Journal of Sciences, Islamic Republic of Iran 22(1): 51-61 (2011) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Newton-Product Integration for a Stefan Problem with Kinetics

B. Babayar-Razlighi[1,2] K. Ivaz,[1,2,*] and M.R. Mokhtarzadeh[3]

[1]*Department of Mathematics, University of Tabriz, Tabriz, Islamic Republic of Iran*
[2]*Research Center of Industrial Mathematics of University of Tabriz, Tabriz, Islamic Republic of Iran*
[3]*Institute for Studies in Theoretical Physics and Mathematics, Tehran, Islamic Republic of Iran*

Received: 15 August 2009 / Revised: 22 December 2010 / Accepted: 8 February 2011

Abstract

Stefan problem with kinetics is reduced to a system of nonlinear Volterra integral equations of second kind and Newton's method is applied to linearize it. Product integration solution of the linear form is found and sufficient conditions for convergence of the numerical method are given. An example is provided to illustrated the applicability of the method.

Keywords: Stefan problem; Kinetic function; Newton's method; Product integration; Weakly singular Volterra integral equations

Introduction

Consider the following modified one-phase Stefan problem in one spatial Variable,

$$u_t = u_{xx} - \gamma u \quad -\infty < x < s(t), \tag{1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=s(t)} = -V(t), \qquad g(u|_{x=s(t)}) = V(t), \tag{2}$$

$$u(x,0) = u^0(x), \tag{3}$$

where $u(x,t)$ is the temperature and $\gamma \geq 0$. The damping term is due to volumetric heat losses. The two boundary conditions determine the problem and make it possible to find the free boundary with position $s(t)$, and velocity $V(t) = \dot{s}(t)$.

Further assume that $g(t)$ is monotonically decreasing differentiable function on $[0,\infty)$ with $|g'| \leq C$ and satisfying

$$-V_0 \leq g(t) \leq -v_0 \quad for \quad some \quad v_0, V_0 > 0. \tag{4}$$

The free boundary problem $(1)-(3)$ arises naturally as a mathematical model of a variety of exothermic phase transition type processes, such as condensed phase combustion [6] also known as self-sustained high-temperature synthesis or SHS [7], solidification with undercooling [5], laser induced evaporation [4], rapid crystallization in thin films [9] etc. These processes are characterized by production of heat at the interface, and their dynamics is determined by the feedback mechanism between the heat release due to the kinetics $g(u|_{x=s(t)})$ and the heat dissipation by the medium. The first boundary condition in (2) (the Stefan boundary condition) expresses the balance between the heat

* Corresponding author, Tel.: +98(411)3293436, Fax: +98(411)3293436, E-mail: ivaz@tabrizu.ac.ir

produced at the free boundary and the heat diffusion through the adjacent medium. Problem $(1)-(3)$ describes propagation of the phase transition front, the second boundary condition (2) is a manifestation of the non-equilibrium nature of the transition; and its analog for the classical Stefan problem is just $u\,|_{x=s(t)}=0$. In the context of condensed phase combustion kinetic boundary condition expresses dependence of propagation velocity on the flame front temperature.

The rest of the paper is organized as follows. In Section 1 a local existence condition is obtained. In Section 2 the Stefan problem with kinetics reduced to a system of nonlinear Volterra integral equations of the second kind and Newton's method is applied to linearize it. A convergence analysis of Newton's method for the problem is provided in the subsections of Section 2. Product integration solution of the linear form is obtained in section 3. Convergence of product integration method is given in Subsection 3.1. Finally in section 4 numerical results of test problem solved by the proposed method is reported.

1. Existence of Local Classical Solution

In order not to clutter formulas with factors of the type $e^{-\gamma t}$, from now on until section 4 we set the damping coefficient $\gamma=0$. The modifications to the $\gamma>0$ case are trivial. A short-time solution of the free boundary problem (1)-(3) will be sought in the form of a superposition of heat potentials,

$$u(x,t)=\int_0^t G(x,s(\tau),t-\tau)\varphi(\tau)d\tau + \int_{-\infty}^0 G(x,\xi,t)u^0(\xi)d\xi, \tag{5}$$

where G is the fundamental solution of the heat equation,

$$G(x,\xi,t-\tau)=\exp\left[-\frac{(x-\xi)^2}{4(t-\tau)}\right]\left[4\pi(t-\tau)\right]^{-\frac{1}{2}} \tag{6}$$

The density of the single layer potential φ and the front position $s(t)$ are to be determined.

Frankel and Roytburd in Ref [3] shown that the single layer potential is continuous up to the boundary and its derivative possesses the standard jump property

$$\lim_{x\to s(t)^-}\frac{\partial}{\partial x}\int_0^t G(x,s(\tau),t-\tau)\varphi(\tau)d\tau = \frac{\varphi(t)}{2}+\int_0^t G_x(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau. \tag{7}$$

This result is, of course, well-known if φ is continuous. It turns out however, that by the nature of the free boundary problem at hand, φ must have a $\frac{1}{\sqrt{t}}$ singularity at $t=0$. Thus a justification of (7) will require an extra effort. If the jump property in (7) holds then for the solution represented by (5), the boundary condition in (2) yield the following equations

$$\begin{aligned} u(s(t),t) &= g^{-1}(V(t)) \\ &= \int_0^t G(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau \\ &+\int_{-\infty}^0 G(s(t),\xi,t)u^0(\xi)d\xi, \end{aligned} \tag{8}$$

$$\begin{aligned} u_x(s(t),t) &= -V(t)=\frac{\varphi(t)}{2} \\ &-\int_0^t G_\xi(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau \\ &-\int_{-\infty}^0 G_\xi(s(t),\xi,t)u^0(\xi)d\xi, \end{aligned} \tag{9}$$

where φ satisfies the balance condition

$$\lim_{t\to 0}\sqrt{t}\,\varphi(t)=\frac{u^0(0)}{\sqrt{\pi}}.$$

We can rewrite the integral equations in (8)-(9) in terms of φ and V [3]:

$$V=K_1(V,\varphi), \tag{10}$$

$$\varphi=-2K_1(V,\varphi)+K_2(V,\varphi), \tag{11}$$

where the nonlinear operators K_1 and K_2 are defined as follows

$$K_1(V,\varphi)=g\left\{\int_0^t G(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau + \int_{-\infty}^0 G(s(t),\xi,t)u^0(\xi)d\xi\right\}, \tag{12}$$

$$K_2(V,\varphi)=2\left\{\int_0^t G_\xi(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau + \int_{-\infty}^0 G_\xi(s(t),\xi,t)u^0(\xi)d\xi\right\}, \tag{13}$$

here as usual,

$$s(t)=\int_0^t V(\tau)d\tau. \tag{14}$$

The equations are supplemented by the initial conditions

$$V(0) = g(u^0(0)), \lim_{t\to 0}\sqrt{t}\varphi(t) = \frac{u^0(0)}{\sqrt{\pi}}. \tag{15}$$

The proof of the following theorem is given in [3]

Theorem 1. Let $g<0$ be continuously differentiable, monotone decreasing function, $u^0 \in C(-\infty,0], u^0>0$. Then the problem in (10)-(11) has a unique solution V,φ such that V and $\sqrt{t}\varphi(t)$ are continuous on $[0,\sigma]$ for some $\sigma>0$, where σ depends only on $Sup\, u^0$. The solution to the free boundary problem is determined by V,φ via the representation (5) with $s(t)=\int_0^t V(\tau)d\tau$.

