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A Study of Entanglement and Squeezing of Multiqubit Cluster Systems

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Abstract

We study entanglement and squeezing of a cluster of spin systems under the influence of the two-axis countertwisting Hamiltonian. The squeezing parameters given by Wineland et al and also by Kitagawa et al. are chosen as the criteria of spin squeezing. The criterion of pairwise entanglement is chosen to be the concurrence and that of the bipartite entanglement the linear entropy. We also define a new squeezing parameter η, which plays a direct role in the investigation of the relationship between squeezing and entanglement. We observe that if the system is squeezed according to the Wineland's criterion, it is squeezed according to the Kitagawa's also, but the reverse is not always true. Moreover, if the system is squeezed according to Kitagawa's criterion, it is pairwise entangled simultaneously and vice versa. It is also observed that the entropy is a linear function of the parameter η^2.

Keywords: Spin squeezing; Entanglement; Two-axis countertwisting Hamiltonian

Introduction

Entanglement [1-11] is closely related to squeezing [12-22] and being considered as an information resource, it is quite relevant to the subject of quantum information and quantum computation [23-26]. Moreover, squeezing has also important applications in quantum measurements and precision spectroscopy [27-31].

In this work [32], we consider a cluster of spin system consisting of several qubits, which are initially in a coherent state and study their time evolution via the well known two-axis countertwisting Hamiltonian [12]

$$H = \frac{\chi}{2i}\left(\hat{S}_+^{\,2} - \hat{S}_-^{\,2}\right) \tag{1}$$

To study the squeezing properties of the system, we use the spin squeezing parameter

$$\xi_K^2 = \frac{2(\Delta S_{n_\perp})^2_{\min}}{S}, \tag{2}$$

introduced by Kitagawa et al and also the spin squeezing parameter

$$\xi_W^2 = 2S\,(\Delta S_{n_\perp})^2_{\min} \Big/ \left|\left\langle \vec{\hat{S}} \right\rangle\right|^2, \tag{3}$$

introduced by Wineland et al [27]. Here $\hat{n}_\perp$ represents a direction perpendicular to the mean spin direction $\hat{n} = \left\langle \vec{\hat{S}} \right\rangle \Big/ \sqrt{\left\langle \vec{\hat{S}} \right\rangle \cdot \left\langle \vec{\hat{S}} \right\rangle}$. We also consider linear entropy

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$$E_L = 1 - Tr(\rho_i^2)\,, \tag{4}$$

as a measure of bipartite entanglement (the entanglement between one spin and all the others)[11], where ρ_i is the reduced density matrix for the i th particle; and the concurrence

$$C = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}\,, \tag{5}$$

as the measure of pairwise entanglement (the entanglement of a pair of spins)[1,2]. Here, λ_i are the eigenvalues of the 4 by 4 matrix

$$\mathrm{P}_{ij} = \rho_{ij}(\sigma_{iy} \otimes \sigma_{jy})\rho_{ij}^*(\sigma_{iy} \otimes \sigma_{jy})\,, \tag{6}$$

where, ρ_{ij} is the reduced density matrix element and ρ_{ij}^* is its complex conjugate.

We shall present our calculations for the prototype 4-qubit system in detail. To demonstrate our procedure; however, the method may be applied to study the entanglement and squeezing of N-qubit systems as well. We shall display our results for a 9-qubit system as an example.

We organize the rest of this paper as follows. In section 2, the initial spin coherent state is introduced and its uncertainty aspects are discussed. Then, its time evolution via Hamiltonian (1) is considered and time dependent spin operators are obtained. Time dependence of the squeezing parameters (2) and (3) are also discussed in section 2. In section 3, entanglement parameters, their time dependence and their relationship to the squeezing parameters, are considered. Finally, section 4 is devoted to discussion and conclusions.

