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Some Probability Inequalities for Quadratic Forms of Negatively Dependent Subgaussian Random Variables

M. Amini,[1] A. Bozorgnia,[2,*] and H. Zarei[3]

[1] Department of Mathematics, Faculty of Science, Sistan and Baluchestan University, Zahedan, Islamic Republic of Iran
[2] Department of Statistics, Faculty of Mathematical Science, Ferdowsi University, Mashhad, Islamic Republic of Iran
[3] Department of Mathematics, Faculty of Science, Zabol University, Zabol, Islamic Republic of Iran

Abstract

In this paper, we obtain the upper exponential bounds for the tail probabilities of the quadratic forms for negatively dependent subgaussian random variables. In particular the law of iterated logarithm for quadratic forms of independent subgaussian random variables is generalized to the case of negatively dependent subgaussian random variables.

Keywords: Negatively dependent; Quadratic forms; Subgaussian random variable; Iterated logarithm

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of subgaussian random variables and let $A = (a_{ij}),\ i, j = 1, 2, ...$ be an array of real numbers and $A_n = (a_{ij}),\ i, j = 1, 2, ..., n$. We define the quadratic forms (Q.F) $Q_n = \mathbf{X_n}^T A_n \mathbf{X_n}$, where $\mathbf{X_n} = (X_1, ..., X_n)^T$. Without loss of generality we may assume that A_n is symmetric. For a matrix $B = (b_{ij}),\ i, j = 1, 2, ..., n$, $\| B \|^2 = \sum_{i,j=1}^{n} b_{ij}^2$ and $\mu^2(B)$ is the largest eigenvalue of $B^T B$. Moreover $tr(B)$ and $r(B)$ stand for trace of B and rank of B, respectively, and $diag(\alpha_1, ..., \alpha_n)$ denotes the diagonal matrix with diagonal elements $\alpha_1, ..., \alpha_n$. Some exponential bounds for the tail probabilities of Q.F's have been studied by Mikosch [7] for the case where $\{X_n, n \geq 1\}$ is an independent subgaussian random variables. The limit behaviors of Q.F's have been studied by many authors such as Krentsberg [6], Gotze and Tikhomirov [5], Giraitis and Taqqu [4]. In this paper we extend some exponential bounds for the tail probabilities of Q.F's for negatively dependent (ND) subgaussian random variables. Then by using these inequalities we obtain the law of the iterated logarithm (LIL) and some probability inequalities for Q.F's.

Definition 1. The random variables $X_1, ..., X_n$ are said to be ND if we have

$$P[\bigcap_{j=1}^{n} (X_j \leq x_j)] \leq \prod_{j=1}^{n} P(X_j \leq x_j),$$

and

$$P[\bigcap_{j=1}^{n} (X_j > x_j)] \leq \prod_{j=1}^{n} P(X_j > x_j),$$

* *E-mail: bozorg@math.um.ac.ir*

for all $x_1,...,x_n \in R$. An infinite sequence $\{X_n, n \ge 1\}$ is said to be ND if every finite subset $X_{i_1},...,X_{i_n}$ is ND.

The following lemmas are listed for reference in obtaining the main results in the next sections. Detailed proofs can be found in Taylor and Chung Hu [9], Buldying and Kozachenco [2], Bozorgnia, Patterson and Taylor [1], Mikosch [7], Chung [3].

Lemma 1.1. ([1]) Let $\{X_n, n \ge 1\}$ be a sequence of ND random variables and $\{f_n, n \ge 1\}$ be a sequence of Borel functions all of which are monotone increasing (or all are monotone decreasing). Then $\{f_n(X_n), n \ge 1\}$ is a sequence of ND random variables.