2. Application of the Newton's Method

Now we apply a Newton's method to linearize of the problem (10)-(11). For this purpose we take

$$U := \left\{\begin{bmatrix} V \\ \varphi \end{bmatrix} : V, \sqrt{.}\varphi(.) \in C[0,\sigma]\right\},$$

with the norm,

$$\|\varphi\|_\sigma = \sup_{0\le t\le\sigma}\sqrt{t}\,|\varphi(t)|,$$

$$\left\|\begin{bmatrix} V \\ \varphi \end{bmatrix}\right\|_U = \max\left\{\|V\|_{C[0,\sigma]}, \|\varphi\|_\sigma\right\}.$$

We know that U is a Banach space with above norm. Introducing an operator $T: U \to U$ through the formula

$$T\begin{bmatrix} V \\ \varphi \end{bmatrix} = \begin{bmatrix} f_1(V,\varphi) \\ f_2(V,\varphi) \end{bmatrix}; \begin{bmatrix} V \\ \varphi \end{bmatrix} \in U, \tag{16}$$

where

$$f_1(V,\varphi) = V - K_1(V,\varphi), \tag{17}$$

$$f_2(V,\varphi) = \varphi + 2K_1(V,\varphi) - K_2(V,\varphi), \tag{18}$$

the problem (10)-(11) can be written in the form

$$T\begin{bmatrix} V \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{19}$$

We suppose that $\left(V^*,\varphi^*\right)^T$ is the exact solution of (19). Then by Taylor expansion of two variables, functions f_1 and f_2 at $\left(\hat{V},\hat{\varphi}\right)^T$ sufficiently close to $\left(V^*,\varphi^*\right)^T$ we have

$$\begin{aligned} 0 &= f_1(V^*,\varphi^*) \\ &= f_1(\hat{V},\hat{\varphi}) + \frac{\partial f_1}{\partial V}(\hat{V},\hat{\varphi})(V^*-\hat{V}) \\ &+ \frac{\partial f_1}{\partial \varphi}(\hat{V},\hat{\varphi})(\varphi^*-\hat{\varphi}) + \frac{1}{2!}\Big[(V^*-\hat{V})\frac{\partial}{\partial V} \\ &+ (\varphi^*-\hat{\varphi})\frac{\partial}{\partial\varphi}\Big]^2 f_1(\hat{V}+\theta_1 h, \hat{\varphi}+\theta_1 k), \end{aligned} \tag{20}$$

Where $h = V^*-\hat{V}, k=\varphi^*-\hat{\varphi}$ and $\theta_1 \in (0,1)$. And,

$$\begin{aligned} 0 &= f_2(V^*,\varphi^*) \\ &= f_2(\hat{V},\hat{\varphi}) + \frac{\partial f_2}{\partial V}(\hat{V},\hat{\varphi})(V^*-\hat{V}) \\ &+ \frac{\partial f_2}{\partial \varphi}(\hat{V},\hat{\varphi})(\varphi^*-\hat{\varphi}) + \frac{1}{2!}\Big[(V^*-\hat{V})\frac{\partial}{\partial V} \\ &+ (\varphi^*-\hat{\varphi})\frac{\partial}{\partial\varphi}\Big]^2 f_2(\hat{V}+\theta_2 h, \hat{\varphi}+\theta_2 k), \end{aligned} \tag{21}$$

$\theta_2 \in (0,1)$.

We approximate above equations by eliminating $O(h^2+k^2)$:

$$\begin{pmatrix} \frac{\partial f_1}{\partial V}(\hat{V},\hat{\varphi}) & \frac{\partial f_1}{\partial \varphi}(\hat{V},\hat{\varphi}) \\ \frac{\partial f_2}{\partial V}(\hat{V},\hat{\varphi}) & \frac{\partial f_2}{\partial \varphi}(\hat{V},\hat{\varphi}) \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} \simeq \begin{pmatrix} -f_1(\hat{V},\hat{\varphi}) \\ -f_2(\hat{V},\hat{\varphi}) \end{pmatrix}. \tag{22}$$

Hence the Newton's method for finding root of (19) is

$$\begin{pmatrix} \frac{\partial f_1}{\partial V}(V_n,\varphi_n) & \frac{\partial f_1}{\partial \varphi}(V_n,\varphi_n) \\ \frac{\partial f_2}{\partial V}(V_n,\varphi_n) & \frac{\partial f_2}{\partial \varphi}(V_n,\varphi_n) \end{pmatrix}\begin{pmatrix} \delta_{n+1} \\ \gamma_{n+1} \end{pmatrix} = \begin{pmatrix} -f_1(V_n,\varphi_n) \\ -f_2(V_n,\varphi_n) \end{pmatrix} \quad n=0,1,..., \tag{23}$$

where

$$\delta_{n+1} := V_{n+1} - V_n, \quad \gamma_{n+1} := \varphi_{n+1} - \varphi_n. \tag{24}$$

