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Maximal Inequalities for Associated Random Variables

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Abstract

In a celebrated work by Shao [13] several inequalities for negatively associated random variables were proved. In this paper we obtain some maximal inequalities for associated random variables. Also we establish a maximal inequality for demimartingales which generalizes and improves the result of Christofides [4].

Keywords: Associated random variables; Complete convergence; Demimartingales

1. Introduction

Definition 1.1. A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be associated if for any two coordinatewise non-decreasing functions *f* and *g* on R^n

$$Cov(f(X_1,...,X_n), g(X_1,...,X_n)) \ge 0,$$

assuming of course that covariance exists. An infinite family of random variables is said to be associated if every finite subfamily is associated.

This definition was introduced by Esary *et al.* [5] as an extension of the bivariate notion of positive quadrant dependence of Lehmann [7]. Associated random variables have found many applications especially in reliability theory. Many authors have studied this concept providing interesting results and applications [2,8-10,12].

Definition 1.2. Let $S_1, S_2, ...$ be a L^1 sequence of random variables. Assume for all $j = 1, 2, ...$

$$E[(S_{j+1} - S_j) f(S_1,...,S_j)] \ge 0, \qquad (1.1)$$

for all coordinatewise nondecreasing functions *f* such that the expectation is defined. Then $\{S_j, j \ge 1\}$ is called a demimartingale. If in addition the function *f* is assumed to be nonnegative, then sequence $\{S_j, j \ge 1\}$ is called a demisubmartingale.

Remark. If the function *f* is not required to be nondecreasing then (1.1) is equivalent to the condition that $\{S_j, j \ge 1\}$ is a martingale with the natural choice of σ-fileds. Similarly, if *f* is assumed to be nonnegative and not necessarily nondecreasing (1.1) is equivalent to the condition that $\{S_j, j \ge 1\}$ is a submartingale.

Demimartingale was introduced by Newman and Wright [10]. Proposition 2 of Newman and Wright shows that partial sum of a sequence of mean zero associated random variables is demimartingale [10]. Chow proved a maximal inequality for submartingales which contains the Hajek-Renyi inequality and other inequalities as special cases [3]. Christofides showed that Chow's maximal inequality for submartingales can be extended to the case of demisubmartingales [4]. We prove a maximal inequality for demimartingales which generalizes and improves the result of [4].

2. Main Results

The main purposes of this paper are to establish some

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maximal inequalities for associated random variables (Theorem 2), also a maximal inequality for demimartingales which generalizes and improves the result of [4]. To prove the main results we will need the following two lemmas. The proof of Theorem 1 is based on the following lemmas.

Lemma 1. Let $S_1, S_2, ...$ be a demisubmartingale (or a demimartingale) and *g* a nondecreasing convex function then $g(S_1), g(S_2), ...$ is a demisubmartingale.

For proof see [4].

The following lemma is in [13].

Lemma 2. For any $x \geq 0$,

$$\ln(1+x) \geq \frac{x}{1+x} + \frac{x^2}{2(1+x)^2}(1+\frac{2}{3}\ln(1+x)).$$

The main results are the following.

Theorem 1. Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of associated random variables with finite second moments. Let $S_j = \sum_{i=1}^{j} X_i$, $ES_n \leq 0$ and $s_n^2 = ES_n^2$.

If $P(S_n \leq c_n) = 1$, $n \geq 1$ where $0 < c_n \uparrow$, then for any $x > 0$,

$$P(\max_{1 \leq j \leq n} S_j \geq x) \leq \exp\{-\frac{x^2}{2(s_n^2 + xc_n)}[1+\frac{2}{3}\ln(1+\frac{xc_n}{s_n^2})]\}. \quad (2.1)$$

Proof. Noting that $(e^x - 1 - x)/x^2$ is a non-decreasing function of *x* on *R*, for any $t > 0$, we have

$$Ee^{tS_n} = 1 + tES_n + E[(\frac{e^{tS_n} - 1 - tS_n}{S_n^2})S_n^2]$$

$$\leq 1 + (\frac{e^{tc_n} - 1 - tc_n}{c_n^2})s_n^2$$

$$\leq \exp\{s_n^2(\frac{e^{tc_n} - 1 - tc_n}{c_n^2})\}.$$

Consequently, by Lemma 1 observing that $\{e^{tS_j}, j = 1, .., n\}$ is a demisubmartingales, the Doob's inequality for demisubmartingale ([4]) guarantees that for any $t > 0$,

