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Ideal Amenability of Banach Algebras and Some Hereditary Properties

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Abstract

Let A be a Banach algebra. A is called ideally amenable if for every closed ideal I of A, the first cohomology group of A with coefficients in I* is trivial. We investigate the closed ideals I for which H1 (A,I*)={0}, whenever A is weakly amenable or a biflat Banach algebra. Also we give some hereditary properties of ideal amenability.

Keywords: Derivation; Weakly amenable; Biflat; Ideally amenable

Introduction

For a Banach algebra A let X be a Banach A-bimodule. Then X^*, the dual space of X, is also a Banach A-bimodule with module multiplications defined by

$$\langle x, a.x^* \rangle = \langle x.a, x^* \rangle,\ \langle x, x^*.a \rangle = \langle a.x, x^* \rangle$$
$$\left(a \in A, x \in X, x^* \in X^*\right).$$

In particular, I and I^* are Banach A-bimodule for every closed ideal I of A. A derivation from A into X is a continuous linear operator D such that

$$D(ab) = a.D(b) + D(a)b \qquad (a,b \in A).$$

We define $\delta_x(a) = a.x - x.a$ for each $x \in X$ and $a \in A$ is a derivation from A into X, which is called an inner derivation. A Banach algebra A is ***amenable*** if every derivation from A into every dual A-bimodule X^* is inner. This definition was introduced by B.E.Johnson in [15], [20] and [14]. A Banach algebra A is ***weakly amenable*** if every derivation from A into A^* is inner. Bade, Curtis and Dales have introduced the concept of weak amenability for commutative Banach algebras [2] (see [17,6,9,10,11]). Let $n \in \mathbb{N}$, a Banach algebra A is called n-weakly amenable if $H^1\left(A, A^{(n)}\right) = \{0\}$, where $A^{(n)}$ is the n-th dual of A [5], [16].

Let $G = SL(2,\mathbb{R})$, the set of elements in $\mathbb{M}_2(\mathbb{R})$ with determinant one, also let $A = L^1(G)$ and I be the augmentation ideal of A, then theorem 5.2 of [18] implies that $H^1(A, I^*) \neq \{0\}$. On the other hand A is weakly amenable. This example guides us to the following definitions.

Let *A* be a Banach algebra and let *I* be a closed ideal of *A*, then *A* is said to be ***I-weakly amenable*** if every derivation from *A* into I^* is inner, in other words $H^1\left(A, I^*\right) = \{0\}$. We call A ***ideally amenable*** if *A* is *I*-

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weakly amenable for every closed ideal I of A [8,7]. Let $n \in \mathbb{N}$, a Banach algebra A is called n-ideally amenable if for every closed ideal I in A, $H^1(A, I^{(n)}) = \{0\}$.

Obviously, an ideally amenable Banach algebra is weakly amenable. Since every closed ideal of A is a Banach A-bimodule, an amenable Banach algebra is ideally amenable. There are examples of Banach algebras show that ideal amenability is not equivalent to the weak amenability or amenability. In the following we give some of them.

Example 1. Let A be the unitization of the augmentation ideal of $L^1(SL(2, \mathbb{R}))$. Then A is weakly amenable but is not ideally amenable [8].

Example 2. Let $A = L^1(SL(2, \mathbb{R}))$. Then A is weakly amenable which is not ideally amenable [18, Theorem 5.2].

Example 3. Let A be a non-nuclear C^*-algebra. Then A is ideally amenable which is not amenable [8].

Example 4. Let A be a commutative weakly amenable Banach algebra which is not amenable. Then, for each $n \in \mathbb{N}$, A is n-ideally amenable [8].

Let A be a Banach algebra, X a Banach A-bimodule and let Y be a closed A-submodule of X. We say that the short exact sequence $\{0\} \to Y \xrightarrow{i} X \xrightarrow{\pi} \frac{X}{Y} \to \{0\}$ of A-bimodules splits if π has a bounded right inverse which is also an A-bimodule homomorphism. The following theorem is well known.

Theorem 1.1. Let A be a Banach algebra, X a Banach A-bimodule and let Y be a closed A-submodule of X. Then the following conditions are equivalent.

i) The short exact sequence $\{0\} \to Y \xrightarrow{i} X \xrightarrow{\pi} \frac{X}{Y} \to \{0\}$ splits.

ii) i has a bounded left inverse which is also an A-bimodule homomorphism.

iii) There exists a continuous projection of X onto Y which is also an A-bimodule homomorphism.

Proof. A more general case, for example, can be found in [3, page 56].