For starting numerical process it is sufficient to evaluate elements of coefficient matrix, $\frac{\partial f_i}{\partial V}, \frac{\partial f_i}{\partial \varphi}, i=1,2.$

$$\frac{\partial f_1}{\partial V}(V,\varphi)u = \lim_{h\to 0} h^{-1}\big[f_1(V+hu,\varphi)$$
$$-f_1(V,\varphi)\big] = \lim_{h\to 0} h^{-1}\big[V+hu$$
$$-K_1(V+hu,\varphi) - V + K_1(V,\varphi)\big]$$
$$= u - \frac{\partial K_1}{\partial V}(V,\varphi)u,$$

where

$$\frac{\partial K_1}{\partial V}(V,\varphi)u = \lim_{h\to 0} h^{-1}\big[K_1(V+hu,\varphi)$$
$$-K_1(V,\varphi)\big] = \lim_{h\to 0} h^{-1}\Big\{g\Big[\int_0^t G(s(t)$$
$$h\sigma(t), s(\tau)+h\sigma(\tau), t-\tau)\varphi(\tau)d\tau$$
$$+\int_{-\infty}^0 G(s(t)+h\sigma(t),\xi,t)u^0(\xi)d\xi\Big]$$
$$-g\Big[\int_0^t G(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau$$
$$+\int_{-\infty}^0 G(s(t),\xi,t)u^0(\xi)d\xi\Big]\Big\}$$
$$= g'(\alpha(t))\lim_{h\to 0}\frac{\Delta\alpha(t)}{h}$$
$$= g'(\alpha(t))\Big\{\int_0^t\big[G_x(s(t),s(\tau),t-\tau)\varphi(\tau)$$
$$\sigma(t) + G_\xi(s(t),s(\tau),t-\tau)\varphi(\tau)\sigma(\tau)\big]d\tau$$
$$+\sigma(t)\int_{-\infty}^0 G_x(s(t),\xi,t)u^0(\xi)d\xi\Big\}$$
$$= g'(\alpha(t))\Big\{\int_0^t G_x(s(t),s(\tau),t-\tau)$$
$$\big[\sigma(t)-\sigma(\tau)\big]\varphi(\tau)d\tau$$
$$+\sigma(t)\int_{-\infty}^0 G_x(s(t),\xi,t)u^0(\xi)d\xi\Big\},$$

where

$$\sigma(t) = \int_0^t u(\tau)d\tau,$$
$$\alpha(t) = \int_0^t G(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau$$
$$+\int_{-\infty}^0 G(s(t),\xi,t)u^0(\xi)d\xi,$$
$$\Delta\alpha(t) = \int_0^t G(s(t)+h\sigma(t), s(\tau)+h\sigma(\tau),$$
$$t-\tau)\varphi(\tau)d\tau + \int_{-\infty}^0 G(s(t)+h\sigma(t),\xi,t)$$
$$u^0(\xi)d\xi - \alpha(t),$$

and note that

$$(G_x + G_\xi)(x,\xi,t) = 0. \qquad (25)$$

Similarly we obtain

$$\frac{\partial f_1}{\partial \varphi}(V,\varphi)u = -\frac{\partial K_1}{\partial \varphi}(V,\varphi)u,$$
$$\frac{\partial f_2}{\partial V}(V,\varphi)u = 2\frac{\partial K_1}{\partial V}(V,\varphi)u - \frac{\partial K_2}{\partial V}(V,\varphi)u,$$
$$\frac{\partial f_2}{\partial \varphi}(V,\varphi)u = u + 2\frac{\partial K_1}{\partial \varphi}(V,\varphi)u$$
$$-\frac{\partial K_2}{\partial \varphi}(V,\varphi)u,$$

where

$$\frac{\partial K_1}{\partial \varphi}(V,\varphi)u = g'(\alpha(t))\int_0^t G(s(t),s(\tau),t-\tau)u(\tau)d\tau,$$
$$\frac{\partial K_2}{\partial V}(V,\varphi)u = 2\int_0^t G_{\xi x}(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau$$
$$(\sigma(t)-\sigma(\tau))d\tau + 2\sigma(t)\int_{-\infty}^0 G_{\xi x}(s(t),\xi,t)u^0(\xi)d\xi,$$
$$\frac{\partial K_2}{\partial \varphi}(V,\varphi)u = 2\int_0^t G_\xi(s(t),s(\tau),t-\tau)u(\tau)d\tau.$$

Substitution above results in (23) gives

$$\delta_{n+1}(t) + g_1^{[n]}(t)\int_0^t \delta_{n+1}(\tau)d\tau$$
$$+\int_0^t \delta_{n+1}(\tau)g_2^{[n]}(t,\tau)d\tau \qquad (26)$$
$$+\int_0^t \gamma_{n+1}(\tau)g_3^{[n]}(t,\tau)d\tau = r_1^{[n]}(t),$$
$$\gamma_{n+1}(t) + g_4^{[n]}(t)\int_0^t \delta_{n+1}(\tau)d\tau$$
$$+\int_0^t \delta_{n+1}(\tau)g_5^{[n]}(t,\tau)d\tau \qquad (27)$$
$$+\int_0^t \gamma_{n+1}(\tau)g_6^{[n]}(t,\tau)d\tau = r_2^{[n]}(t),$$

where

$$g_1^{[n]}(t) = g^{'}(\alpha_n(t))\int_{-\infty}^{0} G_x(s_n(t),\xi,t)u^0(\xi)d\xi,$$

$$g_2^{[n]}(t,\tau) = -g^{'}(\alpha_n(t))\int_0^{\tau} G_x(s_n(t),s_n(\tau),t-\tau)$$

$$\varphi_n(\tau)d\tau,$$

$$g_3^{[n]}(t,\tau) = -g^{'}(\alpha_n(t))G(s_n(t),s_n(\tau),t-\tau),$$

$$r_i^{[n]}(t) = -f_i(V_n,\varphi_n), \quad i = 1,2,$$

$$g_4^{[n]}(t) = \int_{-\infty}^{0}\left[2g^{'}(\alpha_n(t))G_x(s_n(t),\xi,t)\right.$$

$$\left.-2G_{\xi x}(s_n(t),\xi,t)\right]u^0(\xi)d\xi,$$

$$g_5^{[n]}(t,\tau) = \int_0^{\tau}\left[2g^{'}(\alpha_n(t))G_x(s_n(t),s_n(\tau),t-\tau)\right.$$

$$\left.-2G_{\xi x}(s_n(t),s_n(\tau),t-\tau)\right]\varphi_n(\tau)d\tau,$$

$$g_6^{[n]}(t,\tau) = 2g^{'}(\alpha_n(t))G(s_n(t),s_n(\tau),t-\tau)$$

$$-2G_{\xi}(s_n(t),s_n(\tau),t-\tau).$$

Equations (26)-(27) yield the following linear system

$$U^{[n+1]}(t) = F^{[n]}(t) + \int_0^t K^{[n]}(t,\tau)U^{[n+1]}(\tau)d\tau, \qquad (28)$$

where

$$U^{[n]}(t) = \begin{bmatrix}\delta_n(t)\\ \gamma_n(t)\end{bmatrix},$$

$$F^{[n]}(t) = \begin{bmatrix}r_1^{\,n}(t)\\ r_2^{\,n}(t)\end{bmatrix},$$

$$K^{[n]}(t,\tau) = \begin{pmatrix}-g_1^{[n]}(t)-g_2^{[n]}(t,\tau) & -g_3^{[n]}(t,\tau)\\ -g_4^{[n]}(t)-g_5^{[n]}(t,\tau) & -g_6^{[n]}(t,\tau)\end{pmatrix}.$$