Lemma 1.2. ([1]) Let $X_1,...,X_n$ be a finite sequence of ND random variables and $t_1,...,t_n$ be all nonnegative (or nonpositive) then

$$E[e^{\sum_{i=1}^n t_i X_i}] \le \prod_{i=1}^n E[e^{t_i X_i}]$$

Definition 2. A symmetric random variable X is said to be subgaussian (SG) random variable if there exists a nonnegative real number α such that for each real number *t*,

$$Ee^{tX} \le \exp[\frac{\alpha^2 t^2}{2}]. \tag{1.1}$$

The number $\tau(X) = \inf\{\alpha \ge 0 : E(e^{tX}) \le \exp[\frac{\alpha^2 t^2}{2}], t \in R\}$ will be called the Gaussian standard of the random variable X. It is evident that X will be a subgaussian random variable if and only if $\tau(X) < \infty$. Moreover

$$\tau(X) = \sup_{t \ne 0}[\frac{2\ln(E(e^{tX}))}{t^2}]^{1/2},$$

and inequality (1.1) hold for $\alpha = \tau(X)$. A subgaussian random variable X always satisfies the relations $E(X) = 0$ and $E(X^2) \le \tau^2(X)$. If $E(X^2) = \tau^2(X)$, then X is called strictly subgaussian.

Lemma 1.3. ([9]) If X is a subgaussian random variable with $\tau(X) \le \alpha$, then

$$E[e^{t|X|}] \le 2\exp[\frac{\alpha^2 t^2}{2}].$$

Lemma 1.4. ([7]) For a positive matrix B of dimension $k \times k$,

$$\exp(h\mathbf{Z}^T B\mathbf{Z}) = \int_{R^k} e^{\sqrt{2h}\mathbf{Z}^T\mathbf{y}} q(\mathbf{y})d\mathbf{y}, \ \forall \mathbf{Z} \in R^k, \ h > 0,$$

where $q(\mathbf{y})$ is the density of a k-dimensional Gaussian vector with mean zero and variance matrix B.

Lemma 1.5. ([3]) If $\{E_n, n \ge 1\}$ is a sequence of events then

$$P(E_n \quad i.o) \ge \overline{\lim} P(E_n).$$

2. Exponential Bounds for Tail Probabilities

In this section we obtain upper exponential bounds for the probabilities $P[Q_n > x]$ and $P[Q_n^* > x]$, for every $x > 0$. Put $\alpha_i = \tau(X_i)$, $V_n = diag(\alpha_1,...,\alpha_n)$, $B_n = 4\|V_n A_n V_n\|^2$, $\mu_n = \mu(V_n A_n V_n)$ and $Q_n^* = Q_n - 2tr(V_n A_n V_n)$.

Throughout the sections 2 and 3 we suppose that $\{X_n, n \ge 1\}$ is a sequence of ND subgaussian random variables. Since some proofs are the same as Mikosch [7], we abbreviate them. In fact we obtain upper bounds that are greater than upper bounds in Mikosch [7], but these inequalities imply our main results for ND subgaussian random variables. The following lemmas play an essential role in obtaining our results.

Lemma 2.1. Let $\{X_n, n \ge 1\}$ be a sequence of ND subgaussian random variables with $\tau(X_n) \le 1$ for all n. Let A_n be a positive semidefinite matrix and $Q_n = \mathbf{X_n}^T A_n \mathbf{X_n}$. Then

$$E[\exp(hQ_n)] \le \exp(-\frac{1}{2}\sum_{j=1}^n \ln(1-4h\lambda_j)),$$

where $0 \le h \le 1/4\mu_n$ and $\lambda_1,...,\lambda_n$ are the eigenvalues of A_n.

Proof. Similar to the notation Lemma 1.2 of Mikosch [7], let $r(A_n) = k > 0$, there exists an orthogonal matrix U such that $A_n = U^T LU$ and $L = diag(\lambda_1,...,\lambda_k, 0,...,0)$ where $\lambda_1,...,\lambda_k$ are positive eigenvalues of A_n. Then

$$\mathbf{X_n}^T A_n \mathbf{X_n} = V^T L_0 V,$$

where

$V=((U\mathbf{X_n})_1,....(U\mathbf{X_n})_k)$, $L_0=diag(\lambda_1,...,\lambda_k)$.

