for short) if for every pair of disjoint nonempty subsets A_1, A_2 of $\{1, \ldots, n\}$,

(3) $Cov(f_1(X_i, i \in A_1), f_2(X_i, i \in A_2)) \leq 0$

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whenever f_1 and f_2 are coordinatewise increasing such that this covariance exists. Clearly (3) holds if both f_1 and f_2 are decreasing.

An infinite collection of $\{X_n, n \geq 1\}$ is said to be pairwise *ND* (negatively associated) if every finite subcollection is pairwise *ND* (negatively associated).

It can be shown that *NA* implies pairwise *ND* and for $n = 2$, *ND* is equivalent to *NA*.

2. Main Results

In this paper, C stands for a generic constant not necessarily the same at each appearance. Also $\{f(n)\}$ will stand for an increasing sequence such that $f(n) > 0$ for each n, $f(n) \to \infty$ and for $\alpha > 1$, $m(n) = \left[\log_\alpha f(n)\right]$, the integer part of $\log_\alpha f(n)$, is an increasing sequence.

In the following Theorem we present another poof for the theorem of Csörgo *et al*. [4]. (see Chandra and Goswami [3])

Theorem 1. Let $\{X_n, n \geq 1\}$ be a sequence of r.v.'s with finite $Var(X_n)$. Assume that

i) there is a double sequence $\{\rho_{ij}\}$ of non-negative reals such that

$$Var(S(n)) \leq \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} \quad \text{for each } n \geq 1;$$

ii) $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} / (f(i \vee j))^2 < \infty$, $i \vee j = \max(i, j)$.

Then $[S(n) - E(S(n))]/f(n) \to 0$ completely, in the sense of Hsu and Robbins [6] (see also page 225 of Stout [12]).

Proof. Put $Z(n) = \dfrac{1}{f(n)}(S(n) - ES(n))$. It is sufficient to show that $\sum_{n=1}^{\infty} P(|Z(n)| > \varepsilon) < \infty$.

$$\sum_{n=1}^{\infty} P(|Z(n)| > \varepsilon)$$

$$\leq \sum_{n=1}^{\infty} CE(Z^2(n)) = C\sum_{n=1}^{\infty} \frac{E(S(n) - ES(n))^2}{f^2(n)}$$

$$\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\rho_{ij}}{f^2(n)} = C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} \sum_{n \geq i \vee j} \frac{1}{f^2(n)}.$$

The relation $m(n) = [\log_\alpha f(n)]$ now implies $\alpha^{m(n)} \leq f(n) < \alpha^{m(n)+1}$ and $f^{-2}(n) \leq \alpha^{-2m(n)}$. Thus the last sum is

$$\leq C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} \sum_{f(n) \geq f(i \vee j)} \alpha^{-2m(n)}$$

$$\leq C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} \sum_{\alpha^{m(n)+1} \geq f(i \vee j)} \alpha^{-2m(n)}.$$

Let $P = \inf\{n \geq 1, \alpha^{m(n)+1} \geq f(i \vee j)\}$. Then the RHS above is

$$\leq C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} \sum_{n=P}^{\infty} \alpha^{-2m(n)} \leq C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} \sum_{m=m(P)} \alpha^{-2m}$$

$$= C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} \alpha^{-2m(P)} \leq C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\rho_{ij}}{f^2(i \vee j)} < \infty.$$

Then $[S(n) - E(S(n))]/f(n) \to 0$ completely, and the Borel-Cantelli lemma implies that $[S(n) - E(S(n))]/f(n) \to 0$ a.s.

Proposition 1 ([1]). Let $\{X_n, n \geq 1\}$ be a sequence of pairwise *ND* r.v.'s. If $\{f_n, n \geq 1\}$ is a sequence of monotone increasing (or monotone decreasing) functions then $\{f_n(X_n), n \geq 1\}$ is a sequence of pairwise *ND* r.v.'s.

Corollary 1. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise *ND* r.v.'s. Then $\{X_n^+, n \geq 1\}$ and $\{X_n^-, n \geq 1\}$ are two sequences of pairwise *ND* r.v.'s, where X_n^+ and X_n^- are positive and negative parts of a random variable X_n, respectively.

Now we are able to prove the following theorems for pairwise *ND* random variables with finite variances.