2.1. Convergence of Newton's Method

In this Subsection convergence of Newton's method will be proved. We can rewrite (10)-(11) as an operator equation on Banach space U (see beginning of Section 2):

$$T\begin{bmatrix}V\\ \varphi\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}, \qquad (29)$$

where T is defined by (16). For an arbitrary $\begin{bmatrix}V_1\\ \varphi_1\end{bmatrix} \in U$ and $t \in [0,\sigma]$ it can be shown that:

$$T'\begin{bmatrix}V\\ \varphi\end{bmatrix} : U \to U,$$

$$T'\begin{bmatrix}V\\ \varphi\end{bmatrix} = \begin{bmatrix}V_1\\ \varphi_1\end{bmatrix}(t) =$$

$$\begin{bmatrix}\frac{\partial f_1}{\partial V}(V,\varphi) & \frac{\partial f_1}{\partial V}(V,\varphi)\\ \frac{\partial f_2}{\partial V}(V,\varphi) & \frac{\partial f_1}{\partial V}(V,\varphi)\end{bmatrix}\begin{bmatrix}V_1\\ \varphi_1\end{bmatrix}(t) \qquad (30)$$

$$= \begin{bmatrix}V_1 - g'(\alpha)\beta\\ 2g'(\alpha)\beta + \theta\end{bmatrix}(t),$$

where

$$\alpha = \alpha(V,\varphi;t) :=$$

$$\int_0^t G(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau$$

$$+\int_{-\infty}^0 G(s(t),\xi,t)u^0(\xi)d\xi,$$

$$\beta = \beta(V,\varphi,V_1,\varphi_1;t) :=$$

$$\int_0^t G_x(s(t),s(\tau),t-\tau)\varphi(\tau)(s_1(t)-s_1(\tau))d\tau$$

$$+s_1(t)\int_{-\infty}^0 G_x(s(t),\xi,t)u^0(\xi)d\xi$$

$$+\int_0^t G(s(t),s(\tau),t-\tau)\varphi_1(\tau)d\tau.$$

Since $G_x = -G_{\xi}, G_{xx} = -G_{\xi x}$ *so we can write*

$$\theta = \theta(V,\varphi,V_1,\varphi_1;t) =$$

$$2\int_0^t G_{xx}(s(t),s(\tau),t-\tau)\varphi(\tau)(s_1(t)-s_1(\tau))d\tau$$

$$+2s_1(t)\int G_{xx}(S(T),\xi,T)u^0(\xi)d\xi$$

$$+2\int G_x(s(t),s(\tau),t-\tau)\varphi_1(\tau)d\tau,$$

where

$$s(t) = \int_0^t V(\tau)d\tau, \ \ s_1(t) = \int_0^t V_1(\tau)d\tau.$$

Now we prove that for every

$$\begin{bmatrix}V\\ \varphi\end{bmatrix}, \begin{bmatrix}V'\\ \varphi'\end{bmatrix} \in N\left(\begin{bmatrix}V^*\\ \varphi^*\end{bmatrix}\right)$$

there exist $L^* > 0$ such that:

$$\left\| T'\begin{bmatrix}V\\ \varphi\end{bmatrix} - T'\begin{bmatrix}V'\\ \varphi'\end{bmatrix}\right\| \le L^* \left\|\begin{bmatrix}V\\ \varphi\end{bmatrix} - \begin{bmatrix}V\\ \varphi\end{bmatrix}\right\|_U, \qquad (31)$$

where $N\left(\begin{bmatrix} V^* \\ \varphi^* \end{bmatrix}\right)$ is a neighborhood of exact solution $\begin{bmatrix} V^* \\ \varphi^* \end{bmatrix}$. Suppose that for every $\begin{bmatrix} V \\ \varphi \end{bmatrix}, \begin{bmatrix} V' \\ \varphi' \end{bmatrix} \in U$ we define

$$L \equiv T'\begin{bmatrix} V \\ \varphi \end{bmatrix} - T'\begin{bmatrix} V' \\ \varphi' \end{bmatrix}. \quad (32)$$

For arbitrary $\begin{bmatrix} V_1 \\ \varphi_1 \end{bmatrix} \in U$ and $\left\|\begin{bmatrix} V_1 \\ \varphi_1 \end{bmatrix}\right\|_U = 1,$

it is sufficient show that,

$$\left\| L \begin{bmatrix} V_1 \\ \varphi_1 \end{bmatrix} \right\|_U < L^* \left\| \begin{bmatrix} V \\ \varphi \end{bmatrix} - \begin{bmatrix} V' \\ \varphi' \end{bmatrix} \right\|_U. \quad (33)$$

For $t \in [0,\sigma]$ denote $\alpha = \alpha(V, \varphi; t)$,

$\beta = \beta(V, \varphi, V_1, \varphi_1; t),\ \theta = \theta(V, \varphi, V_1, \varphi_1; t),$

$\alpha' = \alpha(V', \varphi', t),\ \beta' = \beta(V', \varphi', V_1, \varphi_1; t)$

and

$\theta' = \theta(V', \varphi', V_1, \varphi_1; t).$

So we can write

$$L\begin{bmatrix} V_1 \\ \varphi_1 \end{bmatrix} = \begin{bmatrix} g'(\alpha')\beta' - g'(\alpha)\beta \\ 2g'(\alpha)\beta + \theta - 2g'(\alpha')\beta' - \theta' \end{bmatrix} \equiv \begin{bmatrix} \tilde{V} \\ \tilde{\varphi} \end{bmatrix}.$$

We define the norm

$$\|\tilde{V}\|_{c[0,\sigma]} := \max\{|\tilde{V}| : t \in [0,\sigma]\}, \quad (35)$$

$$\|\tilde{\varphi}\|_\sigma := \max\{|\tilde{\varphi}|\sqrt{t} : t \in [0,\sigma]\}, \quad (36)$$

Since g is bounded and Lipschitz continuous, these means

$$\exists C_{g'} > 0 |g'(x)| C_{g'} \quad \forall x \in [0,\infty), \quad (37)$$

$$\exists L_{g'} > 0 \quad \forall xy \in [0,\infty) \quad |g'(x) - g'(y)| \le L_{g'}|x - y|, \quad (38)$$

and hence,

$$|\tilde{V}| = |g'(\alpha')\beta' - g'(\alpha)\beta| = |g'(\alpha')(\beta' - \beta) + (g'(\alpha') - g'(\alpha))\beta| \le C_{g'}|\beta' - \beta| + L_{g'}|\alpha' - \alpha||\beta|. \quad (39)$$