Spin Squeezing

A general spin S coherent state, is given by

$$|\theta,\varphi\rangle = \left(1+\gamma^2\right)^{-S} \sum_{m=0}^{2S} \left(e^{i\varphi}\gamma\right)^m \binom{2S}{m}^{1/2} |S, S-m\rangle_z\,, \tag{7a}$$

$$\tan\left(\frac{\theta}{2}\right) = \gamma\,, \tag{7b}$$

where, θ and φ are the polar and the azimuthal angle, respectively and $|S, S-m\rangle_z$ are eigenstates of S^2 and S_z [33]. This may be considered as an ensemble of $N = 2S$ qubits (N one-half spins) with no interaction between them for the moment. Thus, the collective spin operators in the direction $\hat{n}$, may be given by

$$S_n = \sum_{i=1}^{N} \frac{1}{2}\sigma_{i,n}\,. \tag{8}$$

Where, $\sigma_{i,n} = \hat{\sigma}_i . \hat{n}$ is the Pauli matrix in the $\hat{n}$ direction for the i th spin. We consider our prototype four qubit coherent state along the z-direction, as follows

$$|\theta = 0, \varphi\rangle = |S, S\rangle = |2,2\rangle\,. \tag{9}$$

We note that

$$\langle 2,2|\hat{S}_x|2,2\rangle = \langle 2,2|\hat{S}_y|2,2\rangle = 0\,, \tag{10}$$

$$\langle \vec{\hat{S}} \rangle = \langle \hat{S}_z \rangle = \langle 2,2|\hat{S}_z|2,2\rangle = 2\,, \tag{11}$$

$$(\Delta S_x)^2 = (\Delta S_y)^2 = 1\,; \tag{12}$$

meaning that the average spin rests along the z direction and all the spins are upward at $t = 0$. Moreover the uncertainty relation

$$\left(\Delta S_x\right)^2 \left(\Delta S_y\right)^2 \geq \frac{1}{4}\left|\langle \hat{S}_z \rangle\right|^2\,, \tag{13}$$

with the equality sign is satisfied here.

We now study the time evolution of this 4-qubit system via Hamiltonian (1). The nonzero matrix elements of H are given by

$$(H)_{3,1} = (H)_{5,3} = i\sqrt{6}\chi\,,$$

$$(H)_{1,3} = (H)_{3,5} = -i\sqrt{6}\chi\,,$$

$$(H)_{4,2} = -(H)_{2,4} = 3i\chi\,. \tag{14}$$

Thus, our time dependent ket state is found to be

$$|\psi(T)\rangle = e^{-iHt}|2,2\rangle \doteq \begin{pmatrix} (1+\cos(2T))/2 \\ 0 \\ (\sin(2T))/\sqrt{2} \\ 0 \\ (1-\cos(2T))/2 \end{pmatrix}, \tag{15}$$

where, the scaled time, $T = \sqrt{3}\chi t$ has been defined. Using the above dynamically generated state we find

$$\langle \psi(T)|\hat{S}_x|\psi(T)\rangle = \langle \psi(T)|\hat{S}_y|\psi(T)\rangle = 0\,, \tag{16}$$

implying that the dynamic evolution has not changed the average direction of spin and it still stands along the z-axis. We therefore have

$$\left|\langle\psi(T)|\vec{S}|\psi(T)\rangle\right| = 2\left|\cos(2T)\right| . \quad (17)$$

It is now worthwhile to look at the uncertainty relation at time T , we find

$$(\Delta S_x)^2 = \frac{3}{2} - \frac{1}{2}\cos(4T) + \sqrt{3}\sin(2T) , \quad (18)$$

$$(\Delta S_y)^2 = \frac{3}{2} - \frac{1}{2}\cos(4T) - \sqrt{3}\sin(2T) , \quad (19)$$

$$(\Delta S_x)^2(\Delta S_y)^2 = [7 + \cos(8T)]/8 , \quad (20)$$

and

$$\frac{1}{4}\left|\langle\hat{S}_z\rangle\right|^2 = [\cos(2T)]^2 . \quad (21)$$

Comparing (20) and (21) with (12) we observe a redistribution of uncertainties in different directions in this situation. For example, assuming $T = \frac{\pi}{6}$, we find

$$(\Delta S_x)^2 = 3.62 , \quad (\Delta S_y)^2 = 0.26 ,$$

$$(\Delta S_x)^2(\Delta S_y)^2 = 0.95 , \quad \frac{1}{4}\left|\langle\hat{S}_z\rangle\right|^2 = 0.06 ; \quad (22)$$

meaning that uncertainty along the x-axis has increased above the quantum limit, while it has decreased below that limit along the y-axis. The inequality sign in (13) is also satisfied. In fact, we are dealing with a squeezed state at this time, and those are exactly the characteristics that we expect for such states.