Let A be a Banach algebra. Then, the projective tensor product of A is denoted by $A \hat{\otimes} A$. This space is a Banach A-bimodule with module multiplications defined by:

$$a.(b \otimes c) = ab \otimes c, \; (b \otimes c).a = b \otimes ca \; (a, b, c \in A)$$

The corresponding diagonal operator $\Delta : A \hat{\otimes} A \to A$ is defined by $a \otimes b \mapsto ab$, and extended uniquely by linearity. It is clear that Δ is a Banach A-bimodule homomorphism.

Let E be a Banach space and (E) be the space of finite rank operators on E. We say that E has ***the approximation property*** if there is a net $(S_\alpha)_\alpha$ in (E) such that $(S_\alpha) \to id_E$ uniformly on compact subsets of E.

Closed Ideals of Weakly Amnable Banach Algebras

In this section, we find some closed ideals of a weakly amenable Banach algebra A for which $H^1(A, I^*)$ is trivial. Also if we denote the linear span of the set $\{ab : a, b \in A\}$ by A^2, in Theorem 2.14, it is proved that for a closed ideal I satisfies $A^2 \subseteq I$ and $H^1(A, I^*) = \{0\}$, then A^2 is dense in I. This is a generalization of Gronbæk's theorem [3, Theorem 2.8.63].

Let X be a Banach A-bimodule and let Y be a closed A-submodule of X. We give some conditions that $H^1(A, X^*) = \{0\}$ implies $H^1(A, Y^*) = \{0\}$.

Theorem 2.1. Let A be a Banach algebra, X a Banach A-bimodule and let Y be a closed A-submodule of X. If $H^1(A, X^*) = \{0\}$ and the exact sequence

$$\{0\} \to Y^\perp \xrightarrow{i} X^* \xrightarrow{\pi} \frac{X^*}{Y^\perp} \to \{0\} \qquad (1)$$

of Banach A-bimodules splits, then $H^1(A, Y^*) = \{0\}$.

Proof. Let $D : A \to Y^*$ be a derivation. Since the exact sequence (1) splits, π has a bounded right inverse, say ϕ, such that ϕ is also an A-bimodule homomorphism. In this case $\phi \circ D : A \to X^*$ is a derivation, so there exists $f \in X^*$ such that $\phi \circ D = \delta_f$. Thus, we have $\pi \circ \phi \circ D = \pi \circ \delta_f$. This shows that $d_{Y^*} \circ D = \delta_{\pi(f)}$ and therefore $D=\delta_{\pi(f)}$.

The following lemma is in the literature. We give its proof.

Lemma 2.2. The exact sequence (1) splits, if the following splits

$$\{0\}\to Y\overset{i}{\to}X\overset{\pi}{\to}\frac{X}{Y}\to\{0\} \qquad (2)$$

Proof. Since the exact sequence (1) splits, there exists a continuous projection P of X onto Y which is also an A-bimodule homomorphism. Let $Q=id_{X^*}-P^*$, then for each $y\in Y$ and $f\in X^*$ we have

$$\begin{aligned}\langle y,Q(f)\rangle &=\langle y,f\rangle-\langle y,P^*f\rangle\\ &=\langle y,f\rangle-\langle P(y),f\rangle\\ &=0.\end{aligned}$$

So $Q(X^*)\subseteq Y^{\perp}$. On the other hand for each $f\in X^*$ and $x\in X$ we have

$$\begin{aligned}\langle x,Q^2(f)\rangle &=\langle x,Q(f-P^*f)\rangle\\ &=\langle x,(f-P^*f-P^*(f-P^*f))\rangle\\ &=\langle x,f\rangle-\langle P(x),f\rangle-\langle P(x),f\rangle\\ &\quad-\langle P(x),f\rangle+\langle P(x),P^*f\rangle\\ &=\langle x,f\rangle-\langle P(x),f\rangle\\ &=\langle x,Q(f)\rangle\end{aligned}$$

Thus Q is a continuous projection of X^* onto $Y^{\perp}$. Also for $f\in X^*$, $x\in X$ and $a\in A$, we have

$$\begin{aligned}\langle x,Q(a.f)\rangle &=\langle x,a.f-P^*(a.f)\rangle\\ &=\langle x.a,f\rangle-\langle P(x),a.f\rangle\\ &=\langle x.a,f\rangle-\langle P(x).a,f\rangle\\ &=\langle x.a,f\rangle-\langle P(x.a),f\rangle\\ &=\langle x.a,Q(f)\rangle\\ &=\langle x,aQ(f)\rangle\end{aligned}$$

So Q is a left A-module homomorphism. Similarly Q is a right A-module homomorphism and this completes the proof.

Corollary 2.3. Let A, X and Y be as in Theorem 2.1. If the exact sequence (2) splits and

$H^1(A,X^*)=\{0\}$, then $H^1(A,Y^*)=\{0\}$.