By Lemma 1.4 for sufficiently small $h>0$,

$$Ee^{hQ_n}=\int_{R^k}E[e^{\sqrt{2h}V^T\mathbf{y}}]q(\mathbf{y})\mathrm{d}(\mathbf{y}), \qquad (2.1)$$

and

$$Ee^{\sqrt{2h}V^T\mathbf{y}}=E\,\exp(\sqrt{2h}\sum_{j=1}^{n}X_j\sum_{i=1}^{k}y_iu_{ij})$$

$$=E\{\exp(\sqrt{2h}\sum_{j=1}^{n}X_j\gamma_j^+-\sqrt{2h}\sum_{j=1}^{n}X_j\gamma_j^-)\},$$

where $\gamma_j=\sum_{i=1}^{k}y_iu_{ij}$, $\gamma_j^+=\max\{0,\gamma_j\}$ and $\gamma_j^-=\max\{0,\gamma_j\}$. Now by Cauchy Schwarz inequality we have

$$Ee^{\sqrt{2h}V^T\mathbf{y}}$$

$$\le\{E\,\exp(2\sqrt{2h}\sum_{j=1}^{n}X_j\gamma_j^+)E\,\exp(-2\sqrt{2h}\sum_{j=1}^{n}X_j\gamma_j^-)\}^{1/2}$$

$$\le[\prod_{j=1}^{n}E(e^{2\sqrt{2h}X_j\gamma_j^+})\prod_{j=1}^{n}E(e^{-2\sqrt{2h}X_j\gamma_j^-})]^{1/2}$$

$$\le[(\prod_{j=1}^{n}e^{4h\gamma_j^{+2}})(\prod_{j=1}^{n}e^{4h\gamma_j^{-2}})]^{1/2}=e^{2h\mathbf{y}^T\mathbf{y}}. \qquad (2.2)$$

The second inequality holds by Lemma 1.2 and third inequality is true by Lemma 1.3. Now by (2.1) and (2.2), similar to the proof of Lemma 1.2 in [7], we have

$$Ee^{hQ_n}\le\int_{R^k}e^{2h\mathbf{y}^T\mathbf{y}}q(\mathbf{y})d\mathbf{y}$$

$$=(\det L_0)^{-1/2}(2\pi)^{-1/2}\int_{R^k}\exp(\frac{-1}{2}\sum_{i=1}^{k}y_i^2\frac{1-4h\lambda_i}{\lambda_i})d(\mathbf{y})$$

$$\le\exp(\frac{-1}{2}\sum_{i=1}^{n}\ln(1-4h\lambda_i)), \qquad (2.3)$$

where $0<4h\max_{i\le n}\lambda_i<1$.

Lemma 2.2. Let $\{X_n,n\ge1\}$ be a sequence of ND subgaussian random variables and A_n be a positive semidefinite matrix. Then for $0\le h\le1/4\mu_n$,

$$Ee^{hQ_n^*}\le\exp(h^2B_n(1+\frac{2}{3}.\frac{4h\mu_n}{1-4h\mu_n})), \qquad (2.4)$$

where $B_n=4\sum_{i=1}^{n}\lambda_i^2$.

Proof. By transformation $X_j\leftrightarrow X_j/\alpha_j$, we assume that $\alpha_j=1$ for all j .

By Lemma 2.1 we have

$$Ee^{hQ^*_n}\le\exp(-2tr(A_n)-\frac{1}{2}\sum_{i=1}^{n}\ln(1-4h\lambda_i)). \qquad (2.5)$$

In the other hand

$$\exp(-2h\lambda_i-\frac{1}{2}\sum_{i=1}^{n}\ln(1-4h\lambda_i))$$

$$\le\exp(4h\lambda_i^{\,2}(1+\frac{2}{3}(4h\mu_n+(4h\mu_n)^2+...)))$$

$$=\exp(4h^2\lambda_i^2(1+\frac{2}{3}.\frac{4h\mu_n}{1-4h\mu_n})), \qquad (2.6)$$

for $0\le h\le1/4\mu_n$. Now (2.4) obtain of (2.5) and (2.6).

