Journal of Sciences, Islamic Republic of Iran 16(3): 261-265 (2005) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Strong Laws for Weighted Sums of Negative Dependent Random Variables

H.R. Nili Sani,[1,*] M. Amini,[2] and A. Bozorgnia[3]

[1] Department of Statistics, University of Birjand, Birjand, Islamic Republic of Iran
[2] Department of Mathematics, University of Sistan and Baluchestan, Islamic Republic of Iran
[3] Department of Statistics, University of Ferdowsi, Mashhad, Islamic Republic of Iran

Abstract

In this paper, we discuss strong laws for weighted sums of pairwise negatively dependent random variables. The results on i.i.d case of Soo Hak Sung [9] are generalized and extended.

Keywords: Pairwise negatively dependent random variables; Complete convergence; Almost sure convergence; Weighted sums[1]

1. Introduction

For a sequence of independent random variables $\{X_n, n \geq 1\}$ and a double array of constants $\{a_{ni}, n \geq 1, i \geq 1\}$ (called weights), the almost sure (a.s.) limiting behavior of weighted sums $\sum_{i=1} a_{ni} X_i$ was studied by many authors [1]. Sung [9] recently established the following extension of Bai and Cheng [1].

Theorem S [9]: Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables satisfying $E(X) = 0$ and

$$E[\exp(h|X|^{\gamma})] < \infty \text{ for any } h > 0 (\gamma > 0), \qquad (1)$$

and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying

$$A_{\alpha} = \limsup_{n\to\infty} A_{\alpha,n} < \infty, \; A_{\alpha,n}^{\alpha} = \sum_{i=1}^{n} |a_{ni}|^{\alpha} / n \qquad (2)$$

for some $1 < \alpha \leq 2$. Then for $0 < \gamma \leq 1$ and $b_n = n^{1/\alpha} \log^{1/\gamma}(n)$

$$\sum_{i=1}^{n} a_{ni} X_i / b_n \to 0 \quad a.s.,$$

moreover, for $\gamma > 1$ and $b_n = n^{1/\alpha} (\log(n))^{1/\gamma+\delta}$

$$\sum_{i=1}^{n} a_{ni} X_i / b_n \to 0 \quad a.s.,$$

where $\delta = 1 - 1/\gamma - (\gamma - 1)/(1 + \alpha\gamma - \alpha)$.

But, in many stochastic models the assumption of independence among r.v.'s isn't plausible. In fact, increases in some r.v.'s are often related to decreases in other r.v.'s and the assumption of negative dependence is more appropriate than the independence assumption. The main aim of this paper is to try to extend and generalize Theorem S.

2. Negative Dependence

Definition 1. The random variables $X_1, \cdots, X_n (n \geq 2)$ are said to be *(mutually) negatively*

* *E-mail: nilisani@yahoo.com*
[1] AMS: 60F15

dependent (henceforth ND) if both [6]

$$P(X_1 \le x_1, \cdots, X_n \le x_n) \le \prod_{i=1}^{n} P(X_i \le x_i) \qquad (3)$$

and

$$P(X_1 \ge x_1, \cdots, X_n \ge x_n) \le \prod_{i=1}^{n} P(X_i \ge x_i) \qquad (4)$$

for all $x_1, \cdots, x_n \in R$.

The random variables $X_1, \cdots, X_n (n \ge 2)$ are said to be *pairwise negatively dependent* (PND) if (X_i, X_j) is ND for every $i \ne j$, $i, j = 1, \cdots, n$. Events $\{E_n\}$ are said to be PND (or ND) if their indicator functions are.

An infinite sequence is ND (or PND) if every finite subsequence is.

We will need the following result [3,6,8].

Proposition 1. Let $\{X_n\}$ be a sequence of PND(or ND) r.v.'s. Then

(i) $Cov(X_i, X_j) \le 0 \quad i \ne j$,

(ii) If $\{f_n\}$ is a sequence of Borel functions all of which are monotone increasing (or all monotone decreasing) then $\{f_n(X_n)\}$ is a sequence of PND (or ND) r.v.'s.

(iii) The Borel-Cantelli lemma holds for PND (or ND) events.

For other related negative-dependence concepts, we refer to Lehmann [7], and the monograph Joe [5].

Since the conception of PND sequences contains independent and negatively associated sequences, which have a lot of applications, *e.g.* in reliability theory, Percolation theory and multivariate statistical analysis, their limit properties have aroused wide interest. Bozorgnia *et al.* [2] and Taylor *et al.* [10] have studied the strong law of large numbers for weighted sums of negatively dependent r.v.'s.