Corollary 2.4. Let A be a Banach algebra and let $n>1$ be an element of $\mathbb{N}$. If $H^1(A,X^{(n+2)})=\{0\}$, then $H^1(A,X^{(n)})=\{0\}$.

Proof. Let $\wedge_{n-1}:X^{(n-1)}\to X^{(n+1)}$ be the canonical map. Then the exact sequence

$$\{0\}\to X^{(n-1)}\overset{\wedge_{n-1}}{\to}X^{(n+1)}\overset{\pi}{\to}\frac{X^{(n+1)}}{X^{(n-1)}}\to\{0\}$$

Splits, because the adjoint of $\wedge_{n-2},\wedge^*_{n-2}$: $X^{(n+1)}\to X^{(n-1)}$, is a left inverse of $\wedge_{n-1}$ which is also an A-bimodule homomorphism. Now use Corollary 2.3.

First, by Theorem 2.1 and Corollary 2.3, we have the following theorem.

Theorem 2.5. Assume that A is a weakly amenable Banach algebra. If one of the following conditions holds for each closed ideal I in A, then A is ideally amenable.

i) The exact sequence $\{0\}\to I^{\perp}\overset{i}{\to}A^*\overset{\pi}{\to}\frac{A^*}{I^{\perp}}\to\{0\}$, splits.

ii) The exact sequence $\{0\}\to I\overset{i}{\to}A\overset{\pi}{\to}\frac{A}{I}\to\{0\}$, splits.

Theorem 2.6. Let A be a Banach algebra and let I be a closed ideal of A with a bounded approximate identity. Then the following conditions are equivalent

i) I is weakly amenable;

ii) $H^1(A,I^*)=\{0\}$.

Proof. (ii)$\Rightarrow$(i). Let $H^1(A,I^*)=\{0\}$ and let $D:I\to I^*$ be a derivation. Since I is psudo unital A-bimodule, by Proposition 2.1.6. of [20], D has an extension $\bar{D}:A\to I^*$ such that $\bar{D}$ is also a derivation;

by hypothesis $\bar{D}$ and so D is inner. The converse is lemma 2.1 of [8].

Recall that, in a Banach algebra A, a net $(e_\alpha)_\alpha$ is quasi central in A if for each element $a \in A; \lim_\alpha (ae_\alpha - e_\alpha a) = 0$. Obviously, each approximate identity in A is quasi central in A.

Theorem 2.7. Let A be a weakly amenable Banach algebra and let I be a closed ideal of A with a bounded approximate identity which is quasi central in A. Then $H^1(A, I^*) = \{0\}$.

Proof. Let $(e_\alpha)_\alpha$ be a bounded approximate identity in I which is quasi central in A and let J be an ultrafilter on the index set of $(e_\alpha)_\alpha$ such that dominates the order filter. Define

$$P : A^* \to A^*, \phi \mapsto \omega^* - \lim_J (\phi - e_\alpha . \phi)$$

The above limit exists (see [3]). For every $\phi \in A^*$ and $a \in I$ we have

$$a, P\phi = \lim_J a, \phi - e_\alpha . \phi = \lim_J a - ae_\alpha, \phi = 0$$

Thus $PA^* \subseteq I^\perp$. Also for $\phi \in I^\perp$ and $a \in A$, we have

$$\langle a, P\phi \rangle = \lim_J (\langle a, \phi \rangle - \langle ae_\alpha, \phi \rangle) = \langle a, \phi \rangle$$

This shows that P is a projection of A^* onto $I^\perp$. On the other hand, for $a, b \in A$ and $\phi \in A^*$, we have

$$\begin{aligned} \langle b, P(a.\phi) \rangle &= \lim_J \langle b, a.\phi - e_\alpha .(a.\phi) \rangle \\ &= \lim_J \langle ba - be_\alpha a, \phi \rangle \\ &= \lim_J \langle ba - bae_\alpha, \phi \rangle \\ &= \lim_J \langle ba, \phi - e_\alpha . \phi \rangle \\ &= \langle ba, P(\phi) \rangle \\ &= \langle b, a.P(\phi) \rangle \end{aligned}$$

So P is a left A-module homomorphism. Similarly P is a right A-module homomorphism. Therefore the exact sequence $\{0\} \to I^\perp \xrightarrow{i} A^* \xrightarrow{\pi} \frac{A^*}{I^\perp} \to \{0\}$ splits and consequently $H^1(A, I^*) = \{0\}$.

Theorem 2.8. Let A be a weakly amenable Banach algebra and let I be a closed ideal of A with a bounded approximate identity. If one of the following conditions holds,

i) I is Arens regular.
ii) A is Arens regular.

