Maximal Inequalities for Associated Random Variables

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Abstract

In a celebrated work by Shao [13] several inequalities for negatively associated random variables were proved. In this paper we obtain some maximal inequalities for associated random variables. Also we establish a maximal inequality for demimartingales which generalizes and improves the result of Christofides [4].

Keywords: Associated random variables; Complete convergence; Demimartingales

1. Introduction

Definition 1.1. A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be associated if for any two coordinatewise non-decreasing functions *f* and *g* on \mathbb{R}^n

 $Cov(f(X_1,...,X_n),g(X_1,...,X_n)) \ge 0,$

assuming of course that covariance exists. An infinite family of random variables is said to be associated if every finite subfamily is associated.

This definition was introduced by Esary *et al.* [5] as an extension of the bivariate notion of positive quadrant dependence of Lehmann [7]. Associated random variables have found many applications especially in reliability theory. Many authors have studied this concept providing interesting results and applications [2,8-10,12].

Definition 1.2. Let $S_1, S_2, ...$ be a L^1 sequence of random variables. Assume for all j = 1, 2, ...

$$E[(S_{i+1} - S_i)f(S_1, ..., S_i)] \ge 0, \qquad (1.1)$$

for all coordinatewise nondecreasing functions f such that the expectation is defined. Then $\{S_i, j \ge 1\}$ is

called a demimartingale. If in addition the function f is assumed to be nonnegative, then sequence $\{S_j, j \ge 1\}$ is called a demisubmartingale.

Remark. If the function f is not required to be nondecreasing then (1.1) is equivalent to the condition that $\{S_j, j \ge 1\}$ is a martingale with the natural choice of σ -fileds. Similarly, if f is assumed to be nonnegative and not necessarily nondecreasing (1.1) is equivalent to the condition that $\{S_j, j \ge 1\}$ is a submartingale.

Demimartingale was introduced by Newman and Wright [10]. Proposition 2 of Newman and Wright shows that partial sum of a sequence of mean zero associated random variables is demimartingale [10]. Chow proved a maximal inequality for submartingales which contains the Hajek-Renyi inequality and other inequalities as special cases [3]. Christofides showed that Chow's maximal inequality for submartingales can be extended to the case of demisubmartingales [4]. We prove a maximal inequality for demimartingales which generalizes and improves the result of [4].

2. Main Results

The main purposes of this paper are to establish some

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maximal inequalities for associated random variables (Theorem 2), also a maximal inequality for demimartingales which generalizes and improves the result of [4]. To prove the main results we will need the following two lemmas. The proof of Theorem 1 is based on the following lemmas.

Lemma 1. Let S_1, S_2, \dots be a demisubmartingale (or a demimartingale) and g a nondecreasing convex function then $g(S_1), g(S_2), \dots$ is a demisubmartingale.

For proof see [4].

The following lemma is in [13].

Lemma 2. For any $x \ge 0$,

$$\ln(1+x) \ge \frac{x}{1+x} + \frac{x^2}{2(1+x)^2} \left(1 + \frac{2}{3}\ln(1+x)\right).$$

The main results are the following.

Theorem 1. Let $\{X_i, 1 \le i \le n\}$ be a sequence of associated random variables with finite second moments. Let $S_j = \sum_{i=1}^{j} X_i$, $ES_n \le 0$ and $s_n^2 = ES_n^2$.

