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Strong Topological Regularity and Weak Regularity of Banach Algebras

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Abstract

In this article we study two different generalizations of von Neumann regularity, namely strong topological regularity and weak regularity, in the Banach algebra context. We show that both are hereditary properties and under certain assumptions, weak regularity implies strong topological regularity. Then we consider strong topological regularity of certain concrete algebras. Moreover we obtain the following non-commutative analog of a result of Kaplansky. A bounded operator T on a Banach space X whose point spectrum $\sigma_p(T)$ contains a nonzero complex number, is weakly regular.

Keywords: Strongly topologically regular; Weakly regular

Introduction

The notion of weak regularity was introduced by Kaplansky [5] where he showed that the uniform algebra $C(X)$ is weakly regular if and only if every open set in X contains a clopen set. Moreover he showed that every von Neumann regular Banach algebra is finite dimensional. Every semisimple compact Banach algebra is weakly regular. This fact which can be considered as a noncommutative analog of Kaplansky's result follows immediately from [1, Theorem 33.14]. In particular the algebra $K(X)$ of compact operators on the Banach space X is weakly regular. See also [2] for some recent results in this direction. These results have motivated us to run further investigation of the properties of weakly regular Banach algebras and also topological variants of von Neumann regularity which might enforce finite dimensionality.

In this paper we introduce the notion of strong topological regularity as a topological analog of the notion of strong von Neumann regularity of Kaplansky [6], in the category of Banach algebras, and we provide a criterion for this concept. We identify some of algebraic and hereditary properties of such algebras. We show that for a locally compact group G if the measure algebra $M(G)$ or $L^1(G)^{**}$ is strongly topologically regular, then G is discrete. We also characterize strongly topologically regular triangular Banach algebras. Then we study weakly regular elements of a Banach algebra. In particular we extend [1, Theorem 33.14] to those elements of $B(X)$ whose point spectrum contain a nonzero complex number. Moreover we show that under certain conditions, weak regularity implies strong topological regularity.

The following are some of the terminology which are used in this article.

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Throughout A and B are Banach algebras. The left annihilator [resp. right annihilator/ annihilator] of a subset E of A which is denoted by $\ell an(E)$ [resp. $ran(E)/ann(E)$] is the set of all $x \in A$ where $xE = 0$ [resp. $Ex = 0 / Ex = xE = 0$]. When $E = \{a\}$, we denote $\ell an(E)$ [resp. $ran(E)/ann(E)$] simply by $\ell an(a)$ [resp. $ran(a)/ann(a)$]. Recall that A is called reduced if for every $a \in A$, the identity $a^2 = 0$ implies $a = 0$, A is called semiprime, if $\{0\}$ is the only ideal I of A with $I^2 = \{0\}$. If for every $a \in A$ the mapping $x \mapsto axa$ is a compact linear operator, then A is called a compact Banach algebra.

An element $a \in A$ is called von Neumann regular [resp. strongly von Neumann regular] if $a \in aAa$ [resp. $a \in a^2A$]. By [6, Theorem 3.2] the definition of strong von Neumann regularity is symmetric in the sense that $a \in a^2A$ if and only if $a \in Aa^2$. A is von Neumann regular [resp. strongly von Neumann regular] if every $a \in A$ is von Neumann regular [resp. strongly von Neumann regular]. By [7, Theorem 2.4.23] if $a \in A$ is von Neumann regular, then the principal left [resp. right] ideal generated by a has an idempotent generator and hence is closed and complemented.

Results

Strong Topological Regularity

In this subsection we study a natural topological analog of the concept of strong von Neumann regularity.

Definition 1. An element $a \in A$ is called strongly topologically von Neumann regular if $a \in \overline{a^2A}$ where the clousre is taken in the norm topology. A is called strongly topologically von Neumann regular if every element of A is strongly topologically von Neumann regular.

From now on by the suffix "regular" we mean " von Neumann regular".

In the following proposition whose proof is not difficult and is left to the reader, we have collected some basic facts about strong topological regularity.

Proposition 2. Suppose $B_\lambda (\lambda \in \Lambda)$ and B are Banach algebras and A is strongly topologically regular.

(i) If $\varphi : A \to B$ is a continuous epimorphism, then B is strongly topologically regular.

(ii) If $\oplus_\infty B_\lambda$ is strongly topologically regular, then every B_λ is strongly topologically regular.

(iii) The ℓ^1-direct sum $\oplus_1 B_\lambda$ is strongly topologically regular if and only if every B_λ is strongly topological ly regular.

(iv) The unitization $A^\#$ of A is strongly topologically regular if and only if A is.

In the following theorem we provide a sufficient condition for strong topological regularity. Moreover we show that strong topological regularity is a hereditary property.