Then $H^1(A, I^*) = \{0\}$.

Proof. The Arens regularity of A implies the Arens regularity of any closed ideal of A. so we consider the case(i). Let I be Arens regular and let $(e_\alpha)_\alpha$ be quasi central in A. Let $E \in I^{**}$ be a w*-accumulation point of $(e_\alpha)_\alpha$. Then E is the identity of I**. Also for each $a \in A$, we have $aE, Ea \in I^{**}$ and thus aE = E (aE) = (Ea)E = Ea. Therefore $\lim_\alpha (ae_\alpha - e_\alpha a) = 0$ for each $a \in A$, this proves the claim. Now, by the above theorem, $H^1(A, I^*) = \{0\}$.

Corollary 2.9. Let A be a weakly amenable Banach algebra such that each closed ideal of A has a bounded approximate identity which is quasi central in A. Then A is ideally amenable.

It seems that, for a Banach algebra, conditions in the above corollary is much more to exist, but fortunately C*-algberas provide the above conditions.

Corollary 2.10. Every C*-algebra is ideally amenable.

Proof. Let A be a C*-algebra, then A is weakly amenable [14]. Since every closed ideal of A is also a C*- algebra, it has a bounded approximate identity. On the other hand, every C*-algebra is Arens regular. Thus , by Theorem 2.8, for each closed ideal I of A , we have $H^1(A, I^*) = \{0\}$.

Theorem 2.11. Let A be a Banach algebra and let $A^2 \subseteq I$ for a closed ideal I of A. If A is I-weakly amenable, A^2 is dense in I.

Proof. Assume $\bar{A} \neq I$. Then there exists $0 \neq \varphi \in I^*$ such that φ is trivial on $\overline{A^2}$. Let $\Phi \in A^*$ be a Hahn-Banach extension of φ on A. Define $D : A \to I^*$ by $D(a) := \langle a, \Phi \rangle \varphi$. For $a, b \in A$, $D(ab) = ab, \Phi\varphi$. Also for $i \in I$ we have

$$\langle i, D(a)b\rangle = \langle i, (\langle a,\Phi\rangle\varphi)b\rangle$$
$$= \langle a,\Phi\rangle\langle bi,\varphi\rangle$$
$$= 0.$$

So D(a).b = 0. Similarly a.D(b) = 0.

Therefore D is a derivation from A into I^* and by hypothesis there exists $\psi \in I^*$ such that $D = \delta_\psi$. Also for every $i \in I$ we have

$$\langle i, \varphi^2\rangle = \langle i,\varphi\rangle\langle i,\Phi\rangle$$
$$= \langle i, D(i)\rangle$$
$$= \langle i, \delta_\psi(i)\rangle$$
$$= \langle i, i\psi - \psi i\rangle$$
$$= 0.$$

So $\varphi = 0$, which is a contradiction. Thus A^2 is dense in I.

Corollary 2.12. Let A be a Banach algebra and let M be a closed non- maximal modular ideal of A with codimension one. If $H^1(A, M^*) = \{0\}$, then A^2 is dense in M.

Proof. Since the codimension of M is one, there exists $a \in A$ such that $A = \mathbb{C}a + M$. We show that $a^2 \in M$. Assume that a^2 does not belong to M, so there exists $0 \neq \alpha$ and $m \in M$ such that $\alpha^2 = \alpha a + m$. Now let b be an arbitrary element of A, there exist β and $m' \in M$ such that $b = \beta a + m'$, so

$$b - b(\alpha^{-1}a) = (\beta a + m')$$
$$-(\beta a + m')(\alpha^{-1}a)$$
$$= m' - \beta\alpha^{-1}m - \alpha^{-1}m'a$$

Thus $b - b(\alpha^{-1}a)$ belongs to M for each b. This shows that $\alpha^{-1}a$ is a left modular identity for M. Similarly $\alpha^{-1}a$ is a right modular identity for M, so M is a maximal modular ideal which is a contraiction. Therefore a^2 belongs to M and consequently $A^2 \subseteq M$. Now, by the above theorem, A^2 is dense in M.

Closed Ideals of Biflat Banach Algebras

We say that a Banach algebra A is ***biprojctive*** if $\Delta : A \hat{\otimes} A \to A$ has a bounded right inverse which is an A-bimodule homomorphism. Also we say that a Banach algebra A is ***biflat*** if the bounded linear map $\Delta^* : A^* \to (A \hat{\otimes} A)^*$ has a bounded left inverse which is an A-bimodule homomorphism [20]. Obviously by taking adjoints, one sees that every biprojective Banach algebra is biflat. It is well known that every biflat Banach algebra is weakly amenable [3], and a Banach algebra is amenable if and only if it is biflat and has a bounded approximate identity [21].