$$P(\max_{1 \leq j \leq n} S_j \geq x) = P(\max_{1 \leq j \leq n} e^{tS_j} \geq e^{tx}) \leq e^{-tx}(Ee^{tS_n})$$

$$\leq \exp\{-tx + s_n^2(\frac{e^{tc_n} - 1 - tc_n}{c_n^2})\}.$$

Setting

$$t = \frac{1}{c_n}\ln(1+\frac{xc_n}{s_n^2})$$

in the right-hand side of the last inequality, we obtain

$$P(\max_{1 \leq j \leq n} S_j \geq x) \leq \exp\{\frac{x}{c_n} - \frac{x}{c_n}(1+\frac{s_n^2}{xc_n})\ln(1+\frac{xc_n}{s_n^2})\}. \quad (2.2)$$

By Lemma 2, we have

$$\frac{x}{c_n}(1+\frac{s_n^2}{xc_n})\ln(1+\frac{xc_n}{s_n^2})$$

$$\geq \frac{x}{c_n}(1+\frac{s_n^2}{xc_n})\{\frac{xc_n}{(s_n^2+xc_n)} + \frac{1}{2}(\frac{xc_n}{s_n^2+xc_n})^2[1+\frac{2}{3}\ln(1+\frac{xc_n}{s_n^2})]\}$$

$$= \frac{x}{c_n} + \frac{x^2}{2(s_n^2+xc_n)}[1+\frac{2}{3}\ln(1+\frac{xc_n}{s_n^2})],$$

this proves (2.1), by (2.2).

Theorem 2. Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of associated random variables with zero means and finite second moments. Let $S_j = \sum_{i=1}^{j} X_i$ and $s_n^2 = ES_n^2$. Then for all $x > 0$, $a > 0$,

$$P(\max_{1 \leq j \leq n} S_j \geq x) \leq P(\max_{1 \leq j \leq n} X_j > a) + \exp\{-\frac{x^2}{2(s_n^2+nax)}[1+\frac{2}{3}\ln(1+\frac{nax}{s_n^2})]\} \quad (2.3)$$

and

$$P(\max_{1 \leq j \leq n} |S_j| \geq x) \leq 2P(\max_{1 \leq j \leq n} |X_j| > a) + 2\exp\{-\frac{x^2}{2(s_n^2+nax)}[1+\frac{2}{3}\ln(1+\frac{nax}{s_n^2})]\}. \quad (2.4)$$

In particular, we have

$$P(\max_{1\le j\le n}|S_j|\ge x)\le 2P(\max_{1\le j\le n}|X_j|>a)$$
$$+2\exp\{-\frac{x^2}{4s_n^2})+2\left(\frac{s_n^2}{s_n^2+nax}\right)^{\frac{x}{6na}}. \quad (2.5)$$

Proof. Clearly

$$P(\max_{1\le j\le n}|S_j|\ge x)\le P(\max_{1\le j\le n}S_j\ge x)$$
$$+P(\max_{1\le j\le n}(-S_j)\ge x).$$

Since $\{-X_i, 1\le i\le n\}$ is a sequence of associated random variables with zero means and finite second moments, so (2.4) is a direct consequence of (2.3).

(2.5) follows from (2.4) easily, considering whether $s_n^2\le nax$ or $s_n^2>nax$. We need only to prove (2.3). Let $Y_i=\min(X_i,a)$, $i=1,...,n$, $T_j=\sum_{i=1}^{n}Y_j$. We have

$$P(\max_{1\le j\le n}S_j\ge x)\le P(\max_{1\le j\le n}X_j>a)$$
$$+P(\max_{1\le j\le n}X_j\le a,\max_{1\le j\le n}S_j\ge x) \quad (2.6)$$
$$\le P(\max_{1\le j\le n}X_j>a)+P(\max_{1\le j\le n}T_j\ge x).$$

It is easy to show that $\{Y_i, 1\le i\le n\}$ is associated sequence with $EY_i\le 0$ (see $P(4)$ in [5]). Applying Theorem 1 with $c_n=na$, we obtain

$$P(\max_{1\le j\le n}T_j\ge x)\le \exp\{-\frac{x^2}{2(s_n^2+nax)}[1+\frac{2}{3}\ln(1+\frac{nax}{s_n^2})]\},$$

this proves (2.3), by (2.6).

The following theorem is a maximal inequality for demimartingales that generalizes and improves the result of [4].