Theorem 2. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise *ND* r.v.'s with finite $Var(X_n)$. Assume that

$$\sum_{n=1}^{\infty} (f(n))^{-2} Var(X_n) < \infty.$$

Then $\left[S(n)-E(S(n))\right]/f(n)\to 0$ completely.

Proof. Under pairwise *ND* condition we have

$$Var(\sum_{i=1}^{n}X_i)\le\sum_{i=1}^{n}Var(X_i)=\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\qquad\forall n\ge 1,$$

where $\rho_{ii}=Var(X_i)$ for $i=j$ and $\rho_{ij}=0$ for $i\ne j$. It follows from Theorem 1 that $\frac{S(n)-E(S(n))}{f(n)}\to 0$ completely.

Example 1. Let $\{X_n, n\ge 1\}$ be a sequence of iid random variables with finite $Var(X_1)$ and $f(n)=\alpha^n$, $\alpha>1$. It is obvious that conditions of Theorem 2 hold and we have $\frac{S(n)-E(S(n))}{f(n)}\to 0$ completely.

Example 2. Let $\{X_n, n\ge 1\}$ and $f(n)$ be as above, $Y_n=-a_nX_n, a_n>0$ and $a_n=O(n^{\beta})$, $\beta>0$. Put $Z_{2n}=X_n$, $Z_{2n-1}=Y_n$ and $S(n)=\sum_{i=1}^{n}Z_i$. It is obvious that $\{Z_n\}$ is a sequence of pairwise *ND* r.v.'s with finite Variances. Also

$$\begin{aligned}&\sum_{n=1}^{\infty}(f(n))^{-2}Var(Z_n)\\&=\sum_{n=1}^{\infty}(f(2n))^{-2}Var(Z_{2n})\\&\quad+\sum_{n=1}^{\infty}(f(2n-1))^{-2}Var(Z_{2n-1})\\&=\sum_{n=1}^{\infty}(f(2n))^{-2}Var(X_1)\\&\quad+\sum_{n=1}^{\infty}(f(2n-1))^{-2}a_n^2Var(X_1)<\infty\end{aligned}$$

Then, by Theorem 2, $\frac{S(n)-E(S(n))}{f(n)}\to 0$ completely.

The next theorem is an analogue of the three-series theorem of Kolmogorov (1929) for independence r.v.'s. Our intention is to replace the conditions of Chandra and Goswami [3] by suitable weaker conditions of simple nature. (see, in this connection, page 118 of Chung [5]).

Theorem 3. Let $\{X_n, n\ge 1\}$ be a sequence of pairwise *ND* integrable r.v.'s such that there is a sequence $\{B_n, n\ge 1\}$ of Borel subsets of R^1 that are semi intervals $(-\infty, x_n]$ ($(-\infty, x_n)$, $[x_n,\infty)$ or (x_n,∞)), satisfying the following conditions:

(a) $\sum_{n=1}^{\infty}C_nP(X_n\in B_n^c)<\infty$ where $C_n=1\vee(x_n/f(n))^2$;

(b) $\sum_{i=1}^{n}E(X_iI(X_i\in B_i^c))=o(f(n))$;

(c) $\sum_{n=1}^{\infty}(f(n))^{-2}E(X_n^2I(X_n\in B_n))<\infty$;

here B_n^c is the complement of B_n. Then $\left[S(n)-E(S(n))\right]/f(n)\to 0$ almost surely as $n\to\infty$.

Proof. Let $Y_n=X_nI(X_n\in B_n)+x_nI(X_n\notin B_n)$, $n\ge 1$. By Proposition 1, $\{Y_n, n\ge 1\}$ is a sequence of pairwise *ND* r.v.'s. We use Theorem 2 for $\{Y_n, n\ge 1\}$.