Since there is no Hahn-Banach theorem for operators, there is none for bilinear continuous forms. In other words, let E and F be two Banach spaces with G as a subspace of E and let $\phi \in \mathfrak{BL}(G,F;\mathbb{C})$, where $\mathfrak{BL}(G,F;\mathbb{C})$ is the set of all bounded bilinear mappings from G×F into $\mathbb{C}$. In general case, there is no any extension of ϕ to a bilinear map $\tilde{\phi} \in \mathfrak{BL}(E,F;\mathbb{C})$. Since $\mathfrak{BL}(G,F;\mathbb{C}) \approx \mathfrak{L}(G,F^*)$, this situation is equivalent to say that each element $T \in \mathfrak{L}(G,F^*)$ doesn't have any extension to an element $\tilde{T} \in \mathfrak{L}(E,F^*)$ [4,1.5].

However, there is some conditions that Hahn-Banach theorem works for operators as well. Let $\pi(z;E,F)$ be the projective norm of the element $z \in F \hat{\otimes} F$ and G be a subspace of E. Then it is clear that $\pi(z;E,F) \leq \pi(z;G,F)$ for each element $z \in G \hat{\otimes} F$. If there exists $\lambda \geq 1$ such that $\pi(z;G,F) \leq \lambda\pi(z;E,F)$ for each element $z \in G \hat{\otimes} F$, then we say that $.\hat{\otimes} F$ respects G into $E \hat{\otimes} F$ isomorphically. For example, $.\hat{\otimes} F$ respects G into $E \hat{\otimes} F$ isomorphically if G is a complemented subspace of E [4, 3.9].

By the Hahn-Banach theorem, we can extend each element $T \in (G \hat{\otimes} F)^*$ to a continuous linear functional $\tilde{T}$ on $E \hat{\otimes} F$ provided that $.\hat{\otimes} F$ respects G into $E \hat{\otimes} F$ isomorphically [4].

Recall that an ideal I is left essential as a left Banach A-module if the linear span of $\{ai : a \in A, i \in I\}$ is dense in I.

Theorem 3.1. Let A be a biflat Banach algebra and let I be a closed ideal of A which is left essential. If $A \hat{\otimes}$. Respects I into $A \hat{\otimes} A$ isomorphically, then

$H^1\left(A,I^*\right)=\{0\}$.

Proof. Let $D:A\to I^*$ be a derivation. Since A is biflat, $\Delta^*:A^*\to\left(A\hat{\otimes}A\right)^*$ has a bounded left inverse ρ which is an A-bimodule homomorphism. Let $T:\mathfrak{L}\left(A,I^*\right)\to\left(A\hat{\otimes}I\right)^*, S\mapsto T_S$ be the isometric isomorphism which is defined by $\langle a\otimes i,T_S\rangle=\langle i,S(a)\rangle$. Let $\tilde{T}_D$ be a Hahn–Banach extension of T_D on $A\hat{\otimes}A$. We claim that $D=\delta_\phi$, where $\phi=\rho\left(\tilde{T}_D\right)|_I$.

First we show that for each $i\in I$ and $a\in A; i.\left(a\tilde{T}_D-\tilde{T}_D.a\right)=i.\Delta^*(\widetilde{Da})$, where $\widetilde{Da}$ is a Hahn-Banach extension of Da on A. Let $b,c\in A$ we have

$$\langle b\otimes c,i.\left(a\tilde{T}_D-\tilde{T}_D.a\right)\rangle=\langle b\otimes ci,a\tilde{T}_D-\tilde{T}_D.a\rangle$$

$$=\langle b\otimes cia-ab\otimes ci,\tilde{T}_D\rangle$$

$$=\langle b\otimes cia,T_D\rangle-\langle ab\otimes ci,T_D\rangle$$

$$=\langle cia,Db\rangle-\langle ci,D(ab)\rangle$$

$$=\langle ci,a.Db-D(ab)\rangle=\langle ci,Da.b\rangle$$

$$=\langle bci,Da\rangle=\langle bci,\widetilde{Da}\rangle$$

$$=\langle b\otimes ci,\Delta^*(\widetilde{Da})\rangle$$

$$=\langle b\otimes c,i.\Delta^*(\widetilde{Da})\rangle$$

Now, let $a,b\in A$ and $i\in I$. Then we have

$$\langle bi,\delta_\phi(a)\rangle=\langle bi,a.\phi-\phi.a\rangle$$

$$=\langle bi,a.\rho\left(\tilde{T}_D\right)|_I-\rho\left(\tilde{T}_D\right)|_I.a\rangle$$