Theorem 3. Suppose A is strongly topologically regular. Then

(i) For every pair of ideals I and J we have $\overline{IJ} = \overline{I} \cap \overline{J}$.

(ii) A is reduced and semiprime.

(iii) If I is a closed right ideal of A, then I is strongly topologically regular.

(iv) If I is a closed ideal of A, then $\frac{A}{I}$ is strongly topologically regular.

Proof. (i) Suppose A is strongly topologically regular and I, J are as above. It suffices to show that $\overline{I} \cap \overline{J} \subseteq \overline{IJ}$ since the inclusion $\overline{IJ} \subseteq \overline{I} \cap \overline{J}$ holds trivially. Let $a \in \overline{I} \cap \overline{J}$. Then there exists a sequence $\{x_n\}$ in A such that $a^2 x_n \to a$. Thus $a \in \overline{\overline{I}\,\overline{J}} \subseteq \overline{IJ}$.

(ii) Let I be an ideal in A, such that I^2=0. By the fist part, $\overline{I^2} = \overline{I}$. So $I = 0$ and hence A is semiprime. Let $a \in A$, such that $a^2 = 0$. Since A is strongly topologically regular, then there exists a sequence $\{x_n\}$ in A such that $a^2 x_n \to a$. Since $a^2 = 0$, then $a = 0$. Therefore A is reduced.

(iii) Let $a \in I$. By assumption there is a sequence $\{x_n\}$ in A such that $a = \lim_n a^2 x_n$. So $a^2 = \lim_n a^3 x_n$ and hence

$$a = \lim_m a^2 x_m = \lim_m \lim_n a^3 x_n x_m \in \overline{a^2 I}$$

as $a^3 x_n x_m \in \overline{a^2 I}$ for all n and m. Therefore I is strongly topologically regular.

(iv) This statement follows from Proposition 2 (i).

Remark 4. (i) Assume L_a and R_a are left and right multiplication by an element a of A respectively. If a is strongly topologically regular then L_a and R_a are strongly topologically regular elements of $B(A)$.

(ii) In every $*$-Banach algebra B with continuous involution, $b \in B$ is strongly topologically regular if and only if b^* is strongly topologically regular.

In the next theorem G is a locally compact group and $L^1(G)^{**}$ is equipped with the first Arens product. As usual $LUC(G), M(G)$ and $M_c(G)$ denote the Banach algebras of left uniformly continuous functions, bounded regular Borel measures, and continuous measures on G respectively. It is well known that $M_c(G)$ is a closed ideal of $M(G)$.

Theorem 5. Suppose G is metrizable. If $L^1(G)^{**}$ or $M(G)$ is strongly topologically regular, then G is discrete.

Proof. First we show that strong topological regularity of $L^1(G)^{**}$ implies that of $M(G)$. Let E be a right identity for $L^1(G)^{**}$ with E = 1. Then by Theorem 3(iii) $EL^1(G)^{**}$ is strongly topologically regular and hence so is $LUC(G)^*$, as it is isometrically isomorphic to $EL^1(G)^{**}$ (See the proof of the main result of [4]). But by [8, Lemma 4.1] $LUC(G)^*$ is isometrically algebra isometric to $C_0^{\perp}(G) \oplus M(G)$. Thus $M(G)$ is strongly topologically regular by Proposition 2 (iii).

Now suppose G is non-discrete and $M(G)$ is strongly topologically regular. By Theorem 3 (i) we have $\overline{M_c^2(G)} = M_c(G)$ which contradicts [3, Theorem 3.3.39]. Therefore G must be discrete.

Suppose A and B are Banach algebras and X is a Banach A, B-module. The triangular Banach algebra $T = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ is the set of two by two matrices $\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}$, where $a \in A$, $b \in B$ and $x \in X$, with usual matrix operations and ℓ^1-norm. In the next theorem we characterize strongly topologically regular triangular algebras.

Theorem 6. Let T be as above. Then T is strongly topologically regular if and only if A and B are strongly topologically regular and $X = 0$.

Proof. Suppose T is strongly topologically regular and $L = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$. Then L is a closed ideal of T with nilpotency index 2, that is, $L^2 = 0$. So $L = 0$ as T is semiprime by Theorem 3. Thus $X = 0$ and T turns into the ℓ^1-direct sum of A and B. Now the result follows from Proposition 2.

Conversely if A and B are strongly topologically regular and $X = 0$, then T is the ℓ^1-direct sum of A and B and hence by Proposition 2 is strongly topologically regular.

Recall that A is said to be an annihilator Banach algebra if for all closed left ideals L and all closed right ideals R of A the following statements hold.

(i) $\mathrm{r}anL = \{0\}$ if and only if $L = A$.

(ii) $\ell anR = \{0\}$ if and only if $R = A$.

Proposition 7. If A is an annihilator Banach algebra, then every element of A is either a zero divisor or strongly topologically regular.

