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The Almost Sure Convergence for Weighted Sums of Linear Negatively Dependent Random Variables

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Abstract

In this paper, we generalize a theorem of Shao [12] by assuming that $\{X_n\}$ is a sequence of linear negatively dependent random variables. Also, we extend some theorems of Chao [6] and Thrum [14]. It is shown by an elementary method that for linear negatively dependent identically random variables with finite p-th absolute moment $(p \ge 2)$ the weighted sums $\frac{1}{A_n}\sum_{i=1}^{n} a_{ni}X_i$ converge to zero as where $A_n = n^{1/p}(\sum_{i=1}^{n} a_{ni}^2)^{1/2}$ and $\{a_{n,i}\}$ is an array of real numbers. Moreover, we prove the almost sure convergence for weighted sums $\sum_{i=1}^{n} a_{ni}X_i$, $n \ge 1$, when $\{X_i, i \ge 1\}$ is a sequence of pairwise negative quadrant dependence stochastically bounded random variables under some suitable conditions on $a_{n,i}$.

Keywords: Pairwise negatively dependent; Linear negatively dependent; Negative association; Complete convergence

Introduction

Let $\{X_n, n \ge 1\}$ be a sequence of independence random variables and suppose that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is a double array of real numbers, the almost sure convergence of weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ were studied by many authors (see, Chow [6], Thrum [14] and Sung [13]). Thrum [14] established the following extension of Chow [6].

Theorem 1. Suppose that $\{X_n, n \ge 1\}$ is a sequence of *i.i.d* random variables with expectation zero and finite p-th ($p \ge 2$) absolute moment and $\{a_{ni}; 1 \le i \le n, n \ge 1\}$ is a sequence of nonrandom weighting coefficients with $\sum_{i=1}^{n} a_{ni}^2 = 1$ for all $n \ge 1$. Then $\sum_{i=1}^{n} a_{ni}X_i / n^{1/p} \to 0$ $a.s.$ as $n \to \infty$.

But, in many stochastic models the assumption of

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independence among random variables isn't plausible. In fact, increases in some random variables are often related to decreases in other random variables and the assumption of negative dependence is more appropriate than independence assumption. In the case of negative dependence and negative association dependence random variables some these results have been extended by other authors for example: Amini *et.al.* [1], [2], Matula [10] and Nili Sani *et.al.* [11] and Shao [12]. In this paper we generalize some results of Shao [12] and Thrum [14] for sequence $\{X_n, n \ge 1\}$ of LIND random variables. Moreover we prove the almost sure convergence for weighted sums $\sum_{i=1}^{n} a_{ni} X_i$, when $\{X_n, n \ge 1\}$ is a sequence of NQD stochastically bounded random variables under some suitable conditions on a_{ni}. In the following we present some definitions to be used in the proofs of our main results.

Definition 1. *i)* (Lehmann [8]). The random variables X and Y are said negatively quadrant dependent (NQD) if for each $x, y \in R$, $P(X \le x, Y \le y) \le P(X \le x)P(Y \le y)$.

ii) The sequence $\{X_i, i \ge 1\}$ of random variables is said to be pairwise NQD if (X_i, Y_j) is NQD for every $i \ne j$.

iii) (Joag-Dev and Proschan [7]). The random variables $X_1, \cdots, X_n (n \ge 2)$ are said to be negatively associated (*NA*) if for every pair of disjoint nonempty subsets A_1, A_2 of $\{1, \ldots, n\}$,

$$Cov(f_1(X_i, i \in A_1), f_2(X_i, i \in A_2)) \le 0,$$

whenever f_1 and f_2 are coordinatewise increasing and covariance exists.

Definition 2. A sequence $\{X_n, n \ge 1\}$ of random variables is called asymptotically almost negatively associated (AANA) if there is a nonnegative sequence $q(m) \to 0$ such that

$$\begin{aligned} &Cov(f(X_m), g(X_{m+1}, \cdots, X_{m+k})) \\ &\le q(m)(Var(f(X_m))Var(g(X_{m+1}, \cdots, X_{m+k}))^{1/2} \end{aligned} \tag{1}$$

for all $m, k \ge 1$ and for all coordinate increasing continuous functions f and g whenever the right side of (1) is finite.