$$\begin{aligned}\sum_{n=1}^{\infty}(f(n))^{-2}Var(Y_n)&\le\sum_{n=1}^{\infty}f^{-2}(n)E(Y_n^2)\\&\le\sum_{n=1}^{\infty}f^{-2}(n)\{\int_{X_n\in B_n}X_n^2dP(w)+x_n^2P(X_n\in B_n^c)\}\\&=\sum_{n=1}^{\infty}f^{-2}(n)E(X_n^2I(X_n\in B_n))\\&\quad+\sum_{n=1}^{\infty}\frac{x_n^2}{f^2(n)}P(X_n\in B_n^c)\\&\le\sum_{n=1}^{\infty}f^{-2}(n)E(X_n^2I(X_n\in B_n))\\&\quad+\sum_{n=1}^{\infty}C_nP(X_n\in B_n^c)<\infty,\end{aligned}$$

then, Theorem 2 applied to $\{Y_n\}$ yields, $\frac{1}{f(n)}\sum_{i=1}^{n}(Y_i-E(Y_i))\to 0$ a.s. It is easy to show that

$$\frac{1}{f(n)}\sum_{i=1}^{n}(Y_i-E(X_i))=\frac{1}{f(n)}\sum_{i=1}^{n}(Y_i-E(Y_i))$$
$$+\frac{1}{f(n)}\sum_{i=1}^{n}x_iP(X_i\in B_i^c)$$

$$-\frac{1}{f(n)}\sum_{i=1}^{n}(E(X_i I(X_i \in B_i^c)).$$

Since

$$\sum_{n=1}^{\infty}\frac{|x_n|}{f(n)}P(X_n \in B_n^c) =$$

$$\sum_{n:C_n=1}^{\infty}\frac{|x_n|}{f(n)}P(X_n \in B_n^c) + \sum_{n:C_n\neq 1}^{\infty}\frac{|x_n|}{f(n)}P(X_n \in B_n^c)$$

$$\leq \sum_{n=1}^{\infty}C_n P(X_n \in B_n^c) < \infty,$$

then, by Kronecker's lemma we have $\frac{1}{f(n)}\sum_{i=1}^{n}x_i P(X_i \in B_i^c) \to 0$. By (b), we get $\frac{1}{f(n)}\sum_{i=1}^{n}(Y_i - E(X_i)) \to 0.$

Since, by (a), r.v.'s $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are equivalent, then by the first Borel-Cantelli lemma, the desired result follows.

In the next theorem, we use the following lemmas. Lemma 1 can be proved using the summation by parts formula and Lemma 2 is Lemma 15 of Petrov ([11], 277-278).

Lemma 1. If $\sum b_n < \infty$ and b_n is decreasing, then for any bounded $\{\alpha_n\}$ such that $\{n\alpha_n\}$ is increasing, $\sum\left[n\alpha_n - (n-1)\alpha_{n-1}\right]b_n < \infty$.

We denote by ψ_c the set of functions $\psi(x)$ such that (a) $\psi(x)$ is positive and non-decreasing in the interval $x > x_0$ for some x_0 and (b) the series $\sum 1/n\psi(n)$ converges.

Lemma 2. Let $\{a_n\}$ be a sequence on non-negative numbers, $A_n = \sum_{i=1}^{n}a_n, A_n \to \infty$. Then the series $\sum a_n / A_n\psi(A_n)$ converges for any $\psi \in \psi_c$.

We next generalize the SLLN of Chandra and Goswami [3]. The reader should note the naturality of Cesàro uniform integrability in the context of laws of large numbers.

Theorem 4. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise *ND* r.v.'s. Assume that there is a function $\Phi:(0,\infty) \to (0,\infty)$ such that

i) $\inf_{x\geq 1}\Phi(x)/x^2 > 0$;

ii) $t^{-1}\Phi(t)$ is increasing to ∞ as $t \to \infty$;

iii) $\sum_{n=1}^{\infty}(\Phi(n))^{-1} < \infty$;

iv) $\sup_{n\geq 1}[n^{-1}\sum_{i=1}^{n}E(\Phi(|X_i|))] = c(say) < \infty$.

Then $\frac{1}{f(n)}\sum_{i=1}^{n}(X_i - E(X_i)) \to 0$ almost surely as $n \to \infty$.

Proof. We use Theorem 3 with $B_n = (-\infty, n]$ for $n \geq 1$. It is easy to check that $C_n \leq M < \infty$ for each $n \geq 1$. Put $\alpha_n = [n^{-1}\sum_{i=1}^{n}E(\Phi(|X_i|))]$ for $n \geq 1$. We first verify Condition (a);

$$\sum_{n=1}^{\infty}C_n P(X_n \in B_n^c) \leq M\sum_{n=1}^{\infty}P(\Phi(|X_n|) \geq \Phi(n))$$

$$\leq M\sum_{n=1}^{\infty}E(\Phi(|X_n|)/\Phi(n)) < \infty,$$

by Lemma 1 and (iv). To prove Condition (b), let $\varepsilon > 0$. There is an integer $N_1 > 1$ such that for each $\Phi(t) \geq \frac{2t(c+1)}{\varepsilon}$ for $t > N_1$, and so for each $n \geq 1$,

$$n^{-1}\sum_{i=1}^{n}E(|X_i| I(|X_i| > N_1))$$

$$\leq \varepsilon n^{-1}\sum_{i=1}^{n}E(\Phi|X_i| I(|X_i| > N_1))/2(c+1) < \varepsilon/2.$$

