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Umbilicity of (Space-Like) Submanifolds of Pseudo-Riemannian Space Forms

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Abstract

We study $(\nu-)$ umbilic (space-like) submanifolds of pseudo-Riemannian space forms, then define totally semi-umbilic space-like submanifold of pseudo Euclidean space and relate this notion to umbilicity. Finally we give characterization of total semi-umbilicity for space-like submanifolds contained in pseudo sphere or pseudo hyperbolic space or the light cone.A pseudo-Riemannian submanifold M in $\overline{M}$ (a pseudo Riemannian manifold) is called ν-umbilic if the shape operator of M along $\nu \in \mathrm{X}^{\perp}(M)$ is $S = \lambda\, id_{TM}$ for some $\lambda \in C^{\infty}(M)$. A totally semi-umbilical space-like submanifold of a pseudo Euclidean space is an space-like submanifold for which the curvature ellipse degenerates into a segment at every point except perhaps at isolated points. One of our main results says that if M^n is an space-like (immersed) submanifold of $\mathrm{R}_p^{n+m}, n \geq 2$, then M is totally semi-umbilical and C_x^{Π}, B_x^{Π} (some special normal vectors) are in the same direction for every $\Pi \in T_x M$ and all (except isolated) points of M, if and only if there exist linearly independent normal fields $\nu_{n+1},...,\nu_{n+m-1}$ locally defined at every non-umbilical point of M, such that M is ν_α-umbilical, $n+1 \leq \alpha \leq n+m-1$.

Keywords: $(\nu-)$umbilicity; Semi-umbilic submanifolds; Space form

Introduction

A pseudo-Riemannian submanifold M in $\overline{M}$ (a pseduo-Riemannian manifold) is called ν-umbilic if the shape operator of M along $\nu \in \mathrm{X}^{\perp}(M)$ is $S = \lambda\, id_{TM}$ for some $\lambda \in C^{\infty}(M)$. M is totally umbilic if all the shape operators of M are in this form. The study of conditions that hypersurfaces in Lorentzian space forms to be totally umbilic have been studied by many authors. For instance it is known that the only constant mean curvature compact space-like hypersurfaces in S_1^n are the umbilical ones (see [1] or [6]). It is also known that an (n-2)-dimensional submanifold of R^n is contained in S^{n-1} if and only if it is ν-umbilic where ν is the

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restriction of the position vector field. In [5] Izumiya et al consider the same problem for space-like $(n-2)$-submanifolds in Minkowski n-space. They show that umbilicity with respect to a normal parallel vector field ν implies that the submanifold lies in H^{n-1}, S^{n-1} or $(n-1)$-dimensional light cone, if respectively ν is time-like, space-like or light-like.

Izumya et al recently in [4] introduced the curvature ellipses for space-like surfaces in R_1^4, they use in [5] that setting to analyse the total semi-umbilicity of such surfaces. Totally semi-umbilical surfaces are those for which the curvature ellipse degenerates into a segment at every point except perhaps at isolated ones (umbilics) at which it becomes a point. Totally umbilic surfaces are a degenerate case for which the curvature ellipse degenerates everywhere into a point. They prove that as in the Euclidean case, total semi-umbilicity is equivalent to umbilicity with respect to some normal field ν. In this paper we define the notion of semi-umbilicity for n-dimensional space-like (immersed) submanifolds of R_p^{n+m}, $m \ge 1$ and generalize the results of [5] for these submanifolds. Our main results are stated in numbers 1 to 8.

Materials and Methods

Let R_p^n be the n-dimensional vector space R^n with the scalar product defined by

$$< x , y >= -\sum_{i=1}^{p} x_i y_i + \sum_{j>p}^{n} x_j y_j$$

$$\forall x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in \mathrm{R}^n, \quad 0 \le p < n$$

Then R_p^n is a pseudo-Riemannian manifold of signature p, called the pseudo Euclidean space. We define the pseudo sphere, pseudo hyperbolic space and the light cone respectively as follows,

$$S_p^n(a,r) = \{x \in \mathrm{R}_p^{n+1} \,|< x - a, x - a >= r^2, \ \textit{for some } a \in \mathrm{R}_p^{n+1} \textit{ and some } 0 \ne r \in \mathrm{R}\}$$

$$H_p^n(a,r) = \{x \in \mathrm{R}_{p+1}^{n+1} \,|< x - a, x - a >= -r^2, \ \textit{for some } a \in \mathrm{R}_p^{n+1} \textit{ and some } 0 \ne r \in \mathrm{R}\}$$

$$LC_a = \{x \in \mathrm{R}_p^{n+1} \,|< x - a, x - a >= 0, \ \textit{for some } a \in \mathrm{R}_p^{n+1}\}$$

By the simply connected space form $\bar{M}_p^n(c)$ we mean R_p^n, or $S_p^n = S_p^n(0,1)$, $p < n-1$ or $H_p^n = H_p^n(0,1)$, $p > 0$ according to the case $c = 0$, $c = 1$, or $c = -1$, respectively.