Definition 3. The random variables $X_1, X_2, \cdots, X_n$ are said to be linear negatively dependent (LIND) if for any disjoint $A, B \subset \{1, \cdots, n\}$ and $\lambda_j > 0$, $j = 1, \cdots, n$, $\sum_{k \in A} \lambda_k X_k$ and $\sum_{k \in B} \lambda_k X_k$ are NQD.

A sequence $\{X_n, n \ge 1\}$ of random variables is said to be linear negatively dependent (pairwise NQD or negatively associated) if it holds for every finite subsequence.

Obviously, pairwise NQD sequence includes independent random variables and NA sequence, which has wide application in multivariate statistical analysis

The following properties of the NA and NQD random variables and convex functions are based on our results.

(P1) Increasing functions defined on disjoint subsets of a set of negatively associated random variables are negatively associated.(Joag-Dev and Proschan [7])

(P2) If $\{X_n, n \ge 1\}$ is a sequence of pairwise NQD random variables , then

$Cov(X_i, X_j) \le 0, \ \forall \ i \ne j$. (Bozorgnia *et al.* [3]).

(P3) If $\{f_n\}$ is a sequence of Borel functions are all monotone increasing (or are all monotone decreasing) then $\{f_n(X_n)\}$ is a sequence of pairwise NQD random variables (Bozorgnia *et al.* [3]).

(P4) Let (Ω, F, P) be a probability space and $\{A_n, n \ge 1\}$ is a sequence of pairwise NQD events. If $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(\limsup A_n) = 1$. (Matula [10]).

(P5) For any convex function f on R^1, the right derivative $f_+^{'}$ exists and is increasing. Moreover for all a, b, $f(b) - f(a) = \int_a^b f_+^{'}(t)dt$.

The next Theorem due to Shao [12]. We will extend this Theorem by assuming that $\{X_n, n \ge 1\}$ is a sequence of LIND random variables.

Theorem 2. Let $\{X_i, 1 \le i \le n\}$ be a negatively associated sequence, and let $\{X^*_i, 1 \le i \le n\}$ be a sequence of independent random variables such that X^*_i and X_i have the same distribution for each $i = 1, 2, \ldots, n$. Then

$$Ef(\sum_{i=1}^{n} X_i) \le Ef(\sum_{i=1}^{n} X^*_i) \tag{2}$$

For any convex function f on R^1, whenever the expectation on the right hand side of (2) exists.

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Also, in this paper c stands for a generic constant, not necessarily the same at each appearance.

Results

The next Lemma is an important technical tool in the proof of our main result. In fact in this Lemma we extend Theorem 2 for LIND random variables. Set

$$\phi_m(q) =: \sup_{n\ge m} \sup_{A\in F_m^n, B\in F_{n+q}^\infty, P(A)>0} |P(B/A)-P(B)|,$$

$$\phi^* = \lim_{m\to\infty} \phi_m(q).$$

Lemma 1. Let $\{a_{ij}, i=1,\cdots,j,\ j=1,\cdots,n\}$ be a double array of non-negative real numbers and let $\{X_i, i=1,\cdots,n\}$ be a sequence of non-negative LIND r.v's such that $X_n\sum_{j=1}^{n-1} a_{nj}X_j$ and $\sum_{1\le i<j\le n-1} a_{ij}X_iX_j$ are NQD. Assume that $\{X_i^*, i=1,\cdots,n\}$ is a sequence of independent r.v.'s such that X_i and X_i^* have the same distribution for each $i=1,\cdots,n$. Then

$$E(f(\sum_{1\le i<j\le n} a_{ij}X_iX_j)) \le E(f(\sum_{1\le i<j\le n} a_{ij}X_i^*X_j^*)) \quad (3)$$

for any convex function f on R^1, whenever the expectation on the right hand side of (3) exists.