To find the best squeezing direction, we rotate the coordinate system in the $x-y$ plane by angle δ , but keep the z-axis fixed. Obviously, the uncertainties along the new coordinates are functions of δ . Let's define the direction $\hat{n}_\perp = (\cos\delta, \sin\delta, 0)$ in the $x-y$ plain. We may write

$$S_{n_\perp} = S_x \cos\delta + S_y \sin\delta . \quad (23)$$

Therefore we find

$$(\Delta S_{n_\perp})^2 = \frac{3}{2} - \frac{1}{2}\cos(4T) + \sqrt{3}\sin(2T)\cos(2\delta) . \quad (24)$$

Minimizing (24) with respect to δ , we obtain

$$(\Delta S_{n_\perp})^2_{\min} = \frac{3}{2} - \frac{1}{2}\cos(4T) - \left|\sqrt{3}\sin(2T)\right| . \quad (25)$$

This result shows that the minimum uncertainty achievable along a direction in the $x-y$ plain is a periodic function of time. This implies that although the mean spin direction remains along the z-direction, but the distribution of spin directions changes in time. To show this point more vividly, we have illustrated the quasi-probability distributions $Q = \left|\langle\theta,\phi|2,2\rangle\right|^2$ and $Q = \left|\langle\theta,\phi|\psi(T)\rangle\right|^2$ along with their contour plots in Figure 1. The elliptic contours in Figure (1a), in contrast to the circular ones in Figure (1b), represent the redistribution of probabilities and uncertainties clearly.

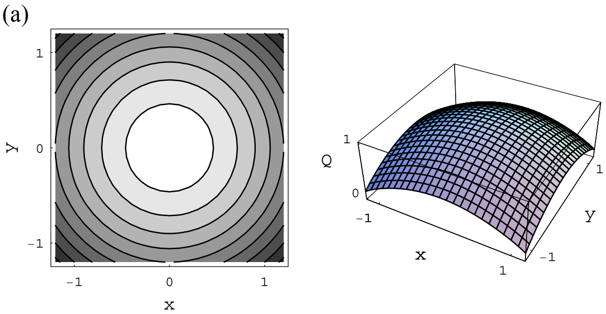
Now, using (2), (3) and (25), we express the squeezing parameters for the $\hat{n}_\perp$ direction as follows.

$$\xi_K^2 = \frac{3}{2} - \frac{1}{2}\cos(4T) - \left|\sqrt{3}\sin(2T)\right| , \quad (26)$$

$$\xi_W^2 = \frac{\frac{3}{2} - \frac{1}{2}\cos(4T) - \left|\sqrt{3}\sin(2T)\right|}{(\cos(2T))^2} . \quad (27)$$

We have plotted these parameters as a function of T in Figure 3. We note that the system becomes squeezed, according to both criteria, at the alternate time intervals, due to the dynamics provided by the Hamiltonian (1). We note that $\xi_W^2 \geq \xi_K^2$; therefore, for the values of $\xi_W^2 < 1$, that the system is squeezed according to Wineland's criterion, it is squeezed according to Kitagawa's also. The reverse is not of course always true.

(a)



(b)

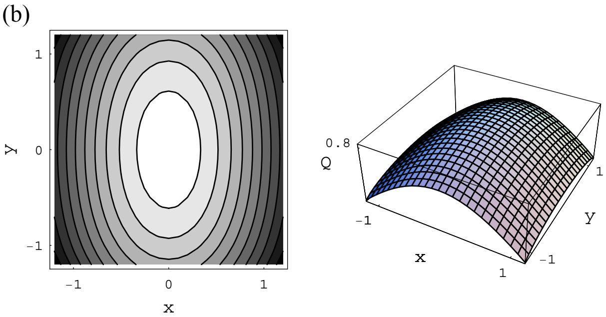


Figure 1. Quasi-probability distribution and its contour plot for *N*=4 at a) *T*=0, b) *T*=0.1776.

Spin Entanglement

First we consider the bipartite entanglement of the system. We are dealing with identical entities, therefore due to exchange symmetry the reduced matrix ρ_i^2 is the same for all the entities. Moreover, the reduced density matrix is just the one-qubit density matrix and we have

$$\rho_i = \frac{1}{2}\begin{pmatrix} 1+\langle\sigma_z\rangle & \langle\sigma_x\rangle - i\langle\sigma_y\rangle \\ \langle\sigma_x\rangle + i\langle\sigma_y\rangle & 1-\langle\sigma_z\rangle \end{pmatrix}, \tag{28}$$

where at the scaled time T we have

$$\langle\sigma_z\rangle = \cos(2\sqrt{3}T)\ ,\ \langle\sigma_y\rangle = 0\ ,\ \langle\sigma_x\rangle = 0\ . \tag{29}$$