Next there is an integer $N > N_1$ such that for each $n \geq N$, $n^{-1}\sum_{i=1}^{N_1}E(|X_i|) < \varepsilon/2$. Then for $n \geq N$

$$\sum_{i=1}^{n}E(X_i I(X_i > i)) \leq \sum_{i=1}^{n}E(|X_i| I(|X_i| > i))$$

$$\leq \sum_{i=1}^{N_1}E(|X_i|) + \sum_{i=1}^{n}E(|X_i| I(|X_i| > N_1)) < n\varepsilon.$$

It is clear that $B_n = C_n \cup D_n \cup E_n$ where

$C_n = (-\infty, -n)$, $D_n = [-n, -n^{1/4}] \cup [n^{1/4}, n]$, and $E_n = (-n^{1/4}, n^{1/4})$. To prove Condition (c), it suffices to show that

(5) $\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(X_n \in C_n)) < \infty$, $\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(X_n \in D_n)) < \infty$

and $\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(X_n \in E_n)) < \infty$

We first show that $\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(X_n \in C_n)) < \infty$. Since $\inf\{y : y = \Phi(|x|)/x^2, x < -1\}$ is positive, then there is a z in the interval $(-\infty, -1)$ such that $\Phi(|z|)/z^2 \le 2 \quad \inf\{y : y = \Phi(|x|)/x^2, x < -1\}$. Hence we have $\Phi(|z|)/z^2 \le 2\Phi(|x|)/x^2$ for each $x < -n$ and $x^2 \le 2\dfrac{z^2}{\Phi(|z|)}\Phi(|x|)$. Hence

$$\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(X_n < -n))$$

$$\le 2\frac{z^2}{\Phi(z)} \sum_{n=1}^{\infty} n^{-2} E(\Phi(|X_n|)) < \infty .$$

To complete the proof that Condition (c) holds, it suffices to show that

(6) $\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(X_n \in D_n)) < \infty$.

For each $n \ge 1$, there is a z_n in the interval $[n^{1/4}, n]$ such that

$\Phi(z_n)/z_n^2 \le 2\inf\{y : y = \Phi(x)/x^2 : n^{1/4} \le x \le n\}$,

note that the right side of the above inequality is positive. Then for $x \in [n^{1/4}, n]$, we have

$$x^2 \le 2nz_n \frac{\Phi(x)}{\Phi(z_n)} \qquad \text{(as } z_n \le n\text{)}$$

$$\le 2n^2 \Phi(x)/t_n \qquad \text{(by } z_n \ge n^{1/4} \textit{and}\ (ii)\text{)}$$

where $t_n = n^{3/4}\Phi(n^{1/4})$ for $n \ge 1$. Observe that

$$\sum_{n=1}^{\infty} n^{-2} E(X_n^2 I(X_n \in D_n)) \le 2\sum_{n=1}^{\infty} E(\Phi(|X_n|))/t_n$$

$$= 2\sum_{n=1}^{\infty} [n\alpha_n - (n-1)\alpha_{n-1}]/t_n .$$

So (5) will follow if we show that $\sum_{n=1}^{\infty} 1/t_n < \infty$ (using Lemma 1). For this purpose, we use Lemma 2 with $a_n = n^{1/4} - (n-1)^{1/4}$ for $n \ge 1$, $\psi(x) = \Phi(|x|)/|x|$; here we are following the notation of Petrov [11] and using Assumptions (ii) and (iii). As $a_n \ge 1/(4n^{3/4})$ for each n and $t_n = n\psi(n^{1/4})$, we get $\sum_{n=1}^{\infty} 1/t_n < \infty$.

Proposition 2. If $\{X_n, n \ge 1\}$ is a sequence of *NA* r.v.'s then Theorems 2, 3, 4 are valid.

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