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Topological Lumpiness and Topological Extreme Amenability

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Abstract

In this paper we give some characterizations of topological extreme amenability. Also we answer a question raised by Ling [5]. In particular we prove that if *T* is a Borel subset of a locally compact semigroup *S* such that $M(S)^*$ has a multiplicative topological left invariant mean then *T* is topological left lumpy if and only if there is a multiplicative topological left invariant mean *M* on $M(S)^*$ such that $M(\chi_T)=1$, where χ_T is the characteristic functional of *T*. Consequently if *T* is a topological left lumpy locally compact Borel subsemigroup of a locally compact semigroup *S*, then *T* is extremely topological left amenable if and only if *S* is.

Keywords: Mean; Topological extreme amenability; Left lumpy

1. Introduction

Let *S* be a locally compact (Hausdorff) semigroup. Let $C_o(S)$ be the subalgebra of $CB(S)$ consisting of functions which vanish at infinity. Let $M(S)^*$ be the Banach space of all bounded regular Borel (signed) measures on *S* with total variation norm.

Let $M_0(S)=\{\mu\in M(S):\mu\geq 0 \text{ and } \|\mu\|=1\}$ be the set of all probability measures in $M(S)$. It is known that $M(S)=C_0(S)^*$ *via* the correspondence $\mu\to\overline{\mu}$ where $\overline{\mu}(f)=\int f\,d\mu$ for any f in $C_o(S)$ [4, § 14]. Consider the continuous dual $M(S)^*$ of $M(S)$. Denote by 1the element 1 in $M(S)^*$ such that such that $1(\mu)=\mu(S)$ for any μ in $M(S)$.

Also if *T* is a Borel subset of *S* we define the Borel characteristic functional χ_T of *T* in $M(S)^*$ by $\chi_T(\mu)=\mu(T)$, $\mu\in M(S)$. An element *M* in $M(S)^{**}$ is called a mean on $M(S)$ if $M(1)=1$ and $M(F)\geq 0$, whenever $F\geq 0$. An equivalent definition for a mean is that

$$\inf\{F(\mu):\mu\in M_o(S)\} \leq M(F)\leq \sup\{F(\mu):\mu\in M_o(S)\}$$

for any *F* in $M(S)^*$. We also note that $M\in M(S)^{**}$ is a mean if and only if $\|M\|=M(1)=1$. Each probability measure μ in $M_o(S)$ is a mean on $M(S)^{**}$ if we put $\mu(F)=F(\mu)$, for any F in $M(S)^*$. An application of Hahn-Banach separation theorem shows that $M_o(S)$ is weak* dense in the set of all means on $M(S)^*$.

Under pointwise operations and supremum norm $C_o(S)$ becomes a Banach algebra. Arens product can thus be defined in $C_o(S)^{**}$. In particular, we have the

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following defining formulas for any f, g in $C_o(S)$, m in $C_o(S)^*$ and θ, φ in $C_o(S)^{**}$.

$$(m \odot f)(g) = m(fg)$$

$$(\varphi \odot m)(f) = \varphi(m \odot f)$$

$$(\theta \odot \varphi)(m) = \theta(\varphi \odot m)$$

This product induces a multiplication in $M(S)^*$ *via* the identification $M(S)=C_o(S)^*$. For F, G in $M(S)^*$ we denote the multiplication of F and G by F × G. In [5] it is shown that F × G is defined via the following three steps:

(i) For any $\mu \in M(S)$ and $f \in C_o(S)$, $\mu_f \in M(S)$ is defined by

$$\int g\, d\mu_f = \int g f\, d\mu \quad \text{for all } g \in C_o(S)$$

(ii) For any $\mu \in M(S)$ and $G \in M(S)^*$, $G \times \mu \in M(S)$ is defined by

$$\int f\, d(G \times \mu) = G(\mu_f) \quad \textit{for all } f \in C_o(S)$$

(iii) For any F, G $\in M(S)^*$, F × G $\in M(S)^*$ is defined by

$$(F \times G)(\mu) = F(G \times \mu) \text{ for all } \mu \in M(S).$$

Then $M(S)^*$ becomes a commutative Banach algebra with identity [5, theorem 1.2.3].

