

Some New Properties of the Searching Probability Used in Design Comparison

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Abstract

Consider search designs for searching one nonzero 2- or 3-factor interaction under the search linear model. In the noisy case, search probability is given by Shirakura et al. (Ann. Statist. 24(6) (1996) 2560). In this paper some new properties of the searching probability are presented. New properties of the search probability enable us to compare designs, which depend on an unknown parameter ρ , for all values of the parameter without needing to check for different values of ρ . Two new quantitative interval scale and parameter free criteria based on search probabilities are proposed for design comparison. These criteria are used to compare given designs. The equivalent search designs is defined based on new proposed criteria and present a class of equivalent designs which are orthogonal arrays of strength 2.

Keywords: Equivalent designs; Interactions; Main effect plans; Search designs; Search models

Introduction

Consider a 2^m factorial experiment with n runs, where $n < 2^m$. A main effects plan allows us to make inference on the general mean and main effects under the assumption that 2-factor and higher order interactions are all zero. Such an assumption may not be valid in reality. There may be a few non-negligible interactions present. Consequently, the estimates of the parameters are biased. Search linear models introduced in [7] permit the search for non-negligible interactions from 2-factor and higher order interactions. Now, we consider the following search linear model for a 2^m factorial experiment:

$$E(\underline{y}) = A_1\xi_1 + A_2\xi_2, \quad \text{var}(\underline{y}) = \sigma^2 I \quad (1)$$

where $\underline{y}(n \times 1)$ is a vector of observations, $\xi_1(v_1 \times 1)$, $v_1 = 1 + m$, is the vector of the general mean and main effects, and $\xi_2(v_2 \times 1)$, $v_2 = \binom{m}{2} + \binom{m}{3}$, is the vector of 2- and 3-factor interactions, A_1 and A_2 are design matrices, and σ^2 is an unknown nonnegative constant. The problem is to search for the one nonzero element of ξ_2 and draw inference on it in addition to the elements of ξ_1 . The designs that resolve this problem are called main effect plus one (MEP.1) plans. When $\xi_2 = 0$, the model is the main effects model. Based on the basic mathematical formulation of the problem introduced by Srivastava [7], the following theorem is fundamental for the given search linear model.

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Theorem 1.1. Consider the model (1) and let $\sigma^2 = 0$. A necessary and sufficient condition that the search and inference problem can be completely solved is that for every submatrix $A_{22}(n \times 2)$ of A_2 ,

$$\text{rank}(A_1 : A_{22}) = v_1 + 2 \quad (2)$$

holds.

Such a design is called a search design. By this theorem, one considers a class of linear models containing the general mean, main effects, and one interaction. There are v_2 models in this class, say M. Each has the general mean and main effects as common parameters, but they are different in one possible interaction; i.e.

$$M = \{M(\zeta) : E(\underline{y}) = A_1 \xi_1 + a(\zeta)\zeta, \zeta \in \xi_2\} \quad (3)$$

where $a(\zeta)$ is the $n \times 1$ column of A_2 corresponding to ζ . The case when $\sigma^2 = 0$ is called the noiseless case. However, in practical experiments we have the noisy case $\sigma^2 > 0$ in which the condition (2) is still necessary. For the noisy case, Srivastava [7] proposed a procedure for solving the search problem, which correspond to minimization of the sum of squares due to error (SSE) of v_2 candidate models in class M. That is, consider models, say M_1 and M_2 , from class M; one says M_1 provides a better fit and is selected over M_2 if $\text{SSE}(M_1) < \text{SSE}(M_2)$. For the noiseless case, this is done with probability one. However, this is not assured for the noisy case and, for a given search design, the stochastic properties of the SSE have to be considered. In this case, the probability

$$P(\text{SSE}(M(\zeta_0)) < \text{SSE}(M(\zeta)) | \zeta_0, \zeta, \sigma^2), \quad (4)$$

where $M(\zeta_0)$ is the possible true model and $M(\zeta)$ for $\zeta (\neq \zeta_0)$ is any other model in M, is not necessarily equal to 1 and, possibly, is less than 1. The value of the probability depends on the value of σ^2 . This probability, called the "searching probability" (SP), also depends on the search design. So, one may be interested in the value of the SP for a given design, using that for design comparison. Based on SPs, several criteria have been developed for comparing search designs, and all depend on the value of the unknown actual quantity of the nonzero interaction effect, $\rho = |\zeta_0|/\sigma$, where ζ_0 is the possible nonzero effect of ξ_2 . Shirakura et al. [6] compared search designs for

different reasonable possible values of ρ . Ghosh and Tschmacher[3] gave modified criteria to perform the comparison once without going through different values of ρ . However, using these criteria, the information is sacrificed due to simplification and reduction of the enormous task of comparison for different values of ρ .