Proof. By the same arguments of Shao [12], we prove (3), by induction on n. Let (Y_1, Y_2) be an independent copy of (X_1, X_2). It follows from (P5),

$$E(f(cX_1X_2)) + E(f(cY_1Y_2))$$
$$-E(f(cX_1Y_2)) - E(f(cY_1X_2))$$
$$= E\{\int_{X_2}^{Y_2} (cY_1f_+'(ctY_1) - cX_1f_+'(ctX_1))dt\}$$
$$= E\{\int_0^\infty (cY_1f_+'(ctY_1) - cX_1f_+'(ctX_1))(I_{(Y_2>t)} - I_{(X_2>t)})dt\}$$

Since $f_+'(x+t)$ and $I_{(x>t)}$ are increasing functions of x for each t, hence $f_+'(X_1+t)$ and $I_{(X_2>t)}$ are NQD. By applying Fubini's theorem, we conclude that

$$2(E(f(cX_1X_2)) - E(f(cX_1^*X_2^*)))$$
$$= (E(f(cX_1X_2)) + E(f(cY_1Y_2))$$
$$-E(f(cX_1Y_2)) - E(f(cY_1X_2)))$$
$$= E\int_{X_2}^{Y_2} (cY_1f_+'(ctY_1) - cX_1f_+'(ctX_1))dt$$
$$= E\int_0^\infty (cY_1f_+'(ctY_1) - cX_1f_+'(ctX_1))(I_{(Y_2>t)} - I_{(X_2>t)})dt$$
$$= 2\int_0^\infty Cov(cX_1f_+'(ctX_1), I_{(X_2>t)})dt \le 0$$

which proves (3) for $n=2$. Let $Z_n = \sum_{1\le i<j\le n} a_{ij}X_iX_j$ and $g(x) = E(f(Z_{n-1}+x))$.

By the induction hypothesis

$$g(x) \le E(f(\sum_{1\le i<j\le n-1} a_{ij}X_i^*X_j^* + x)) \quad (4)$$

Since $X_n\sum_{i=1}^{n-1} a_{in}X_i$ and $\sum_{1\le i<j\le n-1} a_{ij}X_iX_j$ are NQD, and consequence NA, hence Theorems 2 implies that

$$E(f(Z_n)) = E(f(Z_{n-1} + X_n\sum_{i=1}^{n-1} a_{in}X_i))$$
$$\le E(f((X_n\sum_{i=1}^{n-1} a_{in}X_i)^* + Z_{n-1}^*))$$
$$= E(g(X_n\sum_{i=1}^{n-1} a_{in}X_i)^*)$$
$$\le E(f((X_n\sum_{i=1}^{n-1} a_{in}X_i)^* + \sum_{1\le i<j\le n-1} a_{ij}X_j^*X_i^*)),\ by\ \ (4)$$
$$\le E(f(\sum_{i=1}^{n-1}(X_na_{in}X_i)^* + \sum_{1\le i<j\le n-1} a_{ij}X_j^*X_i^*))$$
$$\le E(f((X_n^*\sum_{i=1}^{n-1} a_{in}X_i^*) + \sum_{1\le i<j\le n-1} a_{ij}X_j^*X_i^*))$$
$$= E(f(\sum_{1\le i<j\le n} a_{ij}X_i^*X_j^*)),$$

It is easy to show that $X_na_{i,n}X_i$, $i=1,\cdots,n-1$, are $LIND$ r.v.'s. Therefore, Theorem 2 completes the proof.