Definition 2. The sequence $\{X_n, n \ge 1\}$ of r.v.'s is said to be *stochastically dominated in Cesaro sense* by a r.v. Y providing that there exists such a positive constant K that for all $\lambda \ge 0$ and $n \ge 1$ we have

$$\frac{1}{n}\sum_{i=1}^{n} P(|X_i| \ge \lambda) \le KP(|Y| \ge \lambda). \qquad (5)$$

The sequence $\{X_n, n \ge 1\}$ of r.v.'s is said to be stochastically bounded by r.v. X if there exists such a positive constant K that for all $\lambda \ge 0$ and $n \ge 1$

$$P(|X_n| \ge \lambda) \le KP(|X| \ge \lambda).$$

3. Results

To prove our main result, we'll need the following lemma. That provides some conditions under which the weighted sum converges completely and determinate the rate of convergence. The concept of complete convergence introduced by Hsu and Robbins [4] is as follows. The sequence $\{X_n, n \ge 1\}$ of random variables converges to zero completely (denoted $\lim_{n\to\infty} X_n = 0$ completely), if $\sum_{n=1}^{\infty} P(|X_n| > \varepsilon) < \infty$ for every $\varepsilon > 0$.

In this section $\{a_{ni}\}$ stands for an array of real numbers, $\{l(n)\}$ stands for a non-decreasing sequence of integer numbers such that $l(n) \to \infty$, also $\gamma, \delta, \beta, \alpha, K$ and h stand for positive constants and finally K_1 stands for a generic constant but are not necessarily the same at each occurrence.

Lemma 1. Let $\{X_n, n \ge 1\}$ be a sequence of r.v.'s that are *stochastically dominated in Cesaro sense* by r.v. X that satisfies (1). Let $\{X_{ni}, 1 \le i \le l(n), n \ge 1\}$ be an array of rowwise PND r.v.'s with $E(X_{ni}) = 0$ for $1 \le i \le l(n)$ and $n \ge 1$ that satisfies the following conditions:

(i) $|a_{ni} X_{ni}| \le C |X_i|^{\beta} / \log n$ a.s., for some $0 < \beta \le \gamma$ and some constant $C > 0$.

(ii) $X_{ni}^2 \sum_{i=1}^{l(n)} a_{ni}^2 \le \nu_n |X_i|^{\delta} / \log n$ a.s., for some $\delta > 0$ and some sequence $\{\nu_n\}$ of constants such that $\nu_n l(n) \to 0$.

Then

$$\sum_{n=1}^{\infty} n^{\alpha} P\left(\left|\sum_{i=1}^{l(n)} a_{ni} X_{ni}\right| > \varepsilon\right) < \infty \quad \forall \varepsilon > 0, \ \alpha < 1,$$

thus $\sum_{i=1}^{l(n)} a_{ni} X_{ni}$ converges to zero completely and, hence, a.s.

Proof. Let

$T_n^+ = \sum a_{ni} X_{ni}$ where sum is over i's such that $a_{ni} \ge 0$

$T_n^- = \sum a_{ni} X_{ni}$ where sum is over i's such that $a_{ni} < 0$

then

$$P\left(\left|\sum_{i=1}^{l(n)} a_{ni} X_{ni}\right| > 2\varepsilon\right) \le P(T_n^+ > \varepsilon) + P(T_n^- > \varepsilon)$$

$$+P(T_n^+ < -\varepsilon) + P(T_n^- < -\varepsilon).$$

It is clear that $\frac{1}{n}\sum_{i=1}^{l(n)} E(e^{h|X_i|^\gamma}) \le E(e^{h|X|^\gamma})$. From the inequality $e^x \le 1 + x + \frac{1}{2}x^2 e^{|x|}$ for all $x \in R$, we have

$$E[e^{ta_{ni}X_{ni}}] \le 1 + \frac{1}{2}t^2 a_{ni}^2 E[X_{ni}^2 e^{t|a_{ni}X_{ni}|}]$$

for any $t > 0$. Let $\varepsilon > 0$ be given. By putting $t = 2\log n/\varepsilon$, from condition (i) and condition (ii) we obtain

$$E[e^{ta_{ni}X_{ni}}] \le 1 + \frac{1}{2}\frac{4\log^2 n}{\varepsilon^2} a_{ni}^2 E[X_{ni}^2 \exp(\frac{2\log n}{\varepsilon}|a_{ni}X_{ni}|)]$$

$$\le 1 + \frac{2\log n}{\varepsilon^2}\frac{a_{ni}^2}{\sum_{j=1}^{l(n)} a_{nj}^2} E[\nu_n |X_i|^\delta \exp(\frac{2}{\varepsilon}C|X_i|^\beta)].$$