Using (4), (28) and (29) we finally obtain

$$E_L = 1/2 - \langle\sigma_z\rangle^2/2 = (\sin(2T))^2/2\ . \tag{30}$$

Now, defining the scaled entropy $E_S = 2E_L$ [11] and eliminating time between equations (30), (26) and (27) we find

$$E_S = 2E_L = (1-\eta^2) = (\sin(2T))^2\ . \tag{31}$$

Where, η^2 which may be called squeezing ratio, has been defined by

$$\eta^2 = \frac{\xi_K^2}{\xi_W^2} \le 1\ . \tag{32}$$

This is an interesting result; the scaled entropy which is the criterion of bipartite entanglement is a linear function of the squeezing ratio η^2 and vice versa. The squeezing ratio satisfies the inequalities $0 \le \eta^2 \le 1$; thus, the scaled entropy changes in the range $1 \ge E_S \ge 0$. Moreover, smaller values of η^2 correspond to deeper bipartite entanglement. We have plotted scaled entropy as a function of time in Figure 3. We like to emphasize that E_S is not a simple or a monotonic function of either squeezing parameters ξ_K^2 and ξ_W^2; thus, we can not relate squeezing and bipartite phenomena in a simple manner and that was the reason for introducing the squeezing ratio, in the first place.

We now embark upon studying the pairwise entanglement of the system. First we calculate the reduced initial density matrix ρ_{ij} at $t=0$. In fact, due to the exchange symmetry, it is independent of i and j; thus we drop the indices and call it $\rho(0)$ for simplicity. It has only the nonzero element $\rho(0)_{11} = 1$. The nonzero matrix elements of the dynamically generated time dependent density matrix are given by [34]

$$\rho_{11} = (\cos(T))^2(2+\cos(2T))/3\ ,$$

$$\rho_{44} = (\sin(T))^2(2-\cos(2T))/3\ ,$$

$$\rho_{14} = \rho_{41} = (\sin(2T))/2\sqrt{3}\ ,$$

$$\rho_{22} = \rho_{23} = \rho_{32} = \rho_{33} = (\sin(2T))^2/6\ . \tag{33}$$

We also calculate the nonzero matrix elements of the operator $(\sigma_{iy} \otimes \sigma_{jy})$; we find

$$(\sigma_y \otimes \sigma_y)_{23} = (\sigma_y \otimes \sigma_y)_{32} = 1\ ,$$

$$(\sigma_y \otimes \sigma_y)_{14} = (\sigma_y \otimes \sigma_y)_{41} = -1\ . \tag{34}$$

Using (33) and (34) in (6), we find the nonzero matrix elements of the operator P_{ij}, which we simply call P, as follows

$$P_{11} = [13-\cos(4T)][\sin(2T)]^2/72\ ,$$

$$P_{14} = 2[\cos(T)]^3[\sin(T)][2+\cos(2T)]/3\sqrt{3}\ ,$$

$$P_{41} = 2[\cos(T)][\sin(T)]^3[2-\cos(2T)]/3\sqrt{3}\ ,$$

$$P_{22} = P_{23} = P_{32} = P_{33} = [\sin(2T)]^4/18\ . \tag{35}$$

The square root of the eigenvalues of this matrix in the descending order are found to be

$$\sqrt{\lambda_1} = |\sin(2T)|[1/2\sqrt{3} + \sqrt{4-(\cos(2T))^2}/6]\ ,$$

$$\sqrt{\lambda_2} = |\sin(2T)|[(\sqrt{4-(\cos(2T))^2}/6) - 1/2\sqrt{3}]\ ,$$

$$\sqrt{\lambda_3} = (\sin(2T))^2/3\ ,$$

$$\sqrt{\lambda_4} = 0\ . \tag{36}$$

Finally, application of (36) in (5), gives us the scaled concurrence $C_S = 3C$ as a function of the scaled time T as follows

$$C_S = \sqrt{3}|\sin(2T)| - (\sin(2T))^2\ . \tag{37}$$

Eliminating time between (37) and (26) we find the following linear relation between C_S and ξ_K^2

$$C_S = 3C = [1-\xi_K^2]\ . \tag{38}$$

We have plotted the function C_S in Figure 3. We observe that, if $\xi_K^2 = 1$ we have $C_S = 0$; that is if the system is not squeezed it is not pairwise entangled either. However, if $\xi_K^2 < 1$, the system is squeezed according to Kitagawa's criterion, then we have $C_S > 0$ and the system is also pairwise entangled simultaneously and vice versa.