Lemma 2. Let $\{a_{ij}: i,j=1,2,3,...\}$ be a double array of non-negative real numbers with $a_{ij}=a_{ji}$ for $i\ne j$ and $a_{ii}=0$ for all i. Assume that $\{X_i, i=1,\cdots,n\}$ is a sequence of non-negative LIND random variables with

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$E|X_i|^{2s} < \infty$ for $i \ge 1$ and $s \ge 1$ such that $2X_n \sum_{j=1}^{n-1} a_{nj} X_j$ and $\sum_{1\le i<j\le n-1} a_{i,j} X_i X_j$ are NQD. Then the sequence $\{Z_n, n \ge 1\}$ has the following properties:

$$E|Z_n|^{2s} < \infty \text{ for all } n \ge 1 \quad (5)$$

and

$$E|Z_n|^{2s} \le B(2s)\left(\sum_{i<j}^{n} a_{i,j}^2 (E|X_i|^{2s})^{1/s} (E|X_j|^{2s})^{1/s}\right)^s$$

$$\le (EZ_n^2)^s (B(2s)/2^{2s})$$

$$(\max\{(EX_i^{2s})/(EX_i^2)^2 \,|\, i = 1,\cdots,n\})^2, \quad (6)$$

Where $Z_n = \sum_{i,j=1}^{n} a_{ij} X_i X_j, n \ge 1$ and $B(2s)$ is a constant which depends neither on n nor on the distribution of $(X_1,\cdots,X_n)$.

Proof. We can write the following inequality

$$|Z_n|^{2s} \le 2^{2s} \binom{n}{2}^{2s-1} \sum_{1\le i\le j\le n} |X_i|^{2s} |X_j|^{2s} |a_{i,j}|^{2s}$$

Since X_i and X_j $(i \ne j)$ are non-negative NA random variables and $E|X_i|^{2s} < \infty$ for $i \ge 1$ and $s \ge 1$, hence the above inequality implies that $E|Z_n|^{2s} < \infty$.

We have the following recursive formula

$$Z_n = Z_{n-1} + 2X_n \sum_{j=1}^{n-1} a_{nj} X_j$$

Define $Y_k = \sum_{j=1}^{k-1} a_{kj} X_j$ we have $(\Delta Z)_k = Z_k - Z_{k-1} = 2X_k Y_k$ for all $k = 2,\cdots,n$. Let $\{X_i^*\}$ is a sequence of independent random variables such that X_i and X_i^* have the same distribution for each i. By Lemma 1 and Lemma 2 of Thrum [14] we have

$$E|Z_n|^{2s} \le E|Z_n^*|^{2s} \le (EZ_n^{*2})^s (B(2s)/2^{2s})$$

$$(\max\{EX_i^{*2s} / (EX_i^{*2})^2 \,|\, i = 1,\cdots,n\})^2$$

$$\le (EZ_n^2)^s (B(2s)/2^{2s})$$

$$(\max\{(EX_i^{2s})/(EX_i^2)^2 \,|\, i = 1,\cdots,n\})^2$$

The next Lemma can be obtained from arguments of Thrum [14]. We omit the details.

Lemma 3. Suppose that $X_1,\cdots,X_n,\cdots$ are identically random variables with $E(X_1) = 0$ and $E(|X_1|^p) < \infty$ for some $p \in (0,2)$ that satisfying in Marcinkiewicz and Zygmund's theorem. Assume that nonrandom coefficients $b_{n,i}$ fulfill and $\sup\{\sum_{i=1}^{n} |b_{ni}|, n \ge 1\} < \infty$.

Then

$$\sum_{i=1}^{n} b_{ni} X_i / n^{1/p} \xrightarrow{n\to\infty} 0 \ a.s.$$

Corollary 1. Let $\{X_n, n \ge 1\}$ be a sequence of identically ANNA random variables with $\sum_{m=1}^{\infty} q^2(m) < \infty$ (or pairwise NQD r.v.'s with $\phi^*(1) < 1$). Assume that $E(X_1) = 0$, $E(|X_1|^p) < \infty$ for some $p \in (0,2)$ and nonrandom coefficient $b_{n,i}$ fulfill $\sup\{\sum_{i=1}^{n} |b_{ni}|, n \ge 1\} < \infty$. Then

$$\sum_{i=1}^{n} b_{ni} X_i / n^{1/p} \xrightarrow{n\to\infty} 0 \ a.s.$$

See Theorem 2 of Chandra and Ghosal [5] and Corollary 2.2 of Liang *et.al.* [9].