In this paper, the problem of the dependence of the search probability to ρ , which causes an enormous task or losing information due to simplification for comparing designs, is tackled. To overcome this problem some new properties of searching probabilities are derived and presented to help in simplification and reduction of the number of comparisons task without losing information in design selection.

Materials and Methods

Notation and Preliminary Results

In this section, notation and previous results are presented. They are, mostly, adapted from material given in [6], and are used in subsequent sections of this work. The notation is

$Q = A_1(A_1'A_1)^{-1}A_1'$ is an $n \times n$ matrix of rank $v_1 (< n)$,

$r(\zeta) = a(\zeta)'(I - Q)a(\zeta)$ is a positive constant that depends on the search design,

$b(\zeta) = (I - Q)a(\zeta)/\sqrt{r(\zeta)}$ is an $n \times 1$ vector, for each $\zeta \in \xi_2$,

$x = b(\zeta)'b(\zeta_0)$ is a constant with $-1 < x < 1$,

$\rho = |\zeta_0|/\sigma$ is the absolute value of the actual unknown effect,

$c_1 = \sqrt{\frac{r(\zeta_0)}{2}(1-x)}$, $c_2 = \sqrt{\frac{r(\zeta_0)}{2}(1+x)}$ are positive constants where for $0 \leq x \leq 1$ it is true that $c_1 \leq c_2$, $0 \leq c_1 \leq \sqrt{r(\zeta_0)/2}$, and $\sqrt{r(\zeta_0)/2} \leq c_2 \leq \sqrt{r(\zeta_0)}$.

Assuming \underline{y} has a normal distribution with its mean given as the mean of the true model $M(\zeta_0)$ in M and variance $\sigma^2 I$, then the searching probability of (4) can be written as the following expression

$$G(x, \rho) = 1 - \Phi(c_1 \rho) - \Phi(c_2 \rho) + 2\Phi(c_1 \rho)\Phi(c_2 \rho) \quad (5)$$

where $\Phi(\cdot)$ is the Standard Normal pdf.

Theorem 2.1. [6] The function $G(x, \rho)$ has the following properties:

(i) $G(-x, \rho) = G(x, \rho)$.

(ii) $0.5 \leq G(x, \rho) \leq 1$.

(iii) For a fixed x , $G(x, \rho)$ is continuous and strictly increasing in ρ .

(iv) For a fixed $r(\zeta_0)$ and ρ , $G(x, \rho)$ is monotonically increased as c_1 is increased.

Although the actual value of the effect ρ is restricted to $|\zeta_0|$ and hence positive values but for negative ζ_0 one can show that $G(x, \rho) = G(x, -\rho)$. Therefore, by Theorem 2.1(i), without loss of generality we consider nonnegative $\rho = |\zeta_0|/\sigma$ and $0 < x < 1$. Note also from Theorem 2.1(iii) that, as ρ tends to ∞ , the probability function G increases to 1.

Results

Some New Properties for Search Probabilities

The search probability in (5) depends on the given search design T through c_1 and c_2 . It also depends on the actual value of the effect ρ . So, the SP can be written as $P_T(\rho)$. The following lemma address a property of $P_T(\rho)$.

Lemma 3.1. For a given search design T , and fixed c_1 , there exists a positive point w depending on T , such that $P_T(\rho)$ is convex in $0 \leq \rho \leq w$ and concave for $\rho > w$.

proof. For $P_T(\rho)$ given in (5) we have

$$\frac{\partial^2}{\partial \rho^2} P_T(\rho) = \frac{1}{\pi} e^{-\frac{r(\zeta_0)}{2} \rho^2} (A - B) \tag{6}$$

Such that

$$B = \sqrt{\frac{\pi}{2}} \rho \left[c_1^3 (2\Phi(c_2 \rho) - 1) e^{\frac{1}{2} c_2^2 \rho^2} + c_2^3 (2\Phi(c_1 \rho) - 1) e^{\frac{1}{2} c_1^2 \rho^2} \right]$$

and $A = 2c_1 c_2$ are both positive, and $\frac{1}{\pi} e^{-\frac{r(\zeta_0)}{2} \rho^2}$ is also positive. Note that A is a constant and B is an increasing function of $\rho > 0$. So, for $\rho = 0$ we have

$A > 0$, $B = 0$ and then $A - B > 0$. As ρ increases, $A - B$ tends to zero. Let $\rho = w > 0$ be the point at which $A - B = 0$. Clearly, for $\rho > w$ we have $A - B < 0$. Therefore, by Theorem 2.1(iii), the proof is complete.