We may also write

$$\xi_W{}^2 = \frac{1 - C_S}{1 - E_S}. \quad (39)$$

We have plotted the three functions C_S, E_S, ξ_K^2 and $\xi_W{}^2$ in Figure 3. For $\xi_W{}^2 = 1$ the system is not squeezed and $C_S = E_S = 0$, that is we do not have entanglement either. But, it is squeezed for $\xi_W{}^2 < 1$, which requires $C_S > E_S$; this may be considered as a criterion of the existence of squeezing in this system and vice versa. Similar arguments may be applied to N-qubit systems; we have also plotted the squeezing parameters $\xi_W{}^2$ and $\xi_K{}^2$ and the entanglement parameters C_S and E_S, for a 9-qubit system in Figure 4 as an example.

We now eliminate time between (31) and (37) to obtain the following relation between C_S and E_S which is only applicable to our 4-qubit system

$$C_S = \sqrt{3E_S} - E_S. \quad (40)$$

We have plotted C_S as a function of E_S in Figure 2. Barring one maximum point at $E_S = 0.750$, it is a monotonic function of E_S; increases for the range $E_S = (0, 0.75)$, while decreases for the range $E_S = (0.75, 1)$.

Results and Discussion

We considered multi-qubit cluster systems, initially in coherent states, and studied their time evolution via the two-axis countertwisting Hamiltonian. Our procedure was demonstrated for our prototype 4-qubit system in detail. It was observed that the average spin direction remains along the initial one, but the quasi-probability distribution for spin direction becomes asymmetrical about the z-axis, in contrast to the initial symmetrical one. We showed that the parameters for K-squeezing (defined by Kitagawa et al) and W-squeezing (defined by Wineland et al) are periodic functions of time. Barring some separate instances of time, the system was found to be always K-squeezed, but only W-squeezed in alternate time intervals. It was also noted that if the system is W-squeezed it will be K-squeezed also, but the reverse is not necessarily true.

We observed that the scaled entropy, which is the criterion of bipartite entanglement, is a linear function of the squeezing ratio η^2 and vice versa. η^2 Satisfies the inequalities $0 \le \eta^2 \le 1$; thus, the scaled entropy changes in the range $1 \ge E_S \ge 0$. Moreover, smaller values of η^2 correspond to deeper bipartite entanglement.

We also noted that pairwise entanglement is a linear function of K-squeezing parameter and the system is K-squeezed if it is pairwise entangled and vice versa. Finally, we observed that for $\xi_W{}^2 < 1$ (W-squeezing), the inequality $C_S > E_S$ is satisfied and vice versa; thus, the latter inequality may be considered as the criterion of the existence of W-squeezing and vice versa.

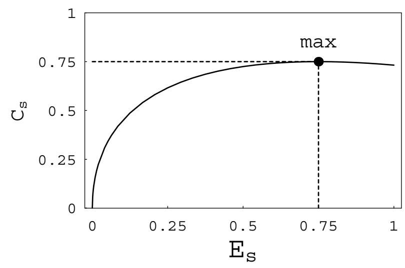


Figure 2. C_S versus E_S for N=4.

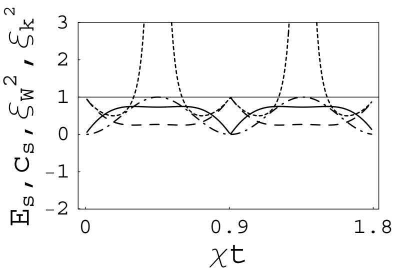


Figure 3. Plots of $\xi_W{}^2$ (dotted line), $\xi_K{}^2$ (dashed line), C_S (solid line) and E_S (dashed-dotted line) versus time for N=4.

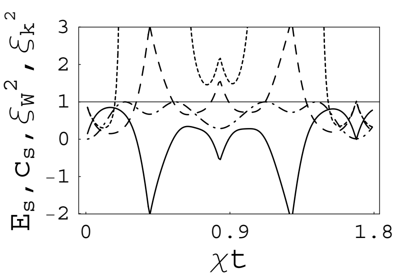


Figure 4. Plots of $\xi_W{}^2$ (dotted line), $\xi_K{}^2$ (dashed line), C_S (solid line) and E_S (dashed-dotted line) versus time for N=9.

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