Theorem 3. Let $S_0,S_1,S_2,...$ be a demimartingale, with $S_0=0$. Let g be a non-decreasing convex function on R^+ with $g(0^+)=0$, $g(xy)\le g(x)g(y)$ for every positive x and y and let $\{c_n, n\ge 1\}$ be a non-increasing sequence of positive numbers. Then for every $x>0$,

$$P(\max_{m\le j\le n}c_jS_j\ge x)\le\frac{1}{g(x)}\{g(c_n)E(g(S_n))^+$$
$$+\sum_{j=m}^{n-1}[(g(c_j)-g(c_{j+1}))E(g(S_j))^+]\}. \quad (2.7)$$

Remark 1. For the case of independent random variables see [11, p. 57].

Remark 2. Taking $g(x)=x$, $m=1$ in (2.7) provides the inequality in Theorem 2.1 of [4].

Proof of Theorem 3. Let $A=\{\max_{m\le j\le n}c_jS_j\ge x\}$. Then A can be written as $A=\bigcup_{j=m}^{n}A_j$, where $A_j=\{c_iS_i<x,\ m\le i<j, c_jS_j\ge x\}$, $m\le j\le n$, the A_j's are disjoint. Therefore,

$$g(x)P(A)=g(x)\sum_{j=m}^{n}P(A_j)=\sum_{j=m}^{n}E[g(x)I(A_j)]$$
$$\le\sum_{j=m}^{n}E[g(c_jS_j)I(A_j)]$$
$$\le\sum_{j=m}^{n}E[g(c_j)g(S_j)I(A_j)]$$
$$=\sum_{j=m}^{n}E[g(c_j)g^+(S_j)I(A_j)].$$

The rest of proof is similar to the proof in Theorem 2.1. of [4].

3. An Application for the Complete Convergence

Complete convergence gives a convergence rate with respect to the strong law of large numbers. One can refer to [1,6] for details. Applying the maximal inequality (2.5), one can get the following result easily.

Theorem 4. Let $1\le p\le 2$, $pr\ge 1$, and let $\{X_n, n\ge 1\}$ be a strictly stationary associated sequence with $EX_n=0$, $E|X_n|^p<\infty$, and

$$\sigma^2:=EX_1^2+2\sum_{j=2}^{\infty}EX_1X_j<\infty.$$

Then for all $\varepsilon>0$,

$$\sum_{n=1}^{\infty}n^{rp-2}P(\max_{1\le j\le n}|S_j|\ge\varepsilon n^{r+1})<\infty.$$

Proof. Note that $s_n^2\le n\sigma^2$. Applying (2.5) with $x=\varepsilon n^{r+1}$, and $a=kn^r$, where $k=\frac{\varepsilon(2r+1)}{6pr}$, we obtain the result.

References

1. Baum L.E., Katz M. Convergence rates in the law of large numbers. *Trans. Amer. Math. Soc.*, **120:** 108-123 (1965).
2. Birkel T. Moment bounds for associated sequences. *Ann. Probab.*, **16:** 1184-1193 (1988).
3. Chow Y.S. A martingale inequality and the law of large numbers. *Proc. Amer. Math. Soc.*, **11:** 107-111 (1960).
4. Christofieds T.C. Maximal inequality for demimartingales and a strong law of large numbers. *Statist. Probab. Lett.*, **50:** 357-363 (2000).
5. Esary J., Proschan F., and Walkup D. Association of random variables with applications. *Ann. Math. Statist.*, **38:** 1466-1474 (1967).
6. Hsu P.L. and Robbins H. Complete convergence and the law of large numbers. *Proc. Nat. Acad. Sci. U.S.A.*, **33:** 25-31 (1947).
7. Lehmann E.L. Some concepts of dependence. *Ann. Math. Statist.*, **37:** 1137-1153 (1966).
8. Matula P. On the almost sure central limit theorem for associated random variables. *Probab. Math. Statist.*, **18:** 411-416 (1998).
9. Newman C.M. and Wright A.L. An invariance principle for certain dependent sequences. *Ann. Probab.*, **9:** 671-675 (1981).
10. Newman C.M. and Wright A.L. Associated random variables and martingale inequalities. *Z. Wahrsch. Verw. Geb.*, **59:** 361-371 (1982).
11. Petrov V.V. *Sums of Independent Random Variables*. Springer-Verlag, NewYork (1975).
12. Prakasa Rao B.L.S. Hajek-Renyi type inequality for associated sequences. *Statist. Probab. Lett.*, **57:** 139-143 (2002).
13. Shao Q.M. A comparison theorem on moment inequalities between negatively associated and independent random variables. *J. Theoretical. Probab.*, **13:** 343-356 (2000).