Theorem 2.1. Let $\{A_n\}$ be a sequence of positive semidefinite symmetric matrices. Then for every $0<\delta<1$ the following inequalities are true for all n :

(A)

$$P[Q_n^*>y]\le\exp(-\frac{y^2}{4B_n}(1-\frac{2}{3}.\frac{2y\mu_n}{B_n}(1-\frac{2y\mu_n}{B_n})^{-1})),$$

for $0\le y\le((1-\delta)/2\mu_n)B_n$.

(B)

$$P[Q_n^*>y]\le\exp(-\frac{y(1-\delta)}{8\mu_n}(1-\frac{2(1-\delta)}{3\delta})),$$

for $y\ge((1-\delta)/2\mu_n)B_n$.

(C)

$$P[Q_n^*>y]\le C(\delta)\exp(-\frac{y(1-\delta)}{2\sqrt{B_n}}),$$

for some constant $C(\delta)>0$ and all $y>0$.

Proof. Method of the proof is the same as proposition 1.1 [7] in the case of independent.

Corollary 2.1. Under the assumptions of Theorem 2.1, i) If $\{y_n\}$ is a sequence of positive real numbers such that $y_n\mu_n/B_n\to0$, then for every $0<\delta<1$ and sufficiently large n ,

$$P[Q_n^* > y_n] \le \exp(-\frac{(1-\delta)y_n^2}{4B_n}).$$

ii) For every $y > 0$,

$$P[Q_n^* > y] \le \exp(-\min(\frac{y}{48\mu_n}, \frac{y^2}{12B_n})).$$

Theorem 2.2. Under the assumptions of Theorem 2.1 the following inequalities are true:

$$P[Q_n - EQ_n > y]$$
$$\le \exp(-\frac{y^2}{4B_n}(1-\frac{2}{3}.\frac{2y\mu_n}{B_n}(1-\frac{2y\mu_n}{B_n})^{-1} + \frac{t_n y}{2B_n})),$$

for $0 \le y \le ((1-\delta)/2\mu_n)B_n$ and

$$P[Q_n - EQ_n > y]$$
$$\le \exp(-\frac{y(1-\delta)}{8\mu_n}(1-\frac{2(1-\delta)}{3\delta}) + \frac{t_n(1-\delta)}{4\mu_n}),$$

for $y \ge ((1-\delta)/2\mu_n)B_n$, where $t_n = 2\sum_{i=1}^n a_{ii}\alpha_i^2 - \sum_{i=1}^n\sum_{j=1}^n a_{ij} Cov(X_i, X_j)$.

Proof. By Markov inequality and (2.4) for every n and $y > 0$, we have

$$P[Q_n - EQ_n > y] \le e^{-hy} Ee^{h(Q_n - EQ_n)}$$
$$= e^{-hy} E\exp(h(Q_n^* + 2tr(V_n A_n V_n) - EQ_n))$$
$$\le \exp(-hy + h^2 B_n(1+\frac{2}{3}.\frac{4h\mu_n}{1-4h\mu_n}) + ht_n). \qquad (2.7)$$

Hence by putting $h = y/2B_n$ and $h = \frac{1-\delta}{4\mu_n}$ for $0 \le y \le ((1-\delta)/2\mu_n)B_n$ and $y \ge ((1-\delta)/2\mu_n)B_n$, respectively, the proof is completed.

3. An Application to the LIL

In this section by using the notations of section 2 and Theorems 2.1 and 2.2, we prove a law of the iterated logarithm for quadratic forms in the case that $\{X_n, n \ge 1\}$ is a sequence of ND subgaussian random variables.