Theorem 3. Let $\{a_{ni}, i = 1,\cdots,n; n \ge 1\}$ be an array of real numbers with $\sum_{i=1}^{n} a_{ni}^2 = 1$ for all $n \ge 1$ and $\{X_n, n \ge 1\}$ be a sequence of LIND identically random variables such that $X_n \sum_{j=1}^{n-1} a_{nj} X_j$ and $\sum_{1\le i<j\le n-1} a_{ij} X_i X_j$ are NQD for every $n \ge 1$ with finite p-th ($p \ge 2$) absolute moment. Then $\sum_{i=1}^{n} a_{ni} X_i / n^{1/p} \xrightarrow{n\to\infty} 0$ a.s.

Proof. Let $X_n^+ = X_n I(X_n > 0)$ and $X_n^- = -X_n I(X_n < 0)$. Then

$$\sum_{i=1}^{n} a_{ni} X_i / n^{1/p} = \frac{1}{n^{1/p}} \{ \sum_{a_{n,i}>0}^{n} a_{ni} X_i^+ - \sum_{a_{n,i}>0}^{n} a_{ni} X_i^- +$$

$$+ \sum_{a_{n,i}<0}^{n} a_{ni} X_i^+ - \sum_{a_{n,i}<0}^{n} a_{ni} X_i^- \} = T_{n1} + T_{n2} + T_{n3} + T_{n4}.$$

Hence without the lose generality we can suppose

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$\{a_{ni}, 1 \le i \le n,\ n \ge 1\}$ is an array of non negative real numbers and $\{X_n; n \ge 1\}$ is a sequence of non negative random variables. The proof follows the same lines as the proof of Theorem 3 of Thrum [14]. It is clear that

$$U_n^2 = (\sum_{i=1}^{n} a_{ni} X_i / n^{1/p})^2 = \sum_{i=1}^{n} a_{ni}^2 X_i^2 / n^{2/p}$$

$$+\sum_{i \ne j}^{n} a_{ni} a_{nj} X_i X_j / n^{2/p} = V_n + W_n .$$

$W_n \xrightarrow{n\to\infty} 0$ a.s. is a result of Lemma 2. Hence it suffices to show that $V_n \xrightarrow{n\to\infty} 0$, a.s. Set $Y_i = X_i^2 - EX_i^2$, $b_{ni} = a_{ni}^2$, $\tilde{a}_{n,i} = a_{n,i}^2 / (\sum_{i=1}^{n} a_{n,i}^4)^{1/2}$, $i = 1,\cdots,n$ and $\tilde{p} = p/2$, then

$\sum_{i=1}^{n} b_{ni} \le 1$ and $\sum_{i=1}^{n} \tilde{a}_{ni}^2 = 1$. Using

$$(\sum_{i=1}^{n} a_{n,i}^4)^{1/2} \le \sum_{i=1}^{n} a_{n,i}^2 \le 1$$

for $\tilde{V}_n = \sum_{i=1}^{n} b_{ni} Y_i / n^{1/\tilde{p}}$ and $\tilde{U}_n = \sum_{i=1}^{n} \tilde{a}_{ni} Y_i / n^{1/\tilde{p}}$ we get $0 \le V_n = \tilde{V}_n + EX_1^2 / n^{1/\tilde{p}}$ and $|\tilde{V}_n| \le |\tilde{U}_n|$. If p belongs to the interval $[2,4)$, then $\tilde{p} \in [1,2) \subseteq (0,2)$ and Lemma 3 can be applied which ensure that $\tilde{V}_n$ and by (1) also V_n tend to zero a.s. In the case $p \in [2^k, 2^{k+1})$ with a natural number $k > 1$ the same argument can be repeated for $\tilde{U}_n$ and $\tilde{p}$ with $\tilde{p} \in [2^{k-1}, 2^k)$. The conclusion now follows by induction.