Now we evaluate some upper bounds for $|\beta' - \beta|, |\alpha' - \alpha|$. By the mean value theorem and noting that for all $\eta > 0$ we have $e^{-\eta} \le 1, \eta e^{-\eta} \le 1$ and $\eta e^{-\eta^2} \le \frac{1}{\sqrt{2e}}$,

$$|\beta' - \beta| \le C_1(t)\left\| \begin{matrix} V - V' \\ \varphi - \varphi' \end{matrix} \right\|_U, \quad (40)$$

where

$$C_1(t) = \left(\sqrt{\frac{\pi}{8}} + 2\sqrt{2}e^{-5/4}\|u^0\|\right)t + \sqrt{\frac{2t}{e\pi}}. \quad (41)$$

$$|\alpha' - \alpha| \le C_2(t)\left\| \begin{bmatrix} V \\ \varphi \end{bmatrix} - \begin{bmatrix} V' \\ \varphi' \end{bmatrix} \right\|_U, \quad (42)$$

where

$$C_2(t) = \frac{\sqrt{\pi}}{2} + \sqrt{\frac{t}{2e\pi}}\|\varphi\|_\sigma + \frac{\sqrt{2t}}{e}\|u^0\|. \quad (43)$$

and the last term is

$$|\theta - \theta'| \le C_3(t)\left\| \begin{bmatrix} V \\ \varphi \end{bmatrix} - \begin{bmatrix} V' \\ \varphi' \end{bmatrix} \right\|_U, \quad (44)$$

where

$$C_3(t) = \sqrt{\pi} + \sqrt{\frac{2t}{\pi}}\|\varphi\|_\sigma + \sqrt{2t}\|u^0\|. \quad (45)$$

By (39),(40) and (42)

$$\|\tilde{V}\|_{C[0,\sigma]} = \max\{|\tilde{V}| : t \in [0,\sigma]\} \le C_4(\sigma)\left\| \begin{bmatrix} V \\ \varphi \end{bmatrix} - \begin{bmatrix} V' \\ \varphi' \end{bmatrix} \right\|_U,$$

where $C_4(t) = C_{g'}C_1(t) + L_{g'}C_2(t)|\beta|$. Similar evaluation lead

$$\|\tilde{\varphi}\|_\sigma = \max\{|\tilde{\varphi}\sqrt{t}| : t \in [0,\sigma]\} \le \sqrt{\sigma}[2C_4(\sigma) + C_3(\sigma)]\left\| \begin{bmatrix} V \\ \varphi \end{bmatrix} - \begin{bmatrix} V' \\ \varphi' \end{bmatrix} \right\|_U.$$

By introduce L^* in the blow form (33) holds.

$$L^* = L^*(\sigma) = Max\left\{C_4(\sigma),\ \sqrt{\sigma}\left[2C_4(\sigma)+C_3(\sigma)\right]\right\} \quad (46)$$

Thus hypothesis of following theorem is satisfy.

Theorem 2. Let X and Y be two Banach spaces, and operator $T: X \to Y$ be Frechet differentiable. Assume x^* is a root of $T(x)=0$ such that $[T'(x^*)]^{-1}$ exists and is a continuous linear map from X to Y. Assume further that $T'(x)$ is locally Lipschitz continuous at $x = x^*$,

$$\|T'(x)-T'(y)\| \le L\|x-y\| \quad \forall x,y \in N(x^*), \quad (47)$$

where $N(x^*)$ is a neighborhood of x^* and $L>0$ is a constant. Then there exists a $\delta > 0$ such that if $\|x_0 - x^*\| \le \delta$, the Newton's sequence $\{x_n\}$ is well-defined and converges to x^*. Furthermore, for some constant M we have the following error bounds

$$\|x_{n+1}-x^*\| \le \mathrm{M}\|x_n - x^*\| \quad (48)$$

$$\|x_n - x^*\| \le \frac{(\mathrm{M}\delta)^{2^n}}{\mathrm{M}} \quad (49)$$

Proof. see [1] pages 155, 156.

3. Product Integration Method

In Eq. (28) g_3 and g_6 are weakly singular kernels in the following form

$$g_3(t,T) = p(t,T)\, g_7(t,T), \quad (50)$$

where

$$p(t,T) := \frac{1}{\sqrt{t-T}}, \quad (51)$$

$$g_7(t,T) := \frac{1}{\sqrt{4\pi}} g'(\alpha(t)) \exp\left\{\frac{-(s(t)-s(T))^2}{4(t-T)}\right\}, \quad (52)$$

$$g_6(t,T) = p(t,T)\, g_8(t,T), \quad (53)$$

$$g_8(t,\tau) := \frac{1}{\sqrt{4\pi}}[2g'(\alpha(t)) - \tfrac{s(t)-s(\tau)}{t-\tau}] \times \exp\left\{\frac{-(s(t)-s(\tau))^2}{4(t-\tau)}\right\}. \quad (54)$$

Now we want to solve the weakly singular integral equation (28) by the method is described in [8].

This method allows us to overcome the difficulty caused by the poor behavior of the solution $U(t)$ at the initial point $t=0$.

Given a relatively short interval $[0,b]$ we first solve the problem

$$U(t) = F(t) + \int_0^t K(t,\tau)U(\tau)d\tau \qquad t\in[0,b], \quad (55)$$

by a Nystrom- type method based upon a whole interval product integration rule of interpolation type, with integrates exactly the kernel $p(t,\tau)$. After the initial interval, the bad behavior of the derivative of U is of less significance. We then solve the problem

$$U(t) = U_1(t) + \int_b^t k(t,T)U(T)\ dT \quad t\in[b,\infty), \quad (56)$$

with

$$U_1(t) = F(t) + \int_0^b k(t,\tau)U(\tau)\ d\tau, \quad (57)$$

by a standard step-by-step method for regular solutions. Since the computation of U_1(t) depends on the starting approximation of U(t) , $t\in[0,b]$, the two methods have to be regarded as paired .