The point w is the inflection point and its value depends on the design T through $r(\zeta_0)$. For $b > w$ the point w partitions this interval into two parts, $[0, w)$ and $[w, b]$ for ρ . For a search design with a high SP value the portion $[0, w)$ is small since both A and B increase as $r(\zeta_0)$ increases. However, this rate in B is sharper than A . It verifies that, for a design with large $r(\zeta_0)$, $A - B$ in Lemma 3.1 switches from positive to negative at a low value of ρ . This is also true for moderate $r(\zeta_0)$ and large to moderate c_1 . These lead us to assume that $P_T(\rho)$ is concave for all ρ greater than a reasonable low value of ρ . This assumption might fail for a design with small $r(\zeta_0)$ and c_1 , which is not of interest due to lack of qualification for comparison. Note that a nonzero effect ζ_0 will generally be nonsignificant if $\rho < 1$. Consequently, our subsequent work will restrict attention to values of $\rho \geq 1$. Under this restriction and by Lemma 3.1, the following results are established.

Corollary 3.2. For a given design T , let $P_T(\rho)$ be a concave function on $[a, b]$. The following properties are true:

(i) $\frac{P_T(\rho) - P_T(a)}{\rho - a} \geq \frac{P_T(b) - P_T(a)}{b - a}$, for $\rho \in [a, b]$ (7)

(ii) For $a \leq \rho_1 \leq \rho_2 \leq b$,

$$\frac{P_T(\rho_1) - P_T(a)}{\rho_1 - a} \geq \frac{P_T(b) - P_T(\rho_2)}{b - \rho_2} \tag{8}$$

proof. This is clear from Lemma 3.1.

By properties (i) and (ii) in corollary 3.2, it is clear that, as ρ increases, the slope of the curve $P_T(\rho)$ decreases. That is, the rate of increase in values of $P_T(\rho)$ will get slower as ρ increases. This means that lower values of ρ have a greater effect on the area under the curve $P_T(\rho)$, denoted by A_T , than the higher values of ρ . With this in mind, the following proposition is presented.

Proposition 3.3. Consider $A_T = \int_a^b P_T(\rho) d\rho$ for the

concave curve $P_T(\rho)$ on $[a, b]$. Let define the area S_T by $S_T = \frac{1}{2}[b - a][P_T(a) + P_T(b)]$. Then $A_T \geq S_T$.

proof. For any $\rho \in [a, b]$, there exists an $0 \leq \alpha \leq 1$ such that ρ can be written as $\rho = \alpha a + (1 - \alpha)b$. The proof is straight forward as $P_T(\rho) \geq \alpha P_T(a) + (1 - \alpha)P_T(b)$ from Lemma 3.1.

Clearly, for a given design T as $P_T(\rho)$ get larger it is more likely that design T identifies the true model from the incorrect one. It is also clear that the value of S_T of a design with higher SP is higher than that of a design with lower SP. With this in mind we suggest approximate A_T by S_T for to simplify design comparison for $\rho \in [a, b]$.

For $a = 1$ and $b \geq 3$, and given search designs T_1 and T_2 , by Theorem 2.1 (i) and (iii) for sufficiently large ρ 's there exists a small ε such that

$$b = \inf \left\{ \rho : \left| P_{T_1}(\rho) - P_{T_2}(\rho) \right| < \varepsilon \right\}. \quad (9)$$

Therefore, for such an a and b , the following theorem is established.

Theorem 3.4. Consider the associated search probabilities $P_{T_i}(\rho)$ for given designs T_1 and T_2 . If $P_{T_1}(\rho) > P_{T_2}(\rho)$ at $\rho = 1$, then $S_{T_1} \geq S_{T_2}$.

proof. By b as defined in (9), $P_{T_1}(\rho) \cong P_{T_2}(\rho)$. Then the proof is clear from proposition 3.3 by substituting $S_{T_i} = [b - 1][P_{T_i}(1)]/2$.

In the next section, these results will be used in comparing search designs.

Design Comparison Criteria

To compare search designs for a given values of ρ , the minimum searching probability criterion is suggested in [6]. Ghosh and Teschmacher [3] proposed a new type of ordinal criteria and generalized it so that a comparison of search designs can be made without requiring a specific value of ρ . However, this is at the cost of losing some information, which may cause difficulty in decision making. The results of the previous section are applicable to these criteria for comparing designs with more possible information when one just uses search probabilities for $\rho = 1$.

In this paper, new criteria are proposed that consider the quantity of paired searching probability differences. It is reasonable since, in comparing designs, one may

come up with a situation in which there is a majority with positive differences in favor of a design, yet with small values compared to the minority of negative differences with large absolute values.