Since $|x|^\delta \le O(e^{C|x|^\beta})$ for all $x \in R$, then the RHS

$$\le 1 + O(1)\frac{1}{2}\nu_n \log n E[\exp(h|X_i|^\beta)],$$

where $h = \frac{\varepsilon + 2}{\varepsilon}C$. Now using inequality $e^x \ge 1 + x$ for all $x \in R$, we have

$$\le \exp(O(1)\frac{1}{2}\nu_n \log n E(e^{h|X_i|^\beta})).$$

Therefore

$$P(\sum_{i=1}^{l(n)} a_{ni}X_{ni} > 2\varepsilon) \le \exp(\frac{-2\log n}{\varepsilon}E)$$

$$[\exp(O(1)\frac{1}{2}\nu_n \log n \sum_{a_{ni}\ge 0} E(e^{h|X_i|^\beta}))$$

$$+\exp(O(1)\frac{1}{2}\nu_n \log n \sum_{a_{ni}<0} E(e^{h|X_i|^\beta}))].$$

Since $2e^{x+y} \ge e^x + e^y$ for $x \ge 0$, $y \ge 0$, then RHS

$$\le 2\exp(\frac{-2\log n}{\varepsilon}\varepsilon)\ [\exp(O(1)\frac{1}{2}\nu_n \log n \sum_{i=1}^{l(n)} E(e^{h|X_i|^\beta}))],$$

and by stochastically domination in Cesaro sense condition we have

$$\le 2\exp(-2\log n + \frac{1}{2}O(1)\nu_n \log n l(n) E(e^{h|X|^\beta}))$$

and for n sufficiently large

$$\le 2\exp(-2\log n + 1/\tau \log n) = 2n^{-2+1/\tau}$$

where $\tau > 1/(1-\alpha)$. Then

$$\sum_{n=1}^\infty n^\alpha P(\sum_{i=1}^{l(n)} a_{ni}X_{ni} > 2\varepsilon) \le K\sum_{n=1}^\infty n^{-2+1/\tau+\alpha} < \infty. \quad (6)$$

By replacing X_{ni} by $-X_{ni}$ from the above statement, we obtain

$$\sum_{n=1}^\infty n^\alpha P(\sum_{i=1}^{l(n)} a_{ni}X_{ni} < -2\varepsilon) \le K\sum_{n=1}^\infty n^{-2+1/\tau+\alpha} < \infty. \quad (7)$$

Hence the result follows by (6) and (7).

Lemma 1 holds for stochastically bounded sequences if in condition (ii) we have only $\nu_n \to 0$.

The following lemma shows that if condition (1) is replaced by the weaker condition

$$E[\exp(h|X|^\gamma)] < \infty \quad \text{for some } h > 0 (\gamma > 0), \quad (8)$$

then condition (i) can be replaced by stronger condition

(iii) $|a_{ni}X_{ni}| \le u_n |X_i|^\beta / \log n$ a.s., for some $0 < \beta \le \gamma$ and some constant $\{u_n\}$ of constants such that $u_n \to 0$.

Lemma 2. Let $\{X_n, n \ge 1\}$, $\{X_{ni}, 1 \le i \le l(n), n \ge 1\}$ and $\{a_{ni}\}$ be as in Lemma 1 expect that (1) and (i) are replaced by (8) and (iii), respectively. Then

$$\sum_{n=1}^\infty n^\alpha P(\left|\sum_{i=1}^{l(n)} a_{ni}X_{ni}\right| > \varepsilon) < \infty \quad \forall \varepsilon > 0,\ \alpha < 1.$$

Thus $\sum_{i=1}^{l(n)} a_{ni}X_{ni}$ converges to zero completely and, hence, a.s.

The proof of Lemma 2 is analogous to the proof of Lemma 1 and hence omitted.