For each μ in $M(S)$ define an operator $l_\mu : M(S)^* \to M(S)^*$ by $l_\mu F(\nu) = F(\mu * \nu)$, $\nu \in M(S)$, we denote $l_\mu F$ by $\mu \odot F$. A mean M on $M(S)^*$ is called topological left invariant (TLIM) if $M(\mu \odot F) = M(F)$ for all $F \in M(S)^*$ and for all $\mu \in M_o(S)$. A topological left invariant mean M on $M(S)^*$ is called a multiplicative topological left invariant mean (MTLIM) if

$$M(F \times G) = M(F) M(G) \text{ for all } F, G \in M(S)^*.$$

If there is a MTLIM on $M(S)^*$ we say that S is extremely topological left amenable (ETLA). For results concerning ETLA semigroups see [5] and [6].

2. Main Results

Note that for elements M, N in $M(S)^{**}$ their Arens product is denoted by $M \odot N$ and is defined by

$$(M \odot N)(F) = M(N_L(F)) \text{ for all } F \text{ in } M(S)^*$$

where $N_L : M(S)^* \to M(S)^*$ is defined by $N_L(\mu) = N(\mu \odot F)$, $\mu \in M(S)$. See [1] and [2].

First we prove two Lemmas.

Lemma 2.1. Suppose M and N are functionals in $M(S)^{**}$.

(i) If M and N are means on $M(S)^*$ then $M \odot N$ is also a mean on $M(S)^*$.

(ii) For each $\mu \in M(S)$ and each $F \in M(S)^*$ we have

$$M_L(\mu \odot F) = \mu \odot M_L(F)$$

(iii) If M is a topological left invariant mean, then $M \odot N$ is also topological left invariant.

Proof. (i) It is easy to see that for each $\mu \in M(S)$ and $1 \in M(S)^*$ we have $\mu \odot 1 = 1(\mu)$, hence

$$(M \odot N)(1) = M(N_L(1)) = M(1) = 1$$

Also $\|M \odot N\| \le \|M\| \|N\|$, hence $M \odot N$ is a mean on $M(S)^*$.

(ii) For each $\nu \in M(S)$

$$\begin{aligned} M_L(\mu \odot F)(\nu) &= M(\nu \odot (\mu \odot F)) \\ &= M((\mu * \nu) \odot F) \\ &= M_L(F)(\mu * \nu) \\ &= (\mu \odot M_L(F))(\nu) \end{aligned}$$

Thus $M_L(\mu \odot F) = \mu \odot M_L(F)$.

(iii) Suppose M is topological left invariant, then for each $\mu \in M_o(S)$ and $F \in M(S)^*$ we have

$$\begin{aligned} (M \odot N)(\mu \odot F) &= M(N_L(\mu \odot F)) \\ &= M(\mu \odot N_L(F)) \\ &= M(N_L(F)) \\ &= (M \odot N)(F) \end{aligned}$$

where we have used (ii) in the second equality. So $M \odot N$ is topological left invariant, whenever M is.

Lemma 2.2. For each $s \in S$, $F \in M(S)^*$ and $M \in M(S)^{**}$ we have

(i) $(\varepsilon_S)_L(F) = F \odot \varepsilon_s$

(ii) $(M \odot \varepsilon_S)(F) = M(F \odot \varepsilon_S)$

(iii) $(\varepsilon_S)_L(F \times G) = (F \times G) \odot \varepsilon_S$

$$= (F \odot \varepsilon_S) \times (G \odot \varepsilon_S)$$

(iv) If M is multiplicative, then $M \odot \varepsilon_s$ is so.

Proof. (i)

$$(\varepsilon_s)_L (F)(\mu) = \varepsilon_s (\mu \odot F) = (\mu \odot F)(\varepsilon_s)$$
$$= F(\mu * \varepsilon_s) = (F \odot \varepsilon_s)(\mu)$$

hence $(\varepsilon_s)_L (F) = F \odot \varepsilon_s$.

(ii) $(M \odot \varepsilon_s)(F) = M((\varepsilon_s)_L (F)) = M(F \odot \varepsilon_s)$ where we have used (i) in the second equality.