Now we describe the Nystrom-type method used to solve equation (55) numerically. We can rewrite (55) as:

$$U(t) = F(t) + \int_0^t \{[\mathrm{K}_3(t)+\mathrm{K}_4(t,\tau)]\ \delta(\tau) + p(t,\tau)\mathrm{K}_5(t,\tau)\gamma(\tau)\}\ d\tau \quad t\in[0,b], \quad (58)$$

where

$$\mathrm{K}_3(t) = -\begin{bmatrix} g_1(t) \\ g_4(t)\end{bmatrix},$$

$$\mathrm{K}_4(t) = -\begin{bmatrix} g_2(t) \\ g_5(t)\end{bmatrix},$$

$$\mathrm{K}_5(t) = -\begin{bmatrix} g_7(t) \\ g_8(t)\end{bmatrix}.$$

Having chosen $N+1$ distinct points $\{t_n\}_{n=0}^N$ in the interval $[0,b]$ we collocate the equation (58) at the nodes $\{t_n\}_{n=0}^{N}$:

$$U(t_n) = F(t_n) + \int_0^{t_n} \{[K_3(t_n) + K_4(t_n,\tau)]\delta(\tau) + p(t_n,\tau)K_5(t_n,\tau)\gamma(\tau)\}d\tau \tag{59}$$

where $n = 0,1,2,...,N$. Thus we use the Lagrange interpolation polynomial.

$$L_N(f;t) = \sum_{j=0}^{N} l_{N,j}(t)f(t_j), \tag{60}$$

to approximate $\delta(\tau), K_5(t_n,\tau)\gamma(\tau)$ and obtain following method:

$$U_{N,n} = F(t_n) + \int_0^{t_n} \{[K_3(t_n) + K_4(t_n,\tau)]\sum_{j=0}^{N} l_{N,j}(\tau)\delta_{N,j} + p(t_n,\tau)\sum_{j=0}^{N} l_{N,j}(\tau)K_5(t_n,\tau)\gamma_{N,j}\}d\tau,$$

or

$$U_{N,n} = F(t_n) + \sum_{j=0}^{N} (\omega_{n,j}^{(1)} + \omega_{n,j}^{(2)})\delta_{N,J} + \sum_{j=0}^{N} \omega_{n,j}^{(3)}\gamma_{N,j} \qquad n = 0,1,2,...,N, \tag{61}$$

where

$$\omega_{n,j}^{(1)} = k_3(t_n)\int_0^{t_n} l_{N,j}(\tau)d\tau, \tag{62}$$

$$\omega_{n,j}^{(2)} = \int_0^{t_n} k_4(t_n,\tau)l_{N,j}(\tau)d\tau, \tag{63}$$

$$\omega_{n,j}^{(3)} = k_5(t_n,t_j)\int_0^{t_n} p(t_n,\tau)l_{N,j}(\tau)d\tau, \tag{64}$$

$$U_{N,j} = \begin{bmatrix} \delta_{N,j} \\ \gamma_{N,j} \end{bmatrix}. \tag{65}$$

To construct the coefficients $\omega_{n,j}^{(1)}$ and $\omega_{n,j}^{(3)}$ we use a Mathematica software. And we use for $\omega_{n,j}^{(2)}$ the Gaussian integration. By solving the linear system (59) we obtain $U_N(t)$ as a Nystrom approximation for $U(t)$:

$$U_N(t) = F(t) + \sum_{j=0}^{N} (\omega_j^{(1)}(t) + \omega_j^{(2)}(t))\delta_{N,j} + \sum_{j=0}^{N} (\omega_j^{(3)}(t)\gamma_{N,j}, \tag{66}$$

where

$$\omega_j^{(1)}(t) = k_3(t)\int_0^t l_{N,j}(\tau)d\tau, \tag{67}$$

$$\omega_j^{(2)}(t) = \int_0^t k_4(t,\tau)l_{N,j}(\tau)d\tau, \tag{68}$$

$$\omega_j^{(3)}(t) = k_5(t,t_j)\int_0^t p(t,\tau)l_{N,j}(\tau)d\tau. \tag{69}$$

Now we are ready to give the convergence of product integration method.

3.1. Convergence of Product Integration for Solving System of Weakly Singular Integral Equations

In our convergence analysis we examine the linear test equation:

$$U(t) = F(t) + \int_0^t p(t,\tau)U(\tau)d\tau \qquad t \in [0,T], \tag{70}$$

where

$$U(t) = \begin{bmatrix} \delta(t) \\ \gamma(t) \end{bmatrix}, \quad F(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}.$$

And assume that the forcing function $g \in C[0,T]$ And p is defined by (51) then the test equation (70) has a unique solution $U \in C[0,T] \times C[0,T]$ That may be expected to have unbounded derivatives at the end point $t = 0$.

If, for a given mesh $\{t_j\}_{j=0}^{N}$ we apply the method of Section 3 to the test equation (70) and obtain as approximate solution $U_N(t)$ the following Nystrom interpolant:

$$U_N(t) = F(t) + \sum_{j=0}^{N} \omega_j(p;t)U_N(t_j), \tag{71}$$

where

$$\omega_j(p;t) = \int_0^t p(t,\tau)l_{N,j}(\tau)d\tau.$$

In order to examine the uniform convergence of the approximate solution $U_N(t)$ to the exact solution $U(t)$ of (70) notice that.

$$U(t) - U_N(t) = \sum_{j=0}^{N} \omega_j(p;t) \times \{U(t_j) - U_N(t_j)\} + t_N(p,U,t). \tag{72}$$

Where $t_N(p,U,t)$ is the local truncate error defined by

$$t_N(p,U,t) = \int_0^t p(t,\tau)U(\tau)d\tau - \sum_{j=0}^{N} \omega_j(p;t)U(t_j). \tag{73}$$

Hence we obtain

$$\|U - U_N\|_\infty \le \left\|(I - A_N)^{-1}\right\|_\infty \|t_N\|_\infty . \quad (74)$$

where A_N is the linear operator defined by

$$A_N : X := C[0,T] \times C[0,T] \to X ,$$

$$A_N U(t) = \sum_{j=0}^{N} \omega_j(p;t) U(t_j),\ U \in X, T \in [0,T] \quad (75)$$

First we investigate the convergence properties of the underling product quadrature rule.

Lemma 1. Let $\{p_i\}_{i=1}^{N}$ is a sequence of orthogonal polynomials on $[-1,1]$ with respect weight function $\omega(t)$, then $\{q_i\}_{i=1}^{N}$ is a sequence of orthogonal polynomials on $[a,b]$ with respect weight function $\tilde{\omega}(t)$ where.

$$q_i(t) = p_i\left(\frac{2}{b-a}\left[t - \frac{b+a}{2}\right]\right),\ t \in [a,b], \quad (76)$$

$$\tilde{\omega}(t) = \omega\left(\frac{2}{b-a}\left[t - \frac{b+a}{2}\right]\right),\ t \in [a,b]. \quad (77)$$

Proof. The proof of this lemma is easy and refer that the reader verify it.