Theorem 1. Let $\{X_n, n \ge 1\}$ be a sequence of PND r.v.'s which $E(X_n) = 0$, and are stochastically bounded by r.v. X that satisfying in (1). Let $\{a_{ni}\}$ be an array of constants satisfying (2) for some $1 < \alpha \le 2$. Then for $0 < \gamma \le 1$ and $b_n = n^{1/\alpha}\log^{1/\gamma}(n)$

$$\sum_{i=1}^n a_{ni}X_i / b_n \to 0 \quad a.s.,$$

moreover, for $\gamma > 1$ and $b_n = n^{1/\alpha}(\log(n))^{1/\gamma+\delta}$

$$\sum_{i=1}^{n} a_{ni} X_i / b_n \to 0 \quad a.s.,$$

where $\delta = 1 - 1/\gamma - (\gamma - 1)/(1 + \alpha\gamma - \alpha)$.

Proof. We first observe that

$$P(\left|\sum_{i=1}^{n} a_{ni} X_i / b_n\right| > \varepsilon) \le P(\left|T_n^{+} / b_n\right| > \varepsilon / 2)$$

$$+ P(\left|T_n^{-} / b_n\right| > \varepsilon / 2),$$

since in the first sentence $\{a_{ni} X_i\}$ and then in the second sentence $\{a_{ni} X_i\}$ are two disjoint sets of PND r.v.'s, by the Markov inequality we have

$$\le \frac{4}{\varepsilon^2 b_n^2} (E(\sum\nolimits_{i: a_{ni} > 0} (a_{ni} X_i))^2 + E(\sum\nolimits_{i: a_{ni} < 0} (a_{ni} X_i))^2)$$

$$\le \frac{K_1}{b_n^2} \sum_{i=1}^{n} E(a_{ni}^2 X_i^2) \le \frac{K_1}{b_n^2} \sum_{i=1}^{n} a_{ni}^2$$

$$\le K_1 \frac{n^{2/\alpha}}{b_n^2} (\frac{1}{n} \sum_{i=1}^{n} \left|a_{ni}\right|^{\alpha})^{2/\alpha} = K_1 A_{\alpha,n}^2 n^{2/\alpha} / b_n^2 \to 0$$

as $n \to \infty$. It follows that

$$\sum_{i=1}^{n} a_{ni} X_i / b_n \to 0 \quad \text{in probability.}$$

We proceed with two cases.

Case 1: $0 < \gamma \le 1$.

Define $X'_{ni} = X_i I(\left|X_i\right| \le \log^{1/\gamma} n) - \log^{1/\gamma} n I(X_i \le -\log^{1/\gamma} n) + \log^{1/\gamma} n I(X_i > \log^{1/\gamma} n)$ and $X''_{ni} = X_i - X'_{ni}$ for $1 \le i \le l(n)$ and $n \ge 1$. It is obvious that $\{X'_{ni}\}$ and $\{X''_{ni}\}$ are stochastically bounded to *X*. Note that $E(e^{|X|^{\gamma}}) < \infty$ implies that $\sum_{n=1}^{\infty} P(\left|X_n\right| > \log^{1/\gamma} n) < \infty$. Hence, by the Borel-Cantelli lemma, $\sum_{i=1}^{n} \left|X''_{ni}\right|$ is bounded a.s. It follows that

$$b_n^{-1} \sum_{i=1}^{n} \left|a_{ni} X''_{ni}\right| \le b_n^{-1} \max_{1 \le i \le n} \left|a_{ni}\right| \sum_{i=1}^{n} \left|X''_{ni}\right|$$

$$\le A_{\alpha,n} \sum_{i=1}^{n} \left|X''_{ni}\right| / \log^{1/\gamma}(n) \to 0 \qquad (9)$$

a.s. as $n \to \infty$.

To complete the proof of Case 1, we will apply Lemma 1 to r.v. X'_{ni} and weight $b_n^{-1} a_{ni}$. We first note that

$$\left|b_n^{-1} a_{ni} X'_{ni}\right| \le b_n^{-1} \left|a_{ni}\right| (\log n)^{(1-\gamma)/\gamma} \left|X_i\right|^{\gamma}$$

$$\le b_n^{-1} A_{\alpha,n} n^{1/\alpha} (\log n)^{(1-\gamma)/\gamma} \left|X_i\right|^{\gamma}$$

$$= A_{\alpha,n} \left|X_i\right|^{\gamma} / \log(n)$$

and

$$b_n^{-2} X'^2_{ni} \sum_{i=1}^{n} a_{ni}^2 \le X_i^2 b_n^{-2} \sum_{i=1}^{n} a_{ni}^2$$

$$\le A_{\alpha,n}^2 X_i^2 / \log^{2/\gamma}(n).$$

Hence conditions (i) and (ii) of Lemma 1 are satisfied, and hence

$$\sum_{i=1}^{n} a_{ni} X'_{ni} / b_n \to 0 \quad a.s., \qquad (10)$$

The result of Case 1 is proved by (9) and (10).