$$=\langle bia-abi,\rho\left(\tilde{T}_D\right)\rangle$$

$$=\langle bi,a.\rho\left(\tilde{T}_D\right)-\rho\left(\tilde{T}_D.a\right)\rangle$$

$$=\langle b,\rho\left(i.\left(a\tilde{T}_D-\tilde{T}_D.a\right)\right)\rangle$$

$$=\langle b,\rho(i.\Delta^*\left(\widetilde{Da}\right))\rangle$$

$$=\langle bi,id_{A^*}(\widetilde{Da})\rangle$$

Since I is left essential and Da, $\delta_\phi(a)$ are both continuous linear functional on I, we have for each $a\in A$, $Da=\delta_\phi(a)$; thus $D=\delta_\phi$.

Corollary 3.2. Let A be a biflat Banach algebra with a left approximate identity. Then A is ideally amenable provided that $A\hat{\otimes}.$ respects all closed ideals into $A\hat{\otimes}A$ isomorphically.

There are a kind of biprojective Banach algebras whose left closed ideals are left essential. These algebras are semiprime biprojctive Banach algebras with the approximation property [22].

Corollary 3.3. Let A be a semiprime, biprojective Banach algebra with the approximation property , and let I be a closed ideal of A. If $A\hat{\otimes}.$ respects I into $A\hat{\otimes}A$ isomorphically, then $H^1\left(A,I^*\right)=\{0\}$. In particular, for each closed ideal I which is complemented as a subspace of A, the assertion holds.

Hereditary Properties

Let A and B be two Banach algebras and let $\phi:A\to B$ be a continuous homomorphism with dense range. We know that B is amenable if A is amenable [15], however this is not true for the weak amenability. In special case, if A is weakly amenable and commutative, then B is weakly amenable [3].

Theorem 4.1. Let A and B be two Banach algebras and let $\phi:A\to B$ be a continuous homomorphism with dense range, also let J be a closed ideal of B. If the following conditions hold:

i) $\phi|_{JC}$ is one to one, where J^c is $\phi^{-1}(J^C)$.

ii) $\phi(J^C)$ is dense in J.

iii) $H^1\left(A,J^{C^*}\right)=\{0\}$.

Then $H^1\left(B,J^*\right)=\{0\}$.

Proof. Let $D:B\to J^*$ be a derivation. Define $T:J^*\to J^{C^*}$, by $f\mapsto T_f$, where $T_f(a):=f(\phi(a))$. Obviously for each $f\in J^*, T_f$ is a continuous linear functional on J^c and so T is well defined. We show that T is onto. Let $g \in J^{C^*}$ and define $\tilde{f} : \phi\left(J^C\right) \to \mathbb{C}$ *by* $\phi(a) \mapsto g(a)$. $\phi(J^C)$ is a subspace of J and $\tilde{f}$ is a bounded linear functional on $\phi(J^C)$. By Hahn-Banach theorem, there exists a continuous linear functional f on J such that $f\mid_{\phi\left(J^C\right)} = \tilde{f}$. It is clear that $T_f = g$.

Now define $\tilde{D} : A \to J^{C^*}$ by $\tilde{D} := T\, o\, D\, o\, \phi.\tilde{D}$ is a bounded linear map. Let $a_1, a_2 \in A$ and $a \in J^C$. Then

$$\langle a, \tilde{D}(a_1, a_2)\rangle = \langle a, T\left(D\left(\phi\left(a_1\right)\right)\right)\rangle$$

$$= \langle a, T\left(D\left(\phi\left(a_1\right)\right).\phi\left(a_2\right) + \phi\left(a_1\right).D\left(\phi\left(a_2\right)\right)\right)\rangle$$

$$= \langle \phi\left(a\right), D\left(\phi\left(a_1\right)\right).\phi\left(a_2\right) + \phi\left(a_1\right).D\left(\phi\left(a_2\right)\right)\rangle$$

$$= \langle \phi\left(a_2 a\right), D(\phi(a_1))\rangle + \langle \phi(a a_1), D(\phi(a_2))\rangle$$

$$= \langle a_2 a, T\, o\, D\, o \phi(a_1)\rangle + \langle a a_1, T\, o\, D\, o\, \phi(a_2)\rangle$$

$$= \langle a, \tilde{D}\left(a_1\right).a_2\rangle + \langle a, a_1.\tilde{D}(a_2)\rangle.$$