Let $M_p^n \subset \mathrm{R}_q^{n+m}$, $n \ge 2$, $m \ge 1$, $0 \le p \le q \le m$ be a pseudo-Riemannian submanifold. Let D, $\tilde{D}$ be the Levi-Civita connections of M and R_q^{n+m} respectively and $D^{\perp}$ be the induced normal connection on M. If $X, Y \in \mathrm{X}(M)$ =the space of tangent vector fields on M, and $\nu \in \mathrm{X}^{\perp}(M)$ =the space of normal vector fields on M, then $(\tilde{D}_Y X)^{\perp} = II(X,Y)$ is the 2nd fundamental form of M, and $S_\nu X = -(\tilde{D}_X \nu)^{\mathrm{T}}$ is the shape operator of M along ν. We have the relation $< II(X,Y), \nu >=< S_\nu X, Y >$. S_ν induces a self adjoint linear map on $T_x M$, $\forall x \in M$.

When M is space-like the distribution defined by the eigenspaces of S_ν is called the ν-principal configuration of M.

Results

Izumiya et al. in lemma 4.1 of [5] proved that a surface in H^{n-1} or S^{n-1} or the light cone is ν-umbilic. The following lemma is a full generalization of it.

Lemma 1. Let M_p^n be an n-dimensional (immersed) pseudo-Riemannian submanifold of $H_q^{n+m}(a,r)$, $S_q^{n+m}(a,r)$ or LC_a for some $a \in \mathrm{R}_{q'}^{n+m+1}$, $1 \le m$, $0 \le p \le q$, $q < n+m$, $q' = q$ or $q+1$. Then M is ν-umbilic, where ν is the position vector field at each point of M.

Proof. We know that the restriction of the position vector field $P \in \mathrm{X}(\mathrm{R}_{q'}^{n+m+1})$ to either of these spaces is normal to them, especially its restriction to M, $\nu = P|_M$ is orthogonal to M. Since the covariant derivative of the position vector field is the identity, we get that $< S_\nu X, Y >= -< D_X^{\mathrm{T}} \nu, Y >= -< X, Y >$ $\forall X, Y \in \mathrm{X}(M)$, so $S_\nu = -I$.

The following lemma generalizes lemma 4.2 of [5] with a much simpler proof.

Lemma 2. *Suppose that M_p^n is a pseudo-Riemannian submanifold of $\bar{M}_q^{n+m}(c)$, $p \le q$, such that M is ν-*

umbilic for some normal field ν on M. Let λ be the eigen function of the shape operator S_ν, then we have the following.

a) *If ν is normal parallel, then λ is constant.*

b) *Suppose that ν is normal parallel (i.e. $D_X^\perp \nu = 0$, $\forall X \in \mathfrak{X}(M)$), then $\lambda = 0$ if and only if ν is parallel.*

Proof. a) We follow the proof of lemma 4.35 of [7]. If $u \in T_x M$, x is an arbitrary point of M, choose $U, W \in \mathfrak{X}(M)$, so that $U(x) = u$, $W(x)$ are independent. Since $S_\nu = \lambda I$ we have (write S for S_ν)

$$(D_U S)(W) = D_U(SW) - S(D_U W)$$
$$= D_U(\lambda W) - \lambda D_U W = (U.\lambda) W.$$

We show that

$$(D_U S)(W) = (D_W S)(U), \ \forall U, W \in \mathfrak{X}(M)$$

so we get that

$$(U.\lambda) W = (W.\lambda) U.$$

In particular at the point x, $(U.\lambda) W(x) = (W.\lambda) U(x)$, hence $(U.\lambda)(x) = 0$, since $U(x) = u$, $W(x)$ are independent. Now we prove the claim.