Case 2: $\gamma > 1$.

Define for each $1 \le i \le l(n)$, $n \ge 1$

$$X_{ni}^{1} = X_i I(\left|X_i\right| \le (\log n)^{\delta_1}) + (\log n)^{\delta_1} I(X_i > (\log n)^{\delta_1})$$

$$-(\log n)^{\delta_1} I(X_i < -(\log n)^{\delta_1})$$

$$X_{ni}^{2} = (X_i - (\log n)^{\delta_1}) I((\log n)^{\delta_1} < X_i \le (\log n)^{1/\gamma}) +$$

$$((\log n)^{1/\gamma} - (\log n)^{\delta_1}) I(X_i > ((\log n)^{1/\gamma})$$

$$X_{ni}^{3} = (X_i - (\log n)^{1/\gamma}) I(X_i > (\log n)^{1/\gamma})$$

$$X_{ni}^{4} = (X_i + (\log n)^{\delta_1}) I(-(\log n)^{1/\gamma} \le X_i < -(\log n)^{\delta_1}) +$$

$$((\log n)^{\delta_1} - (\log n)^{1/\gamma}) I(X_i < -(\log n)^{1/\gamma})$$

$$X_{ni}^{5} = (X_i + (\log n)^{1/\gamma}) I(X_i < -(\log n)^{1/\gamma}),$$

where $\delta_1 = \alpha\delta + \alpha/\gamma - \alpha + 1 = 1/(1 + \alpha\gamma - \alpha)$. Define $a'_{ni} = a_{ni} I(\left|a_{ni}\right| \le n^{1/\alpha} / (\log n)^{\delta_2})$ and $a''_{ni} = a_{ni} - a'_{ni}$ for $1 \le i \le l(n)$ and $n \ge 1$, where $\delta_2 = 1 - 1/\gamma - \delta$. We rewrite

$$b_n^{-1} \sum_{i=1}^{n} a_{ni} X_i = b_n^{-1} \sum_{i=1}^{n} [a'_{ni} X_{ni}^{1} + a''_{ni} X_{ni}^{1}$$

$$+ a_{ni} X_{ni}^{2} + a_{ni} X_{ni}^{3} + a_{ni} X_{ni}^{4} + a_{ni} X_{ni}^{5}]$$

$$=: A_n + B_n + C_n + D_n + E_n + F_n.$$

Since $\left|X_{ni}^{j}\right| \le \left|X_i\right|$, $j = 1, \cdots, 5$, then for every $j = 1, \cdots, 5$, $\{X_{ni}^{j}\}$ are stochastically bounded to *X*. For

A_n, we will apply Lemma 1 to the r.v. X_{ni}^1 and weight $b_n^{-1}a'_{ni}$. Observe that

$$\left|b_n^{-1}a'_{ni}X_{ni}^1\right| \le b_n^{-1}\left|a'_{ni}\right|\left|X_i\right| \le \frac{n^{1/\alpha}}{b_n(\log n)^{\delta_2}}\left|X_i\right| = \frac{1}{\log n}\left|X_i\right|$$

and so condition (i) is satisfied. Moreover, we have

$$(X_{ni}^1)^2\sum_{i=1}^n b_n^{-2}a'^2_{ni} \le \frac{n^{(2-\alpha)/\alpha}\sum_{i=1}^n\left|a_{ni}\right|^\alpha}{b_n^2(\log n)^{\delta_2(2-\alpha)}}(X_{ni}^1)^2 \le \frac{n^{2/\alpha}A_{\alpha,n}^\alpha}{b_n^2(\log n)^{\delta_2(2-\alpha)}}X_i^2 \le \frac{A_{\alpha,n}^\alpha}{(\log n)^{\delta_2(2-\alpha)+2\delta+2/\gamma}}X_i^2.$$

Since $2/\gamma+2\delta+\delta_2(2-\alpha)=(2+\alpha\gamma-\alpha)/(1+\alpha\gamma-\alpha)>1$, condition (ii) is satisfied. Hence $A_n\to 0$ a.s. by Lemma 1.