Thus $\tilde{D}$ is a derivation. Since $H^1\left(A, J^{C^*}\right) = \{0\}$, there exists $g \in J^{C^*}$ such that $\tilde{D} = \delta_g$. But T is onto, so there exists $f \in J^*$ such that $T_f = g$. We claim that $D = \delta_f$. Let $a \in A$ and $a_1 \in J^C$, we have

$$\langle \phi\left(a_1\right), D\left(\phi\left(a\right)\right)\rangle = \langle a_1, \tilde{D}(a)\rangle$$

$$= \langle a_1, a.g - g.a\rangle$$

$$= \langle a_1 a - a a_1, g\rangle$$

$$= \langle a_1 a - a a_1, T_f\rangle$$

$$= \langle \phi\left(a_1 a - a a_1\right), f\rangle$$

$$= \langle \phi\left(a_1\right), \phi\left(a\right) f - f.\phi\left(a\right)\rangle$$

$$= \langle \phi\left(a_1\right), \delta_f\left(\phi\left(a\right)\right)\rangle$$

Since $\phi(J^C)$ is dense in J, we have $D\left(\phi\left(a\right)\right) = \delta_f\left(\phi\left(a\right)\right)$ for each $a \in A$. Again since $\phi(A)$ is dense in B, $D = \delta_f$

Let A and B be two Banach algebras and let $\phi : A \to B$ be a continuous homomorphism with dense range. In general, we assert that the ideal amenability of A does not imply the ideal amenability of B.

We know that, the approximation property is not necessary for the weak amenability of the algebra of approximable operators on a Banach space [1, Corollary 3.5]. Also there are some Banach spaces E with the approximation property such that (E) is not weakly amenable. In [1], Ariel Blanco gives one of these examples; let $\left(e_n\right)_n$ be a sequence of positive numbers such that $\sum_n e_n < \infty$. Also let $\left(p_n\right)_n \subseteq \,]1,2[$ and $\left(k_n\right)_n \subseteq \mathbb{N}$ be two strictly increasing sequences satisfying the following inequalities

$$K_n^{\frac{1}{P_n} - \frac{1}{2}} \geq e_n^{-1}, \; K^{\frac{1}{P_{n+1}} - \frac{1}{2}} \leq 2$$

Now set $E = \left(\sum \oplus_n l_{p_n}^{k_n}\right)_{l_2}$, where l_2-norm is defined on $\sum \oplus_n l_{p_n}^{k_n}$ by

$$\| \left(f_1, f_2, \ldots\right) \|_{l_2} = \left(\sum_{i=1}^{\infty} \| f_i \|_{p_i}^{\,2}\right)^{\frac{1}{2}}$$

Indeed, E is the space of element $(f_1, f_2, \ldots)$ for each $i \in \mathbb{N}; f_i \in l_{p_i}^{k_i}$ and $\sum_{i=1}^{\infty} \| fi \|_{p_i}^{\,2} < \infty$. Then E is a Banach space with the approximation property such that for which, by [1, Proposition 5.3], $\mathfrak{A}(E)$ is not a weakly amenable Banach algebra. On the other hand, the nuclear algebra (E) of E is biprojective and so is weakly amenable. Since (E) is topologically simple, then (E) is ideally amenable. But (E) is not ideally amenable and the inclusion map $i : \mathfrak{N}\left(E\right) \to \mathfrak{A}(E)$ is a continuous homomorphism with dense range. This proves the assertion.

Theorem 4.2. Suppose Y and Z are closed subspaces of a Banach space X, and suppose that there is a collection $\Lambda \subset \mathfrak{L}(X)$ with the following properties:

i) Every $\phi \in \Lambda$ maps X into Y.

ii) Every $\phi \in \Lambda$ maps Z into Z.

iii) sup $\{\|\phi\|: \phi\in \Lambda\} < \infty$.

iv) To every $y \in Y$ and to every $\varepsilon > 0$ corresponds a $\phi\in \Lambda$ such that $\| y - \phi y \| < \varepsilon$.

Then $Y + Z$ is closed.

Proof. [19, Theorem 1.2].

Corollary 4.3. Suppose A is a Banach algebra. Let I be a right closed ideal and J be a left closed ideal of A. If I has a bounded approximate identity, then $I + J$ is closed.

Proof. Let $(e_\alpha)_\alpha$ be a bounded approximate identity for . Set

$$\Lambda = \{L_{e_\alpha} : A \to A \mid L_{e_\alpha}(a) = e_\alpha a\}$$

Obviously $\Lambda \subset \mathfrak{L}(A)$. For each α, we have $L_{e_\alpha}(A) = \{e_\alpha a : a \in A\} \subset I$ and $L_{e_\alpha}(J) = \{e_\alpha j : j \in J\} \subset J$. Also we have

$$\sup\{\| L_{e_\alpha} \in \Lambda\} \le \sup_\alpha\{\| e_\alpha \|\} < \infty$$

Now let $\varepsilon > 0$ is given and $i \in I$. There exists α_0 such that

$$\| i - L_{e_{\alpha_0}}(i) \| = \| i - e_{\alpha_0} i \| < \varepsilon$$

Thus by the above theorem, $I + J$ is closed.