(iii) the first equality follows from (i) and the second one follows from [5, p.27]

(iv) Suppose $M \in M(S)^{**}$ is multiplicative. Then:

$$(M \odot \varepsilon_s)(F \times G) = M((\varepsilon_s)_L (F \times G))$$
$$= M((F \odot \varepsilon_s) \times (G \odot \varepsilon_S))$$
$$= M(F \odot \varepsilon_s) M(G \odot \varepsilon_s)$$
$$= ((M \odot \varepsilon_s)(F))((M \odot \varepsilon_S)(G))$$

where we have used (iii) in the second equality and (ii) in the last equality.

The following theorem is an extension of [5, theorem 3.2.1]. But first we need a definition.

Definition 2.3. Let S be a locally compact semigroup and T a Borel subset of S. T is said to be topological left lumpy in S if it satisfies the following condition.

(TLL) For each $\delta > 0$ and $\mu \in M_o(S)$ with compact support, there exists $a \in S$ such that $\mu * \varepsilon_a (T) > 1 - \delta$.

It is known that (TLL) is equivalent to each of the following conditions:

$(TLL)_1$ For any $\delta > 0$ and $\nu \in M_o(S)$ with compact support, there exists $\mu \in M_o(S)$ with compact support such that

$$\mu(T) > 1 - \delta \qquad \text{and} \qquad (\nu * \mu)(T) > 1 - \delta$$

$(TLL)_2$ For any $\delta > 0$ and $\nu \in M_o(S)$ with compact support, there exists $\mu \in M_o(S)$ with compact support such that

$$\mu(T) > 1 - \delta \qquad \text{and} \qquad (\nu * \mu)(T) > 1 - \delta$$

See [7, pp. 571-574 and addendum on p.585] for more details. See also [3].

Theorem 2.4. Suppose T is a Borel subset of a locally compact semigroup S. Suppose $M(S)^*$ has a MTLIM then the following statements are equivalent:

(i) T is topological left lumpy.

(ii) There is a MTLIM on $M(S)^*$ such that $M(\chi_T) = 1$.

Proof. (i) $\Rightarrow$ (ii). Let $F = \{\mu_1, ..., \mu_k\}$ be a finite subset of $M_0^c(S)$ (The elements in $M_o(S)$ with compact support). For each $\varepsilon > 0$ there is $s = s_{(F,\varepsilon)} \in S$ such that $\frac{\mu_1 + ... + \mu_k}{k} * \varepsilon_s (T) > 1 - \frac{\varepsilon}{2}$ (by TLL), in particular $\mu_i * \varepsilon(T) > 1 - \varepsilon,\ 1 \le i \le k$.

Let F be the collection of all finite (nonempty) subsets of $M_0^c(S)$. Put $\Delta = F \times (0, \infty)$ and order Δ as following:

$$(F_1, \alpha_1) \ge (F_2, \alpha_2) \Leftrightarrow F_2 \subseteq F_1 \quad \text{and} \quad \alpha_1 < \alpha_2$$

By above discussion there is a net $\{s_\alpha\}$ of elements of S with $\gamma = (F, \alpha) \in \Delta$. Since the set of means on $M(S)^*$ is weak* compact the net $\{\varepsilon_{s_\alpha}\}$ has a subnet $\{\varepsilon_{s_\beta}\}$ which converges weak* to a mean N on $M(S)^*$ and also for each $\mu \in M_0^c(S)$ we have

$$N(\mu \odot \chi_T) = \lim_\beta (\mu \odot \chi_T)(\varepsilon_{s_\beta}) = \lim_\beta (\mu * \varepsilon_{s_\beta})(T) = 1 \tag{1}$$

Now suppose M is MTLIM on $M(S)^*$. Since the Arens product is weak* continuous in the second variable and using Lemma 2.2 (iv) we conclude that $M \odot N$ is multiplicative. Also since M and N are means and M is topological left invariant, by using Lemma 2.1 we conclude that $M \odot N$ is a MTLIM on $M(S)^*$. Now since $M_0^c(S)$ is weak* dense in the set of means on $M(S)^*$, by using (1) we obtain $(M \odot N)(\chi_T) = 1$.