For B_n, we obtain

$$\left|B_n\right| \le b_n^{-1}(\log n)^{\delta_1}\sum_{i=1}^n\left|a''_{ni}\right| \le \frac{(\log n)^{\delta_1+\delta_2(\alpha-1)}\sum_{i=1}^n\left|a_{ni}\right|^\alpha}{b_n n^{(\alpha-1)/\alpha}} = A_{\alpha,n}^\alpha$$

and so $\limsup_{n\to\infty}\left|B_n\right| \le A_\alpha^\alpha$.

For C_n, we will again apply Lemma 1 to the random variable X_{ni}^2 and weight $b_n^{-1}a_{ni}$. Noting that

$$\left|b_n^{-1}a_{ni}X_{ni}^2\right| \le \frac{\left|a_{ni}\right|}{b_n(\log n)^{\delta_1(\gamma-1)}}\left|X_i\right|^\gamma \le \frac{A_{\alpha,n}}{(\log n)^{\delta_1(\gamma-1)+\delta+1/\gamma}}\left|X_i\right|^\gamma = \frac{A_{\alpha,n}}{\log n}\left|X_i\right|^\gamma$$

and

$$(X_{ni}^2)^2\sum_{i=1}^n b_n^{-2}a_{ni}^2 \le \frac{\sum_{i=1}^n a_{ni}^2}{b_n^2(\log n)^{2\delta_1(\gamma-1)}}(\left|X_i\right|)^{2\gamma} \le \frac{A_{\alpha,n}^2 n^{2/\alpha}}{b_n^2(\log n)^{2\delta_1(\gamma-1)}}(\left|X_i\right|)^{2\gamma}$$

$$= \frac{A_{\alpha,n}^2}{(\log n)^{2/\gamma+2\delta+2\delta_1(\gamma-1)}}(\left|X_i\right|)^{2\gamma} = \frac{A_{\alpha,n}^2}{\log^2 n}(\left|X_i\right|)^{2\gamma},$$

we have $C_n\to 0$ $a.s.$ by Lemma 1.

It can be shown that $E_n\to 0$ $a.s.$ by the same method as in C_n and $D_n\to 0$ $a.s.$ and $F_n\to 0$ $a.s.$ by the same method as in Case 1.

Accordingly, we obtain that $\limsup_{n\to\infty} b_n^{-1}\left|\sum_{i=1}^n a_{ni}X_i\right| \le A_\alpha^\alpha$ a.s. By replacing X_i by ϖX_i, ϖ as an arbitrary positive number, we have

$$\limsup_{n\to\infty} b_n^{-1}\left|\sum_{i=1}^n a_{ni}X_i\right| \le \frac{A_\alpha^\alpha}{\varpi}\ \ a.s.$$

By letting $\varpi\to\infty$, the proof of Case 2 is completed.

ND random variables will lose the property of negative dependence after we truncate them by usual indicators. Only monotone functions preserve the property of negative dependence, as it mentioned in Proposition 1 (ii). This is the reason why the authors need to use *monotone truncation*, that is, a sum of indicators.

References

1. Bai Z.D. and Cheng P.E. Marcinkiewicz strong laws for linear statistics. *Statist. Probab. Lett.*, **46**: 105-112 (2000).
2. Bozorgnia A., Patterson R.F., and Taylor R.L. Limit theorems for negatively dependent random variables. Technical Report, University of Georgia (1993).
3. Bozorgnia A., Patterson R.F., and Taylor R.L. Limit theorems for dependent random variables. *World Congress Nonlinear Analysts*, **92**, Vol. I-IV (Tampa, FL, 1992), 1639-1650, de Gruyter, Berlin (1996).
4. Hsu P.L. and Robbins H. Complete convergence and the law of large numbers. *Proc. Nat. Acad. Soc., U.S.A.*, **33**: 25-31 (1947).
5. Joe H. *Multivariate Models and Dependence Concepts*. Chapman and Hall (1997).
6. Joag-Dev K. and Proschan F. Negative association of random variables with applications. *Ann. Statist.*, **11**: 286-295 (1983).
7. Lehmann E.L. Some concept of dependence. *Ann. Math. Statist.*, **37**: 1137-1153 (1966).
8. Matula P. A note on the almost sure convergence of sums of negatively dependent random variables. *Statist. Probab. Lett.*, **15**: 209-213 (1992).
9. Sung S.H. Strong laws for weighted sums of i.i.d. random variables. *Ibid.*, **52**: 413-419 (2001).
10. Taylor R.L., Bozorgnia A., and Patterson R.F. A strong law of large numbers for arrays of rowwise negatively dependent random variables. *J. Appl. Math. and Stoch. Anal.* **20**: 643-656 (2002).