For a design T_i , let SPM_i be the $v_2 \times v_2$ matrix of searching probabilities for $\rho = 1$. That is, the columns of SPM_i correspond to all ζ_0 representing the possible true nonzero effect and the rows correspond to all ζ representing other effects in ξ_2 . Note that, for a given $\zeta_0 \in \xi_2$, the elements in the corresponding column are the searching probabilities, each of which represents how likely it is that the true effect ζ_0 is distinguishable from another effect $\zeta (\neq \zeta_0) \in \xi_2$. Averaging the SPs in a given column over $\zeta (\neq \zeta_0)$ is the distinguishable mean search probability of ζ_0 . Since $\zeta \neq \zeta_0$, the diagonal elements of SPM are not of interest.

Global Mean Criterion

For two search designs, T_i and T_j , define

$$g_{ij} = \frac{1}{v_2(v_2 - 1)} tr \left[(\mathbf{J} - \mathbf{I})(SPM_i - SPM_j) \right] \quad (10)$$

where \mathbf{J} is a $v_2 \times v_2$ matrix of ones and \mathbf{I} is the identity matrix of order v_2 . Consequently, T_i is said to be better than T_j if $g_{ij} > 0$. We call this the Global Mean criterion and denote it by G-criterion.

Effectwise Mean Criterion

In the class of models considered in section 1 it is known that only one of the effects, say ζ_0 , is the true nonzero interaction effect in ξ_2 . So, we focus on each of the columns in SPM. Now, for given designs T_i and T_j , we consider the $v_2 \times v_2$ diagonal matrix D_{ij}

$$D_{ij} = diag \left[\frac{1}{v_2 - 1} (\mathbf{J} - \mathbf{I})(SPM_i - SPM_j) \right] = (d_{it}). \quad (11)$$

A $d_{it} > 0$ means that the design T_i distinguishes the t -associated nonzero $\zeta_0 \in \xi_2$ with a higher average search probability than does T_j . Now, let $d_{ij}^+(d_{ij}^-)$ be the number of positive (negative) diagonal elements d_{it} , $t = 1, 2, \dots, v_2$. Let d_{ij}^0 be the number of d_{it} with value zero. Clearly, $0 \leq d_{ij}^+ \leq v_2$, and $d_{ij}^+ + d_{ij}^- + d_{ij}^0 = v_2$. The following rules discriminate designs.

1. If $d_{ij}^+ = \nu_2$, then T_i is *efficiently* better than T_j . If $d_{ij}^- = \nu_2$, T_j is *efficiently* better than T_i .

2. If the majority of d_{ij} 's are positive(negative); i.e. $d_{ij}^+ > \nu_2/2$ ($d_{ij}^- > \nu_2/2$), then T_i (T_j) is considered *relatively* better than T_j (T_i).

In the case that ties exist (and hence zero diagonal elements) if neither the positive nor negative number of diagonal elements achieve a majority, then under the condition $d_{ij}^+ = d_{ij}^- > 0$, the comparison between T_i and T_j might be inconclusive. We call this the Effectwise Mean criterion and denote it by E-criterion.

Isomorphic Designs

It can happen that two or more search designs have the same searching probability efficiency in searching for a nonzero interaction effect with respect to the criterion which is applied to them for comparison purposes.

Definition 4.1. Let T_i and T_j be two search designs with n runs. They said to be equivalent or isomorphic in the G-criterion (are *G-isomorphic*) if $g_{ij} = 0$. They are also said to be equivalent in the E-criterion (are *E-isomorphic*) if $d_{ij}^0 = \nu_2$. Obviously, if designs are E-isomorphic, then they are G-isomorphic, but the converse is not true.

Implementation

The utility of the G- and E-criteria defined above in design comparison will be presented in this section. First, the criteria are applied to MEP.1 plans for $m = 7$. The following design T_1 given in [2] is a MEP.1 plan with $n = 15$ runs. It is a balanced array of strength 2, composed of T_{11} and T_{12} , i.e. $T_1' = [T_{11}' : T_{12}']$ where for the vector of ones $J_{7 \times 1}$, $T_{11}' = [J_{7 \times 1} : I_7]$ and T_{12}' is the incidence matrix of a symmetric BIB design with parameters $b = \nu = 7$ and $r = k = 4$.

$$T_1' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad (12)$$

Let us take the MEP.1 plan for $m = 7$ with $n = 15$ runs given in [5] as T_2 . Using the G-criteria for

comparing T_1 and T_2 gives $g_{12} = 0.033154 > 0$. Therefore, T_1 is better than T_2 . Now, use the E-criterion to determine d_{12}^+ . It gives $d_{12}^+ = 56$ which means T_1 is *efficiently* better than T_2 .