If $P(S_n \le c_n) = 1$, $n \ge 1$ where $0 < c_n \uparrow$, then for any x > 0,

$$P(\max_{1 \le j \le n} S_j \ge x)$$

$$\le \exp\{-\frac{x^2}{2(s_n^2 + xc_n)} [1 + \frac{2}{3} \ln(1 + \frac{xc_n}{s_n^2})]\}.$$
 (2.1)

Proof. Noting that $(e^x - 1 - x)/x^2$ is a non-decreasing function of x on R, for any t > 0, we have

$$Ee^{tS_n} = 1 + tES_n + E[(\frac{e^{tS_n} - 1 - tS_n}{S_n^2})S_n^2]$$

$$\leq 1 + (\frac{e^{tC_n} - 1 - tC_n}{c_n^2})s_n^2$$

$$\leq \exp\{s_n^2(\frac{e^{tC_n} - 1 - tC_n}{c_n^2})\}.$$

Consequently, by Lemma 1 observing that $\{e^{iS_j}, j = 1,..,n\}$ is a demisubmartingales, the Doob's inequality for demisubmartingale ([4]) guarantees that for any t > 0,

$$P(\max_{1 \le j \le n} S_j \ge x) = P(\max_{1 \le j \le n} e^{tS_j} \ge e^{tx}) \le e^{-tx} (Ee^{tS_n})$$
$$\le \exp\{-tx + S_n^2 (\frac{e^{tC_n} - 1 - tC_n}{c^2})\}.$$

Setting

$$t = \frac{1}{c_n} \ln(1 + \frac{xc_n}{s_n^2})$$

in the right-hand side of the last inequality, we obtain

$$P(\max_{1 \le j \le n} S_j \ge x) \le \exp\{\frac{x}{c_n} - \frac{x}{c_n}(1 + \frac{s_n^2}{xc_n})\ln(1 + \frac{xc_n}{s_n^2})\}.$$
(2.2)
By Lemma 2, we have

$$\frac{x}{c_n} (1 + \frac{s_n^2}{xc_n}) \ln(1 + \frac{xc_n}{s_n^2})$$

$$\geq \frac{x}{c_n} (1 + \frac{s_n^2}{xc_n}) \{\frac{xc_n}{(s_n^2 + xc_n)} + \frac{1}{2} (\frac{xc_n}{s_n^2 + xc_n})^2 [1 + \frac{2}{3} \ln(1 + \frac{xc_n}{s_n^2})]\}$$

$$=\frac{x}{c_n}+\frac{x^2}{2(s_n^2+xc_n)}[1+\frac{2}{3}\ln(1+\frac{xc_n}{s_n^2})],$$

this proves (2.1), by (2.2).

Theorem 2. Let $\{X_i, 1 \le i \le n\}$ be a sequence of associated random variables with zero means and finite second moments. Let $S_j = \sum_{i=1}^{j} X_i$ and $s_n^2 = ES_n^2$. Then for all x > 0, a > 0,

$$P(\max_{1 \le j \le n} S_{j} \ge x) \le P(\max_{1 \le j \le n} X_{j} > a) + \exp\{-\frac{x^{2}}{2(s_{n}^{2} + nax)} [1 + \frac{2}{3} \ln(1 + \frac{nax}{s_{n}^{2}})]\}$$
(2.3)

and

$$P(\max_{1 \le j \le n} |S_j| \ge x) \le 2P(\max_{1 \le j \le n} |X_j| > a) + 2\exp\{-\frac{x^2}{2(s_n^2 + nax)} [1 + \frac{2}{3}\ln(1 + \frac{nax}{s_n^2})]\}.$$
(2.4)

In particular, we have

$$P(\max_{1 \le j \le n} |S_j| \ge x) \le 2P(\max_{1 \le j \le n} |X_j| > a) + 2\exp\{-\frac{x^2}{4s_n^2}\} + 2\left(\frac{s_n^2}{s_n^2 + nax}\right)^{\frac{x}{6na}}.$$
(2.5)

Proof. Clearly

$$P(\max_{1 \le j \le n} | S_j| \ge x) \le P(\max_{1 \le j \le n} S_j \ge x) + P(\max_{1 \le j \le n} (-S_j) \ge x).$$

Since $\{-X_i, 1 \le i \le n\}$ is a sequence of associated random variables with zero means and finite second moments, so (2.4) is a direct consequence of (2.3).

(2.5) follows from (2.4) easily, considering whether $s_n^2 \le nax$ or $s_n^2 > nax$. We need only to prove (2.3).