Theorem 3. Let $\{t_j\}_{j=0}^{N}$ be the zeros of the $(N+1)$ st-degree member of a set of polynomials that are orthogonal on $[0,T]$ with respect to the weight function.

$$\omega(t) = u\left(\frac{2t}{T} - 1\right)\left(2 - \frac{2t}{T}\right)^{\alpha}\left(\frac{2t}{T}\right)^{\beta}, \quad -1 < \alpha \le \frac{3}{2}, \beta > -\frac{1}{2}. \quad (78)$$

Here $u(t)$ is positive and continuous in $[0,T]$ and the modulus of continuity φ of u satisfies $\int_0^1 \varphi(u,\delta)\frac{d\delta}{\delta} < \infty$ Let $L_N(U;t)$ denote the vector of interpolating polynomial of degree $\le N$ that coincides with the vector function $U(t) = (\delta(t), \gamma(t))^T$ at the nodes $\{t_j\}_{j=0}^{N}$. Then for every vector function U with

$U(.)(.)^{-\sigma} \in X := C[0,T] \times C[0,T], \sigma > -\frac{1}{2}$ (not an integer) there holds

$$\lim_{N \to \infty} \|t_N(p,U,t)\|_\infty = 0 \quad (79)$$

In particular we have the bounds

$$\left\|t_N\left(|t - T|^{-\frac{1}{2}}, U, t\right)\right\|_\infty = O\left\{(N+1)^{-2\sigma-1} Log(N+1)\right\} \quad (80)$$

Proof. Note that for all

$$U \in X := C[0,T] \times C[0,T], U(t) = (u_1(t), u_2(t))^T$$

$$\| U \|_\infty = \max\left\{\| u_1\|_\infty, \| u_2\|_\infty\right\} \quad (81)$$

And apply relations (14),(15) in theorem 1 of [8] in the vector case. Apply Lemma 1 for balance of interval of orthogonality. The bound (80) is an immediate consequence of theorem 5 in [2]. □

Now we investigate the behavior of the first term $\left\|(I - A)^{-1}\right\|_\infty$ in the right hand side of (74).

Theorem 4. Let the operator A_N be defined as in (75) and the nodes $\{t_j\}_{j=1}^{N}$ chosen as in theorem 3. Then for all N sufficiently large, there exist a constant $C > 0$ independent of N such that

$$\left\|(I - A)^{-1}\right\|_\infty \le C. \quad (82)$$

Proof. Conditions of lemmas 1,2 of [8] are satisfy and hence by theorem 2 of [8] the result arrive.

Theorem 5. Let U be the exact solution of the equation (70). Let U_N be the approximate solution obtained by discretizing the integral term of (70) by a product quadrature rule of interpolatory type constructed on a set of distinct nodes $\{t_j\}_{j=1}^{N}$. If the nodes $\{t_j\}_{j=1}^{N}$ are the zeros of the $(N+1)$ st-degree member of a set of polynomials the weight function (78) with $-\frac{1}{2} < \alpha, \beta < \frac{3}{2}$ then U_N converges uniformly to U. Moreover, the rate of convergence of U_N to U coincides with the one of the basic quadrature rule we choose to approximate the integral term of (70).

The proof follows immediately from the estimate (74) together with Theorems 3 and 4. The bound (80) supply an estimate of the rate of convergence.

4. Numerical Examples

Substituting $v = e^{\gamma t}u$, in $u_t = u_{xx} - \gamma u$ yields $v_t = v_{xx}$, since

$$v_t = \gamma e^{\gamma t}u + e^{\gamma t}u_t = e^{\gamma t}(\gamma u + u_t) = v_{xx},$$

hence we can put $\gamma = 0$. Now consider the following test problem.

$$u_t = u_{xx} \quad -\infty < x < s(t), \tag{83}$$

$$\left(\frac{\partial u}{\partial x}\right)|x = s(t) = -V(t) \tag{84}$$

$$u(x,0) = \exp(\alpha x). \quad -\infty < x < 0, \quad \alpha > 0, \tag{85}$$

$$g(u\big|_{x=s(t)}) = V(t), \tag{86}$$

where

$$g(t) = \exp(-t) - \alpha - \frac{1}{e}. \tag{87}$$

It is not difficult to verify that for $\alpha > 0$ the functions $s(t) - \alpha t, u(x,t) = \exp(\alpha x + a^2 t)$ are an exact solution of test problem.

Without loss of generality we can suppose $\alpha = 1$. For this problem we can write (61) in the form of linear system,

$$AX = b, A = (a_{ij}) \in \mathbb{R}^{(2N+2)\times(2N+2)},$$

$$X, b \in \mathbb{R}^{(2N+2)\times 1},$$

For an arbitrary $i, j \in \{1, 2, ..., N+1\}$ we have

$$a_{ii} = 1 + g_1(t_{i-1})\int_0^{t_{i-1}} l_{N,i-1}(\tau)d\tau$$

$$+\int_0^{t_{i-1}} g_2(t_{i-1},\tau)l_{N,i-1}(\tau)d\tau,$$

$$a_{ij} = g_1(t_{i-1})\int_0^{t_{i-1}} l_{N,j-1}(\tau)d\tau$$

$$+\int_0^{t_{i-1}} g_2(t_{i-1},\tau)l_{N,j-1}(\tau)d\tau, i \neq j,$$

$$a_{i,j+N+1} = g_7(t_{i-1},t_{j-1})$$

$$\times\int_0^{t_{i-1}} p(t_{i-1},\tau)l_{N,j-1}(\tau)d\tau,$$

$$a_{i+N+1,j} = g_4(t_{i-1})\int_0^{t_{i-1}} l_{N,j-1}(\tau)d\tau$$

$$+\int_0^{t_{i-1}} g_5(t_{i-1},\tau)l_{N,j-1}(\tau)d\tau,$$

$$a_{i+N+1,i+N+1} = 1 + g_8(t_{i-1},t_{i-1})$$

$$\times\int_0^{t_{i-1}} p(t_{i-1},\tau)l_{N,i-1}(\tau)d\tau,$$

$$a_{i+N+1,j+N+1} = g_8(t_{i-1},t_{j-1})$$

$$\times\int_0^{t_{i-1}} p(t_{i-1},\tau)l_{N,j-1}(\tau)d\tau, i \neq j,$$

$$X = \left(\delta_{N,0},...,\delta_{N,N},\gamma_{N,0},...,\gamma_{N,N}\right)^T,$$

$$b = \left(r_1(t_0),...,r_1(t_N),r_2(t_0),...,r_2(t_N)\right)^T.$$

Now we evaluate the exact φ for the test problem corresponding to $\alpha = 1$. For this problem we have

$$u(x,t) = \exp(x+t), s(t) = -t,$$
$$u^0(x) = \exp(x). \tag{88}$$

Substituting this values in (5) tends to

$$2\sqrt{\pi}\exp(x+t) = \int_0^t \frac{1}{\sqrt{t-\tau}}\exp\left\{\frac{-(x+\tau)^2}{4(t-\tau)}\right\}$$

$$\times\varphi(\tau)d\tau + \frac{1}{\sqrt{t}}\int_{-\infty}^0 \exp\left\{\xi - \frac{(x-\xi)^2}{4t}\right\}d\xi. \tag{89}$$