Next, we consider the case of $m = 4$ with $n = 12$ runs. For this case designs are compared in two parts according to the designs which are selected for comparison.

Part 1. In this part we repeat the comparison of the designs D_1 , D_2 and D_3 given in [3] on the basis of their criterion I. However, we enjoy the advantage of Theorem 3.4 in decision making. The G- and E-criteria is also considered in comparing these designs. The results are presented in Table 1. All of the criteria confirm that D_1 is better than D_2 and D_3 , and, in fact, it is better than D_2 *efficiently* by E-criterion. Using Theorem 3.4 in comparing D_2 and D_3 , the last row of Table 1 shows that D_2 is better than D_3 under criterion I. The G-criterion and E-criterion give the same result, though D_2 is *relatively* better than D_3 .

Part 2. In this part, consider the class of plans for $m = 4$ obtained as follow from T_1' in (12). Consider $T_{11}' = [J_{4 \times 1} : I_4]$ and, for $m = 4$, let T_{12} be obtained by deleting any three of the seven columns from T_{12}' of $m = 7$ in (12). Ghosh and Talebi [2] showed that the resulting $T' = [T_{11}' : T_{12}']$ are isomorphic 12-run plans which are MEP.1 plans for $m = 4$. We compute the SPM for all 35 designs in this class. These SPMs are equivalent in having two distinct elements $f = 0.967265$ and $h = 0.944356$ as off-diagonal elements. The values f and h occurs 6 and 3 times respectively in each column of SPM. For any pair of designs, say T_i and T_j , $1 \leq i < j \leq 35$, in this class the g_{ij} of G-criterion and d_{ij}^+ s of E-criterion are determined. The result is $g_{ij} = 0$, $d_{ij}^+ = d_{ij}^- = 0$, and $d_{ij}^0 = 10$, for $1 \leq i < j \leq 35$. It means, by definition 4.1, that all of these MEP.1 plans are E-isomorphic and

Table 1.

Comparison	Criterion I			G-Criterion	E-Criterion		
	n_{ij}^+	n_{ij}^-	n_{ij}^0		g_{ij}	d_{ij}^+	d_{ij}^-
D_1 vs D_2	78	0	12	0.01170	10	0	0
D_1 vs D_3	73	17	0	0.01661	9	1	0
D_2 vs D_3	47	43	0	0.00491	7	3	0

hence G-isomorphic search designs. Note also that the average searching probability of the columns are equal, which means that the designs are invariant in distinguishing any one of the interaction effect as the possible nonzero $\zeta_0 \in \zeta_2$.

Discussion

For a 2^m factorial experiment, search designs are considered for searching and making inference on one interaction from 2- or 3-factor interactions under the search linear model. The condition given in (2) is necessary and sufficient for searching and estimating the problem in the noiseless case. However, it is only a necessary condition in the noisy case. Therefore, the search designs obtained in the noiseless case may not identify the true interaction and discriminate the correct model perfectly for the noisy case. In this case, the search probability given in (4) to discriminate between the correct and incorrect models is less than one, so it is most desirable to have a design with as high a search probability as possible. By now, it has been known that most of the search designs obtained for the noiseless case, for example, [1], [4] and [3], are all desirable for the noisy case with a relatively high SP. The positive value w proposed, as the inflection point in Lemma 3.1 is guaranteed to be less than 1 as long as $r(\zeta_0)$ is greater than 2, which is true for almost all of these designs. It means that the search probability $P_T(\rho)$ for comparable search designs is a concave function for $\rho > w$ with $w < 1$. That is, $P_T(\rho)$ has a sharp slope for $w \leq \rho \leq 1$ which causes it to tend to a value close to 1 rapidly. Under this situation, comparing the designs for lower values of ρ , at which the significant nonzero interaction effect is small compared to probably a large noise, is more difficult to be identified yet, important than at higher values of ρ where $P_T(\rho)$ is flat. Under this scenario, the properties of the searching probability, which are obtained, in this paper are quite helpful in design comparison. Keeping this in mind, the G- and E-

criteria based on search probabilities are presented and applied to candidate designs to determine the design which is most likely to identify the nonzero interaction and discriminate the correct model. In addition to these criteria, it was shown that, by using the new properties of the SP, criterion I given in [3] is not inconclusive in determining the better design between D_2 and D_3 any more. G- and E-isomorphic (equivalent) search designs are defined and a class of the equivalent search designs, all are of orthogonal arrays of strength 2, is presented for $m = 4$.

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