Let
$$Y_i = \min(X_i, a)$$
, $i = 1, ..., n$, $T_j = \sum_{i=1}^{n} Y_j$. We have

$$P(\max_{1 \le j \le n} S_j \ge x) \le P(\max_{1 \le j \le n} X_j > a)$$

$$+ P(\max_{1 \le j \le n} X_j \le a, \max_{1 \le j \le n} S_j \ge x)$$
(2.6)

$$\le P(\max_{1 \le j \le n} X_j > a) + P(\max_{1 \le j \le n} T_j \ge x).$$

It is easy to show that $\{Y_i, 1 \le i \le n\}$ is associated sequence with $EY_i \le 0$ (see P(4) in [5]). Applying Theorem 1 with $c_n = na$, we obtain

$$P(\max_{1 \le j \le n} T_j \ge x) \le \exp\{-\frac{x^2}{2(s_n^2 + nax)} [1 + \frac{2}{3} \ln(1 + \frac{nax}{s_n^2})]\},\$$

this proves (2.3), by (2.6).

The following theorem is a maximal inequality for demimartingales that generalizes and improves the result of [4].

Theorem 3. Let $S_0, S_1, S_2, ...$ be a demimartingale, with $S_0 = 0$. Let g be a non-decreasing convex function on R^+ with $g(0^+) = 0$, $g(xy) \le g(x)g(y)$ for every positive x and y and let $\{c_n, n \ge 1\}$ be a non-increasing sequence of positive numbers. Then for every x > 0,

$$P(\max_{m \le j \le n} c_{j} S_{j} \ge x) \le \frac{1}{g(x)} \{g(c_{n}) E(g(S_{n}))^{+} + \sum_{j=m}^{n-1} [(g(c_{j}) - g(c_{j+1})) E(g(S_{j}))^{+}]\}.$$
(2.7)

Remark 1. For the case of independent random variables see [11, p. 57].

Remark 2. Taking g(x) = x, m = 1 in (2.7) provides the inequality in Theorem 2.1 of [4].

Proof of Theorem 3. Let $A = \{\max_{m \le j \le n} c_j S_j \ge x\}$. Then A can be written as $A = \bigcup_{j=m}^n A_j$, where $A_j = \{c_i S_i < x, m \le i < j, c_j S_j \ge x\}, m \le j \le n$, the A_i 's are disjoint. Therefore,

$$g(x)P(A) = g(x)\sum_{j=m}^{n} P(A_{j}) = \sum_{j=m}^{n} E[g(x)I(A_{j})]$$

$$\leq \sum_{j=m}^{n} E[g(c_{j}S_{j})I(A_{j})]$$

$$\leq \sum_{j=m}^{n} E[g(c_{j})g(S_{j})I(A_{j})]$$

$$= \sum_{j=m}^{n} E[g(c_{j})g^{+}(S_{j})I(A_{j})].$$

The rest of proof is similar to the proof in Theorem 2.1. of [4].

3. An Application for the Complete Convergence

Complete convergence gives a convergence rate with respect to the strong law of large numbers. One can refer to [1,6] for details. Applying the maximal inequality (2.5), one can get the following result easily.

Theorem 4. Let $1 \le p \le 2$, $pr \ge 1$, and let $\{X_n, n \ge 1\}$ be a strictly stationary associated sequence with $EX_n = 0$, $E |X_n|^p < \infty$, and

$$\sigma^2 := EX_1^2 + 2\sum_{j=2}^{\infty} EX_j X_j < \infty .$$

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p-2} P\left(\max_{1 \le j \le n} |S_j| \ge \varepsilon n^{r+1}\right) < \infty$$

Proof. Note that $s_n^2 \le n\sigma^2$. Applying (2.5) with $x = \varepsilon n^{r+1}$, and $a = kn^r$, where $k = \frac{\varepsilon(2r+1)}{6pr}$, we obtain the result.

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