For $x = s(t)$ in (89) we have

$$2\sqrt{\pi} - \frac{1}{\sqrt{t}}\int_{-\infty}^0 \exp\left\{\xi - \frac{(t+\xi)^2}{4t}\right\}d\xi$$

$$= \int_0^t \frac{1}{\sqrt{t-\tau}}\exp\left\{\frac{-(t-\tau)}{4}\right\}\times\varphi(\tau)d\tau. \tag{90}$$

If we define

$$\mu(t) := \frac{1}{\sqrt{t}}\exp\left(-\frac{t}{4}\right), \tag{91}$$

Then we can write

$$(\mu * \varphi)(t) = \sqrt{\pi}\left(1 + erf\left(\frac{\sqrt{t}}{2}\right)\right). \tag{92}$$

Now we take Laplace transform from (92) and obtain

$$L\{\varphi(t)\} = \frac{1}{2s}\left\{1 + \sqrt{1+4s}\right\}. \tag{93}$$

And hence we obtain the exact φ for our problem

$$\varphi(t) = \frac{1}{2}\left\{1 + \frac{2e^{-t/4}}{\sqrt{\pi t}} + erf\left(\frac{\sqrt{t}}{2}\right)\right\}. \tag{94}$$

4.1. Discussion and Conclusion

In the test problem we set $b=0.01, N=3$. The nodal points are zeros of

$$q_8(t)=1-72b^{-1}t+1260b^{-2}t^2-9240b^{-3}t^3$$
$$+34650b^{-4}t^4-72072b^{-5}t^5+84084b^{-6}t^6$$
$$-51480b^{-7}t^7+12870b^{-8}t^8,$$

where $\{q_n\}_{n=0}^{\infty}$ are orthogonal with respect $\omega(t)=1$ on $t\in[0,b]$. Suppose $\psi(.)=\sqrt{.}\varphi(.)$ then ψ is continuous [3] and (94) yield

$$\psi(t)=\frac{\sqrt{t}}{2}\left\{1+erf\left(\frac{\sqrt{t}}{2}\right)\right\}+\frac{2e^{-t/4}}{\sqrt{\pi}}. \quad (95)$$

Table1. Crude data

i	$\lvert V(t_i)-\tilde{V}(t_i)\rvert$	$\lvert\psi(t_i)-\tilde{\psi}(t_i)\rvert$
1	0.001	0.000858958
2	0.001	0.000717929
3	0.001	0.000576911
4	0.001	0.000435904
5	0.001	0.00029491
6	0.001	0.000153927
7	0.001	0.0000129561
8	0.001	0.000128003
9	0.001	0.000268951
10	0.001	0.000409887

Table 2. Data after one step of Newton's method

i	$\lvert V(t_i)-\tilde{V}(t_i)\rvert$	$\lvert\psi(t_i)-\tilde{\psi}(t_i)\rvert$
1	2.15257×10^{-6}	9.29878×10^{-6}
2	2.96036×10^{-6}	0.0000131859
3	3.57996×10^{-6}	0.0000161803
4	4.11035×10^{-6}	0.0000186981
5	4.58619×10^{-6}	0.000020923
6	5.01559×10^{-6}	0.0000229382
7	5.41556×10^{-6}	0.0000247808
8	5.79135×10^{-6}	0.0000264993
9	6.14684×10^{-6}	0.0000281118
10	6.48531×10^{-6}	0.0000296337

For this problem the exact value of V is $V(t)=1$. Now with initial guess

$$\psi_0(t)=\frac{1}{\sqrt{\pi}}+\frac{1}{2}\sqrt{t}+0.001, \quad (96)$$

$$V_0(t)=-1-0.001, \quad (97)$$

the absolute errors of the solutions using original data at points $t_i=0.001i, i=1,2,...,10$ are given in Table 1. Table 2 gives the same quantities using $\tilde{V}$ and $\tilde{\psi}$ 1-step approximated values of V and ψ respectively.

First of all Table 1 shows the precision of the method is considerable so it can be applied to many practical problems. Secondly we show in Table 2 that further improvements in precision are possible by using better approximate values for V and ψ. Therefore this method can be applied to wide range of problems in different applications.

Acknowledgments

The authors are thankful to the referees for their valuable suggestions to improve this paper.

References

1. Atkinson, K. and Han, W. Theoretical numerical analysis: A functional analysis framework, Springer (2001).
2. Criscuolo, G., Mastroianni, G. and Monegato, G. Convergence properties of a class of product formulas for weakly singular integral equations, Math. Comp. **55:** 213-230 MR 90m:65230, (1990).
3. Frankel, M. and Roytburd, V. Compact attractors for a Stefan problem with kinetics, EJDE **15:** 1-27 (2002).
4. Golberg, S. M. and Tribelskii, M. I. On laser induced evaporation of nonlinear absorbing Media, Zh. Tekh. Fiz. (Sov. Phys.-J. Tech. Phys) **55**: 848-857 (1985).
5. Langer, J. S. Lectures in the theory of pattern formation, in: Chance and Matter, J. Souletie, J. Vannimenus and R. Stora, eds., Elsevier Science Publishers (1987).
6. Matkowsky, B. J. and Sivashinsky, G. I. Propagation of a pulsating reaction front in solid fuel combustion, SIAM J. Appl. Math. **35:** 230-255 (1978).
7. Munir, Z. A. and Anselme-Tamburini, U. Self-propagation exothermic reactions: the synthesis of high-temperature materials by combustion, Mater. Sci. Rep. **3:** 277-365 (1989).
8. Orsi, A. P. Product integration for Volterra integral equations of the second kind with weakly singular kernels, Math. Comp. **65**: 1201-1212 (1996).
9. Weeks, J. and Van Saarloos, W. Surface undulations in explosive crystallization: a nonlinear analysis of a thermal instability, Physica D **12:** 279-294 (1984).