The following Example is evidence of random variables X_1, X_2, X_3 that $X_n \sum_{j=1}^{n-1} a_{n,j} X_j$ and $\sum_{1 \le i < j \le n-1} a_{i,j} X_i X_j$ are NQD.

Example 1. Let X_1, X_2 and X_3 have joint probability distribution as given in the following table

X_3	0		1	
X_2	0	1	0	1
X_1				
0	0	3/11	1/11	4/11
1	1/11	2/11	0	0

Also, assume that $n = 3$ and $a_{ij} = 1, i \ne j$. It is easy to show that $X_n \sum_{j=1}^{n-1} a_{nj} X_j$ and $\sum_{1 \le i < j \le n-1} a_{ij} X_i X_j$ are NQD.

In the following, we obtain the almost sure convergence for weighted sums $\sum_i a_{n,i} X_i$, where $\{X_n, n \ge 1\}$ is a sequence of pairwise NQD stochastically bounded random variables and $\{a_{ni}\}$ is an array of real numbers under some suitable conditions on $a_{n,i}$.

Theorem 4. Let $\{X_n, n \ge 1\}$ be a sequence of pairwise NQD random variables such that for all $n \ge 1$

$$P(|X_n| \ge x) \le c \int_x^{\infty} e^{-\gamma t^2} dt \qquad \gamma > 0 ;$$

let $\{a_{nj}, 1 \le j \le n\}$ be a triangular array of real numbers with $\sum_{i=1}^{n} a_{ni}^2 = O(n^{-\beta})$, $\beta > 1$. Then

$$\sum_{i=1}^{n} a_{ni} X_i \xrightarrow{n\to\infty} 0 \qquad a.s$$

Proof. by Cauchy Schwartz's inequality we have

$$\left|\sum_{j=1}^{n} a_{nj} X_j\right|^2 \le (\sum_{j=1}^{n} a_{nj}^2)(\sum_{j=1}^{n} X_j^2) \le cn^{-\beta} (\sum_{j=1}^{n} X_j^2),$$

Choosing an integer k such that $2^{k-1} \le n < 2^k$, we get

$$\limsup \left|\sum_{j=1}^{n} a_{nj} X_j\right|^2 \le c \limsup \frac{1}{(2^{k-1})^{\beta}} (\sum_{j=1}^{2^k} X_j^2).$$

Therefore

$$\sum_{k=1}^{\infty} P((2^{k-1})^{-\beta} \sum_{j=1}^{2^k} X_j^2 > \varepsilon)$$

$$\le \frac{1}{\varepsilon} \sum_{k=1}^{\infty} (2^{k-1})^{-\beta} (\sum_{j=1}^{2^k} EX_j^2)) \le c \sum_{k=1}^{\infty} (2^{\beta})^{-k} < \infty$$

Now (P4) complete the proof.

Corollary 3. Under the assumptions of Theorem 4

i) If $\beta > \alpha + 1$, and for every $\alpha > 0$, $\sum_{j=1}^{n} E(X_j^2) = O(n^{\alpha})$, then

$$\sum_{i=1}^{n} a_{ni} X_i \xrightarrow{n\to\infty} 0. \qquad a.s.$$

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ii) If $a_{nj} = \dfrac{1}{n+j}, 1 \le j \le n, n \ge 1$ and $\sum_j a_{nj}^2 = O(n^{2\alpha-2})$, then for every $\alpha > 3/2$,

$$\sum_{i=1}^{n} a_{ni} X_i \xrightarrow{n\to\infty} 0. \quad a.s.$$

Theorem 5. Let $\{X_n, n \ge 1\}$ be a sequence of pairwise NQD random variables with $EX_n = 0$, $n \ge 1$, that are stochastically bounded by $Z \sim N(0,1)$ and let $\{a_{nj}, 1 \le j \le n, n \ge 1\}$ be a triangular array of real numbers such that

$$\sum_{i=1}^{n} \left| a_{nj} - a_{n(j+1)} \right| = O(n^{-\beta}), \quad \beta > 1$$. Then