Proof. Suppose $a \in A$ is not a zero divisor. Then so is a^4. If $b \in \ell an(\overline{a^2 A})$ then $ba^4 = 0$ implies that $b = 0$ as a^4 is not a zero divisor. Thus $\ell an(\overline{a^2 A}) = \{0\}$ and hence $\overline{a^2 A} = A$. Therefore $a \in \overline{a^2 A}$, that is, a is strongly topogically regular.

Weak Regularity

Recall that a Banach algebra is called weakly regular if for any $0 \neq a \in A$ there exists a non-zero $x \in A$ such that $xax = x$. Weak regularity is equivalent to the following condition: Every non-zero one sided ideal contains a non-zero idempotent. Every regular algebra is weakly regular. By [1, Theorem 33.14] every semisimple compact Banach algebra is weakly regular. In particular the algebra of compact operators on a Banach space is weakly regular. In the next theorem we extend this result to certain, not necessarily compact operators.

Theorem 8. Let X be a Banach space and $T \in B(X)$ be such that the point spectrum $\sigma_p(T)$ contains a nonzero complex number. Then T is weakly regular.

Proof. It is easy to see that T is weakly regular if and only if αT is weakly regular for every nonzero $\alpha \in \mathbb{C}$. Moreover if $0 \neq \lambda \in \sigma_P(T)$, then $1 \in \sigma_p(\lambda^{-1}T)$. So without loss of generality we can assume that $1 \in \sigma_p(T)$. By the functional calculus $1 \in \sigma_p(T^2)$. Let M be a non-zero finite dimensional subspace of the eigenspace $\ker(I - T^2)$ and P be the projection on M. Since $(I - T^2)P = 0$, then $P = T^2 P$. So if

$a = TP$, then $aTa = TPTTP = TP^2 = TP = a$. Thus T is weakly regular. Now for the general case, there is an $a \in A$ such that $a\lambda^{-1}Ta = a$ and hence $\left(\lambda^{-1}a\right)T\left(\lambda^{-1}a\right) = \lambda^{-1}a$.

Now we consider hereditary properties of weakly regular Banach algebras and some conditions which force a weakly regular Banach algebra to be strongly topologically regular.

Proposition 9. Suppose A is weakly regular.

(i) If I is a closed ideal of A, then I is weakly regular.

(iii) If A is weakly regular and $e \neq 0$ is an idempotent in A, then eAe is weakly regular.

Proof. (i) Let $a \in I$. By assumption there exists an $x \in A$ such that $xax = x$. Now $xax \in I$ and $(xax)a(xax) = xax$. Thus I is weakly regular.

(ii) Let $a \in eAe$. Then there exists $x \in A$ such that $xax = x$. Since $ae = a = ea$, then $exeaexe = exe$. Therefore a is weakly regular as an element of eAe.

Remark 10. If A is weakly regular, then every non-zero one sided ideal of A is not nilpotent and hence A is semiprime.

Theorem 11. If A is weakly regular, commutative and has a bounded approximate identity, then it is strongly topologically regular.

Proof. Let $a \in A$ and $B = \dfrac{A}{\overline{a^2A}}$. By Proposition 9, B is weakly regular and by the preceding remark it is semiprime. Now if $b \in B$ is such that $b^2 = 0$, then $(bB)^2 = 0$ and hence $bB = 0$. This implies that $b = 0$, since B has a bounded approximate identity. Therefore B is reduced. On the other hand $a^2 \in \overline{a^2A}$, as A is approximately unital. So if $b = a + \overline{a^2A}$, then $b^2 = 0$ and hence $b = 0$ as B is reduced. Therefore $a \in \overline{a^2A}$.

Example 12. (i) Assume L_a and R_a are left and right multiplication by an element a of A, respectively. If a is weakly regular then L_a and R_a are weakly regular.

(ii) In every $*$-Banach algebra, $a \in A$ is weakly regular if and only if a^* is weakly regular.

(iii) Let A be a C^*-algebra. If every positive element of A is weakly regular, then A is weakly regular. Since if $a \in A$, then there exists a non zero $x \in A$ such that $x = xa^*ax$. So $xa^* = \left(xa^*\right)a(xa^*)$ which means that a is weakly regular.

Discussion

In the present paper and in [3] we studied several weaker forms of von Neumann regularity to see whether they force the underlying algebra to be finite dimensional. However it seems that these assumptions are not that restrictive to lead to such a strong consequence. Besides there are examples of infinite dimensional weakly regular Banach algebras. Strong topological regularity is a topological generalization of von Neumann regularity, while weak regularity is an algebraic one. But both properties imply semiprimeness. We infer from theorems 5 and 8, that both of these properties are more consistent with the operator norm than the ℓ^1-norm and its extensions. The problem of weak regularity for arbitrary operators on a Banach space is still open.

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