By corollary 4.34 of [7] we have that $(\tilde{D}_U II)(W, X) = (\tilde{D}_W II)(U, X)$, $\forall U, W, X \in \mathfrak{X}(M)$. where

$$(\tilde{D}_U II)(W, X) = D_U^\perp(II(W, X))$$
$$- II(D_U W, X) - II(W, D_U X).$$

So

$$< (\tilde{D}_U II)(W, X), \nu > = U. < II(W, X),$$
$$\nu > - < S(D_U W), X > - < SW, D_U X > =$$
$$< (\tilde{D}_W II)(V, X), \nu > = W. < II(U, X),$$
$$\nu > - < S(D_W U), X > - < SU, D_W X >$$
$$\Rightarrow U. < SW, X > - < S(D_U W), X > - < SW, D_U X >$$
$$= W. < SU, X > - < S(D_W U), X > - < SU, D_W X >$$
$$\Rightarrow < D_U(SW), X > - < S(D_U W), X > = < D_W(SU),$$
$$X > - < S(D_W U), X >$$
$$\Rightarrow < (D_U S) W, X > = < (D_W S) U, X >.$$

Hence $(D_U S) W = (D_W S) U$.

(b) We have $\lambda = < SX, X >$ $\forall X \in \mathfrak{X}(M)$, for $< X, X > = +1$ so

$$\lambda = < SX, X > = < II(X, X), \nu > = < D_X^\perp X,$$
$$\nu > = X. < X, \nu > - < X, (\tilde{D}_X \nu)^{\mathrm{T}} >,$$

but $X. < X, \nu > = 0$, since $< X, \nu > = 0$. Hence, if ν is parallel, i.e. $\tilde{D}_X \nu = 0$, then $\lambda = 0$.

Conversely if $\lambda = 0$ then $S_\nu X = -(\tilde{D}_X \nu)^{\mathrm{T}} = 0$, since ν is normal parallel,i.e. $D_X^\perp \nu = (\tilde{D}_X \nu)^\perp = 0$ we get that $\tilde{D}_X \nu = 0$.

Remark: In case (b), $\bar{M}_q^{n+m}$ can be replaced by any $(n+m)$-dimensional pseudo- Riemannian manifold of index q without any restriction on its curvature.

Theorem 3. Let $M = M_p^n$ be a pseudo-Riemannian submanifold of $\mathbb{R}_q^{n+m}, m \geq 2$, $p < q$. If M is ν-umbilic for some normal parallel time-like (respectively space-like or light-like) vector field $\nu \in \mathfrak{X}^\perp(M)$, then either M is contained in some $H_{q-1}^{n+m-1}(a, r)$ (respectively, some $S_q^{n+m-1}(a, r)$, or the light cone LC_a) if $S_\nu \neq 0$ or M lies in a hyperplane of index $q-1$ (respectively, of index q, or degenerate) if $S_\nu = 0$.

Proof. See the proof of Theorem 4.3 of [5].

Remark: 1) Similarly one can prove that if $M_p^n \subset S_q^{n+m}$, $0 < p < q$, then M_p^n is ν-umbilic, for some normal parallel time-like vector field $\nu \in \mathfrak{X}^\perp(M) \subset \mathfrak{X}(S_q^{n+m})$ if and only if M lies in the intersection of S_q^{n+m} with an (affine) hyperplane $P \subset \mathbb{R}_q^{n+m+1}$ of index $q-1$, so $M \subset S_{q-1}^{n+m-1}(a, r)$ for some $0 \neq a \in \mathbb{R}_q^{n+m+1}$, $1 \neq r \in \mathbb{R}_+$ if $S_\nu \neq 0$ and $M \subset S_{q-1}^{n+m-1}(0,1)$ if $S_\nu = 0$.

If ν is space-like or null, analogous statements hold. The case $M_p^n \subset H_q^{n+m}$ is similar (see Proposition 3.2 of [2] for the proof of positive definite case, the proof for the indefinite metric is a slightly modified version of it).

2) If $M_p^n \subset \mathbb{R}_q^{n+m}$ is ν_i-umbilic with respect to

$\nu_1,\ldots,\nu_r$ ($\{\nu_i\}$ are everywhere linearly independent and normal parallel) then M lies in the intersection of r (pseudo) spheres or r_λ (pseudo) spheres and r_μ (pseudo) hyperbolic spaces ($r_\lambda + r_\mu = r$) or r (pseudo) hyperbolic spaces, and the intersection is an ($n+m-r$)-dimensional (pseudo) sphere or (pseudo) hyperbolic space.

From Lemma 1 and Theorem 3 we get the following corollary.

Corollary 4. *Let M_p^n be a pseudo-Riemannian (immersed) submanifold of $\mathbb{R}_q^{n+m}$, $p<q$. Then M is ν-umbilic for some normal parallel time-like (respectively space-like or light-like) vector field ν if and only if either M is contained in some $H_{q-1}^{n+m-1}(a,r)$ (respectively $S_q^{n+m-1}(a,r)$ or the light cone LC_a) if $S_\nu \neq 0$ or M lies in a hyperplane of index $q-1$ (respectively of index q or degenerate) provided that $S_\nu = 0$.*

Remark: Analogous result can be stated for a submanifold M_p^n of S_q^{n+m} or H_p^{n+m}.