Theorem 4.4. Let A be a Banach algebra and let I be a closed ideal of A with a bounded approximate identity. If I and $\frac{A}{I}$ are ideally amenable, then A is ideally amenable.

Proof. Let J be an arbitrary closed ideal of A and let $D : A \to J^*$ be a derivation. Consider $\iota : I \cap J \to J$ as an inclusion map. Obviously $\iota^* : J^* \to (I \cap J)^*$ is an A-bimodule homomorphism and so, $\iota^* o D : A \to (I \cap J)^*$ and consequently $\iota^* o D|_I : I \to (I \cap J)^*$ are drivations. Since I is ideally amenable, there exists $\phi_1 \in (I \cap J)^*$ such that $\iota^* o D|_I = \delta_{\phi_1}$. Let Φ_1 be a Hahn-Banach extension of ϕ_1 on J. Define $\tilde{D} : D - \delta_{\Phi_1}$. So $\tilde{D}$ is a derivation from A into J^*. We show that $\tilde{D}|_I = 0$. Let $i \in I$ and $j \in J$, we have

$$\begin{aligned}\langle j, \tilde{D}(i)\rangle &= \langle j, D(i)\rangle - \langle j, \delta_{\Phi_1}(i)\rangle \\ &= \langle j, D(i)\rangle - \langle ji - ij, \Phi_1\rangle \\ &= \langle j, D(i)\rangle - \langle ji - ij, \phi_1\rangle \\ &= \langle j, D(i)\rangle - \langle j, \iota^*(\mathrm{D(i)})\rangle \\ &= \langle j, D(i)\rangle - \langle \iota(\mathrm{j}), \mathrm{D(i)}\rangle \\ &= 0.\end{aligned}$$

In this case, $\tilde{D}|_I = 0$ and $\tilde{D}$ induces a map from $\frac{A}{I}$ into J^* which is a derivation and we call it $\tilde{D}$ itself. Since $I \subset Ann(\frac{J}{J\cap I})$, the annihilator of $\frac{J}{J\cap I}$, then $\frac{J}{J\cap I}$ is a Banach $\frac{A}{I}$ -bimodule.

On the other hand, let $(e_\alpha)_\alpha$ be a bounded approximate identity in I. Then for each $x \in I \cap J$ and $a \in A$ we have

$$\begin{aligned} x, \tilde{D}(a + I) &= \lim_\alpha x e_\alpha, \tilde{D}(a) \\ &= \lim_\alpha e_\alpha, \tilde{D}(a).x \\ &= \lim_\alpha e_\alpha, \tilde{D}(ax) - a.\tilde{D}(x) \\ &= 0\end{aligned}$$

So, $\tilde{D}\left(\frac{A}{I}\right) \subseteq (I \cap J)^\perp \cong \left(\frac{J}{J\cap I}\right)^*$. Since I has a bounded approximate identity, then by Corollary 5.3 , $J + I$ is a closed ideal of A, thus $\frac{J+I}{I}$ is a Banach space. Now define $\psi : \frac{J}{J\cap I} \to \frac{J+I}{I}$ by $j + J\cap I \mapsto j + I$. It is clear that ψ is an algebra isomorphism, therefore ψ is an $\frac{A}{I}$-bimodule isomorphism. Also

$$\| \psi(j + J\cap I) \| = \| j + I \|$$

$$= \inf\{\| j + i \| : i \in I\}$$

$$\leq \inf\{\| j+i \| : i \in I \cap J\}$$

$$= \| j + J \cap I \|$$

So, ψ is bounded. By the open mapping theorem Ψ is a homeomorphism, in other words, $\left(\frac{J}{J \cap I}\right)^* \cong \left(\frac{J+I}{I}\right)^*$. Therefore there exists $\Phi_2 \in (I \cap J)^{\perp}$ such that $\tilde{D} = \delta_{\Phi_2}$. So we have $D = \delta_{\Phi_1 + \Phi_2}$, that is, D is inner and A is ideally amenable.

We recall that the above theorem is true for weak amenability and amenability even if I does not have any hounded approximate identity [3].

Results

In this paper, We investigate the closed ideals I of Banach algebra A for which H1 (A,I*)={0}, whenever A is weakly amenable or a biflat Banach algebra. Also we give some hereditary properties of ideal amenability.

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