Theorem 3.1. Let $\{X_n, n \ge 1\}$ be a sequence of ND subgaussian random variables and $\{A_n\}$ be a sequence of positive semidefinite symmetric matrices, $B_n \to \infty$ and $Q_n = \mathbf{X}_\mathbf{n}^T A_n \mathbf{X}_\mathbf{n}$. Then

i)

$$\overline{\lim} \frac{Q_n^*}{\chi_1(n)} \le 1. \qquad w.p.1.$$

ii) If

$$\mu_n = o((B_n / \log_2 B_n)^{1/2}), \qquad (3.1)$$

and

$$t_n = o((B_n / \log_2 B_n)^{1/2}), \qquad (3.2)$$

then

$$\overline{\lim} \frac{Q_n - EQ_n}{\chi_2(n)} \le 1. \qquad w.p.1.$$

where

$$\chi_i = (2B_n)^{1/2} \log_2^{1/i} B_n \quad i = 1,2. \qquad (3.3)$$

With $\log_2 x = \log\log x$ and $\log x = \max\{1, \ln x\}$, for any $x > 0$.

Proof. i) By Theorem 2.1 for $y = \chi_1(n)$ and $\delta \in (0,1)$ we have

$$P[Q_n^* > \chi_1(n)] \le C(\delta)\exp(-\frac{(1-\delta)}{\sqrt{2}}\log_2 B_n),$$

and

$$P[Q_n^* \le \chi_1(n)] \ge 1 - C(\delta)\exp(-\frac{(1-\delta)}{\sqrt{2}}\log_2 B_n).$$

Since $B_n \to \infty$, we have

$$\overline{\lim_n} P[Q_n^* \le \chi_1(n)] = 1.$$

Hence by Lemma 1.5,

$$P[\frac{Q_n^*}{\chi_1(n)} \le 1 \quad i.o] = 1.$$

ii) Let $y = \chi_2(n)$ and $\delta \in (0,1)$. Since $\mu_n = o((\frac{B_n}{\log_2 B_n})^{1/2})$, then there exists $N > 0$ such that for every $n \ge N$ and $\varepsilon > 0$,

$$\frac{\mu_n(\log_2 B_n)^{1/2}}{\sqrt{B_n}} \le \varepsilon .$$

Since ε is arbitrary, by (3.3) we can assume $0 \le y \le ((1-\delta)/2\mu_n)B_n$.

Now with substituting the value y of (3.3) into Theorem 2.2 we have

$$P[Q_n - EQ_n > \chi_2(n)]$$

$$\le \exp(-\frac{\log_2 B_n}{8}(1-\frac{2\sqrt{2}}{3\sqrt{B_n}}\mu_n(\log_2 B_n)^{1/2}$$

$$(1-\frac{\sqrt{2}\mu_n(\log_2 B_n)^{1/2}}{\sqrt{B_n}})^{-1})$$

$$+\frac{(\log_2 B_n)^{1/2}}{4\sqrt{B_n}}t_n).$$

Hence by (3.1) and (3.2),

$$P[Q_n - EQ_n > \chi_2(n)]$$

$$\le \exp(-\frac{\log_2 B_n}{8}(1-o(1)(1-o(1))^{-1}) + o(1))$$

Since $B_n \to \infty$ as $n \to \infty$, the right side of the last inequality tends to zero and so

$$\overline{\lim_n} P[Q_n - EQ_n \le \chi_2(n)] = 1 .$$

Hence by Lemma 1.5,

$$P[\frac{Q_n - EQ_n}{\chi_2(n)} \le 1 \quad i.o] = 1 .$$

This completes the proof.

For subgaussian sequence of ND random variables we have the following example.

Example. Let $\{X_n, n \ge 1\}$ be a sequence of standard normal distributions such that $Cov(X_i, X_j) \le 0$, for each $i \ne j$, then it is a sequence of ND strictly subgaussian random variables with $\tau(X_i) = 1$, hence

$$\overline{\lim} \frac{\sum_{i=1}^{n} X_i^2 - 2n}{(8n)^{1/2}\log_2 4n} \le 1. \qquad w.p.1 .$$

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