Semi-Umbilicity

Given an immersed space-like submanifold $M^n \subset \mathbb{R}_p^{n+m}$, $1 \le p \le m$ take a (local) tangent frame field $\{e_i\}_{i=1}^n$ and a local normal frame field $\{e_\alpha\}_{\alpha=n+1}^{n+m}$ on a neighborhood U of $x \in M$.

Take a two dimensional subspace $\Pi = span_{\mathbb{R}}$ $\{e_k(x), e_l(x)\} \subset T_xM$, ($1 \le k, l \le n$), take $u \in \Pi$, $\|u\|=1$, u is represented by $\theta \in S^1 \subset \Pi$. Let $P \subset \Pi$ be a small enough open set in Π about O_x, then $exp_x(P)$ is a 2-dimensional submanifold of M (whose Gaussian curvature at x is $K_M(\Pi)$, where K_M is the sectional curvature of M). Denote by γ_θ the curve obtained by intersecting $exp_x(P)$ with the plane defined by the direct sum of the normal subspace N_xM and the straight line in the tangent direction represented by u.

Such a curve is called the normal section of M with respect to Π and ν. Consider a parameterization of γ_θ by its arc length s such that $\gamma_\theta(0) = x$. The curvature vector $\eta(\theta)$ of γ_θ at x, i.e. the second derivative of γ_θ with respect to s at 0 lies in N_xM. Varying θ from 0 to 2π, the vector $\eta(\theta)$ describes an ellipse in N_xM, called the curvature ellipse of M with respect to Π at x. It can be seen that this ellipse is the image of the map $\eta : S^1 \subset \Pi \subset T_xM \to N_xM$ given by

$$\eta(\theta) = -\sum_{\alpha=n+1}^{n+p} [\cos\theta \ \sin\theta] \begin{bmatrix} s_\alpha^{kk} & s_\alpha^{kl} \\ s_\alpha^{kl} & s_\alpha^{ll} \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} e_\alpha$$

$$+ \sum_{\beta>n+p} [\cos\theta \ \sin\theta] \begin{bmatrix} s_\beta^{kk} & s_\beta^{kl} \\ s_\beta^{kl} & s_\beta^{ll} \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} e_\beta$$

Where $S_\alpha = (s_\alpha^{kl})$ is the shape operator of M along e_α and $s_\alpha^{kl} = \langle s_\alpha e_k, e_l \rangle$.

So $\eta(\theta) = H_x^\Pi + B_x^\Pi \cos 2\theta + C_x^\Pi \sin 2\theta$

With

$$H_x^\Pi = \frac{1}{2}\sum_{\alpha=n+1}^{n+p} (s_\alpha^{kk} + s_\alpha^{ll})\, e_\alpha - \frac{1}{2}\sum_{\beta>n+p}^{n+m} (s_\beta^{kk} - s_\beta^{ll})\, e_\beta$$

$$B_x^\Pi = \frac{1}{2}\sum_{\alpha=n+1}^{n+p} (s_\alpha^{kk} - s_\alpha^{ll})\, e_\alpha - \frac{1}{2}\sum_{\beta>n+p}^{n+m} (s_\beta^{kk} - s_\beta^{ll})\, e_\beta$$

$$C_x^\Pi = \sum\nolimits_{\alpha=n+1}^{n+p} s_\alpha^{kl}\, e_\alpha - \sum\nolimits_{\beta>n+p}^{n+m} s_\beta^{kl}\, e_\beta$$

This ellipse contained in the indefinite m-space N_xM may degenerate into a segment for certain points of M and certain $\Pi \subset T_xM$. One can see that in this case, H_x^Π, B_x^Π and C_x^Π are in the direction of the segment, then we call M semi-umbilic at x with respect to Π. If M is semi-umbilic with respect to all $\Pi \subset T_xM$, we say that M is semi-umbilic at x. There may also exist points x at which the curvature ellipse becomes a point for each $\Pi \subset T_xM$. These are degenerate semi-umbilics called umbilic. If all (except isolated) points of $M^n \subset \mathbb{R}_p^{n+m}$ are semi-umbilic, M is said to be totally semi-umbilical.