$$\sum_{i=1}^{n} a_{ni} X_i \xrightarrow{n\to\infty} 0. \quad a.s.$$

Proof. By extension of Rademacher and Mensov's inequality (Chandra and Chatterjee [4]) for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} P(n^{-\beta} \max_{1\le i\le n} |S_i| > \varepsilon)$$

$$\le c \sum_{n=1}^{\infty} n^{-2\beta} (\log n)^2 \sum_{j=1}^{n} EX_j^2$$

$$\le c \sum_{n=1}^{\infty} n^{-2\beta+1} (\log n)^2 < \infty ,$$

Also, applying Abel's summation rule we get

$$\left| \sum_{i=1}^{n} a_{ni} X_i \right| \le \max_{1\le i\le n} \left| \sum_{i=1}^{j} X_i \right| \left(\sum_{i=1}^{j} \left| a_{n,j} - a_{n,(j+1)} \right| \right)$$

$$\le c n^{-\beta} \max_{1\le i\le n} \left| \sum_{i=1}^{j} X_i \right|,$$

Therefore

$$\sum_{n=1}^{\infty} P\left(\sum_{j=1}^{n} \left| a_{nj} X_j \right| > \varepsilon\right)$$

$$\le \sum_{n=1}^{\infty} P(n^{-\beta} \max_{1\le j\le n} |S_j| > \varepsilon) < \infty,$$

This and (P4) imply that

$$\sum_{i=1}^{n} a_{ni} X_i \xrightarrow{n\to\infty} 0. \quad a.s.$$

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References

1. Amini M., Bozorgnia A. and Zarei H. Some strong limit theorems of weighted sums for negative dependent generalized Gaussian random variables. *Statist. Probab. Lett.*, **77**: 1106-1110 (2007).
2. Amini M., Azarnoosh H.A, and Bozorgnia A. The almost sure convergence of weighted sums of ND r.v.s. *J. Sci. I. R. of Iran,* **10**(2): 112-116 (1999).
3. Bozorgnia A., Patterson R.F. and Taylor R. L. Limit theorems for dependent random variables. *World Congress Nonlinear Analysts,* **92**, Vol. I-IV(Tampa, FL, 1992): 1639-1650, de Gruyter, Berlin (1996).
4. Chandra T.K. and Chatterjee D. *A first course in Probability*. Alpha Science Int. Ltd, 467 p. (2001).
5. Chandra T.K. and Ghosl S. Extensions of the strong law of large numbers for marcinkiewicz and Zygmund for dependent variables. *Acta Math. Hungar.* **71**: 327-336 (1996).
6. Chow Y.S. Some convergence theorems for independent random variables. *Annals of Math.Statistics.*, **37**: 1482-1492 (1966).
7. Joag-Dev K. and Proschan F. Negative association of random variables with applications. *Ann. Statist.,* **11**: 286-295 (1983).
8. Lehmann E.L. Some concept of dependence, *Ann. Math. Statist.,* **37**: 1137-1153 (1966).
9. Liang H.Y., Chen Z.J. and Su C. Convergence of Jamison_type weighted sums of pairwise negatively quadrant dependent random variables. *Acta Math. Appl. Sinicia, English Series,* **18**: 161-168 (2002).
10. Matula P. A note on the almost sure convergence of sums of negatively dependent random variables. *Statist. Probab. Lett.*, **15**: 209-213 (1992).
11. Sani H.R., Amini M. and Bozorgnia A. Strong Laws for Weighted Sums of PND Random Variables. *J. Sci. I. R. of Iran*, **16**(3): 261-265 (2005).
12. Shao Q.M. A comparison theorem on moment inequalities between negatively associated and independent random variables. *J. Theoretical Probability*, **13**: 343-356(2000).
13. Sung S.H. Strong laws for weighted sums of i.i.d. random variables. *Statist. Probab. Lett.,* **52**: 413-419 (2001).
14. Thrum R. A remark on almost sure convergence of weighted sums. *Probab. Th. Rel.Fields,* **75**: 425-430 (1987).

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