(ii) $\Rightarrow$ (i) Suppose M is a MTLIM on $M(S)^*$ such that $M(\chi_T)$. If $\{\mu_\alpha\}$ is a net in $M_0^c(S)$ which converges to M in weak* topology, then for each $\nu \in M_0^c(S)$ we have

$$\omega^* - \lim_\alpha (\nu * \mu_\alpha - \mu_\alpha) = \nu \odot M - M = 0$$

Since $\lim_{\alpha} \mu_\alpha (T) = M(\chi_T) = 1$ and for each $\nu \in M_0^C(S)$

$$(\nu * \mu_\alpha)(\chi_T) = \chi_T(\nu * \mu_\alpha) = (\nu * \mu_\alpha)(T)$$

We conclude that for each $\nu \in M_0^c(S)$, $\lim_\alpha (\nu * \mu_\alpha)(T) = 1$.

So for each $\nu \in M_0^c(S)$ and each $\delta > 0$ there is $\mu = \mu_\alpha \in M_0^c(S)$ such that $(\nu * \mu)(T) > 1 - \delta$. Therefore by $(TLL)_2$, *T* is topological left lumpy.

Let *S* be a locally compact semigroup and *T* a locally compact Borel subsemigroup of *S*. We recall some of the constructions in [8] and [9].

Let $B(S)$ be the σ-algebra of Borel subsets of *S*.

(1) Let $\mu \in M(S)$, then μ_T is the restriction of μ to $B(T)$ and $\mu_T \in M(T)$.

(2) Let $F \in M(T)^*$, then $F' \in M(S)^*$ is well-defined by $F'(\mu) = F(\mu_T)$ for any $\mu \in M(S)$.

(3) Let $M \in M(S)^{**}$, then $M_0 \in M(T)^{**}$ is well-defined by $M_0(F) = M(F')$

For any $F \in M(T)*$.

Lemma 2.5. (a) $F \times \mu_T = (F' \times \mu)_T$ for $F \in M(T)^*$ and $\mu \in M(S)$.

(b) $(F \times G)' = F' \times G'$ for any $F, G \in M(T)^*$.

Proof. (a) For any $A \in B(T)$ we denote ξ_A for characteristic function of *A* in *T* and χ_T for characteristic function of *A* is *S*.

$$\begin{aligned}(F \times \mu_T)(A) &= \int \xi_A d(F \times \mu_T) = F((\mu_T)_{\xi_A}) \\ &= F((\mu_{\chi_A})_T) = F'(\mu_{\chi_A}) \\ &= \int \chi_A d(F' \times \mu) \\ &= (F' \times \mu)(A) = (F' \times \mu)_T (A)\end{aligned}$$

(b) For any $\mu \in M(S)$ by (a) we have

$$\begin{aligned}(F' \times G')(\mu) &= F'(G' \times \mu) = F((G' \times \mu)_T) \\ &= F(G \times \mu_T) \\ &= (F \times G)(\mu_T) = (F \times G)'(\mu)\end{aligned}$$

We now state the main result of this paper which answers a question raised by J.M. Ling, See [5, then P. 51].

Theorem 2.6. Let *T* be a topological left lumpy locally compact Borel subsemigroup of a locally compact semigroup *S*. Then *T* is ETLA if and only if *S* is ETLA.

Proof. Suppose *T* is ETLA, then by [5, Theorem 3.2.3] *S* is ETLA.

Conversely suppose *S* is ETLA, by theorem 2.4 there is a MTLIM on $M(S)^*$ such that $M(\chi_T) = 1$. Then $M_0(F) = M(F')$ defines a TLIM on $M(T)^*$, we show that M_0 is multiplicative

$$\begin{aligned}M_0(F \times G) &= M((F \times G)') = M(F' \times G') \\ &= M(F')M(G') \\ &= M_0(F)M_0(G)\end{aligned}$$

Corollary 2.7. Let *T* be a left ideal of a locally compact semigroup *S*, Then $M(T)^*$ has a MTLIM if and only if $M(S)^*$ has a MTLIM.

Proof. It suffices to show that every left ideal is topological left lumpy. Let $t \in T$. If $K \subseteq S$ is compact then $Kt \subseteq ST \subseteq T$. Consider the Dirac measure ε at t. For any $\mu \in M_0(S)$ with $\mu(K) = 1$, we have $\mu * \varepsilon_t(T) = \int \chi_T(xt) d\mu(x) = \int_K \chi_T(xt) d\mu(x) = \mu(K) = 1$, hence *T* is topological left lumpy.

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