Proposition 5. *Let M^n be a space-like (immersed) submanifold of $\mathbb{R}_p^{n+m}$ and let $\nu \in \mathfrak{X}^\perp(M) - \{0\}$. A point $x \in M$ is ν-umbilic if and only if $\nu(x)$ is orthogonal to the vectors B_x^Π, C_x^Π defined above for all $\Pi \subset T_xM$.*

Proof. We can write $\nu = \sum_{\alpha=n+1}^{n+m} \nu_\alpha e_\alpha$, then

$$\langle \nu, C_x^\Pi \rangle = \sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha \nu_\alpha s_\alpha^{kl} - \sum_{\beta>n+1}^{n+m} \varepsilon_\beta \nu_\beta s_\beta^{kl},$$

$$\varepsilon_\lambda = \langle e_\lambda, e_\lambda \rangle \ , \ \lambda \geq n+1$$

$$\langle \nu, B_x^\Pi \rangle = \frac{1}{2}\sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha \nu_\alpha (s_\alpha^{kk} - s_\alpha^{ll}) - \frac{1}{2}\sum_{\beta>n+p}^{n+m} \varepsilon_\beta \nu_\beta (s_\beta^{kk} - s_\beta^{ll}).$$

x is a ν-umbilic point if and only if $S_\nu |_\Pi = \lambda id_\Pi$ for all 2-dimensional subspace $\Pi \subset T_x M$, i.e. $s_\nu^{kl} = 0$, $1 \leq k,l \leq n$, $k \neq l$ and $s_\nu^{kk} = s_\nu^{ll} \quad \forall\ 1 \leq k,l \leq n$, but $s_\nu^{kl} = \sum_{\alpha=n+1}^{n+m} \nu_\alpha s_\alpha^{kl}$, hence the result follows.

Using Proposition 5 we are able to express total semi-umbilicity in terms of ν-umbilicity as follows.

Theorem 6. *Let M^n be a space-like (immersed) submanifold of $\mathbb{R}_p^{n+m}$, $n \geq 2$, then M is total semi-umbilical and C_x^Π, B_x^Π are in the same direction for every $\Pi \subset T_x M$ and all (except isolated) points of M if and only if there exist linearly independent normal fields $\nu_{n+1}, \ldots, \nu_{n+m-1}$ locally defined at every non-umbilical point of M, such that M is ν_α-umbilical, $n+1 \leq \alpha \leq n+m-1$.*

Proof. Suppose that there exist ν_α as defined such that M is ν_α-umbilic, following the proof of Theorem 5.3 of [5], by using Proposition 5 we get that for all $\Pi \subset T_x M$, B_x^Π and C_x^Π are in the same direction, since $span_{\mathbb{R}}\{\nu_{n+1}(x), \ldots, \nu_{n+m-1}(x)\} \subset N_x M$ is $(m-1)$-dimensional and $\nu_\alpha \perp B_x^\Pi, C_x^\Pi$, $n+1 \leq \alpha \leq n+m-1$. Since B_x^Π, C_x^Π generate the curvature ellipse at $x \in M$ with respect to Π, this ellipse must degenerate for all $\Pi \subset T_x M$. Conversely suppose that M is totally semi-umbilic and B_x^Π, C_x^Π are parallel for all $\Pi \subset T_x M$, then $(B_x^\Pi)^\perp$ is $(m-1)$-dimensional, so we can choose $(m-1)$ locally defined normal vector fields ν_α, such that $\nu_\alpha(x)$ is orthogonal to $B_x^\Pi \parallel C_x^\Pi$ and $\{\nu_\alpha(x) \mid n+1 \leq \alpha \leq n+m-1\}$ is linearly independent. By Proposition 5 M is ν_α-umbilic.

As a corollary of Lemma 1 and Theorem 6 we get

Corollary 7. *Given a space-like n-dimensional submanifold $M^n \subset \mathbb{R}_p^{n+m}$, then $M \subset H_{p-1}^{n+m-1}(a,r)$ (respectively $M \subset S_p^{n+m-1}(a,r)$ or $M \subset LC_a^{n+m-1}$) if and only if every point of M is umbilical with respect to some time-like (respectively space-like or light-like) normal vector or semi-umbilical and the curvature ellipses define a parallel normal field on M.*

A modified version of the proof of corollary 5.7 of [5] proves that.

Corollary 8. *Suppose that $M^n \subset \mathbb{R}_p^{n+m}$, $n \geq 2$ is a non-totally umbilical space-like submanifold. Then M is totally semi-umbilical and the curvature ellipses are parallel if and only if it has a unique principal configuration, i.e. for any two non umbilic normal fields ν, η; S_ν, S_η have the same eigenspaces (and eigenvectors).*

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