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An Entanglement Study of Superposition of Qutrit Spin-Coherent States

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Abstract

Considering generalized concurrence as the criterion of entanglement, we study entanglement properties of superposition of two qutrit coherent states, as a function of their amplitudes. These states may attain maximum entanglement or no entanglement at all, depending on the choice of the parameters involved. The states revealing maximum entanglement also display the maximum violations of the Bell-CHSH inequalities.

Keywords: Spin Coherent State; Entanglement; Qutrit; Bell-CHSH inequality

Introduction

It is well known that entanglement is the cornerstone of several exciting non-classical phenomena including quantum computation, quantum teleportation and quantum cryptography [1-5]. Concurrence and its extension through the convex-roof method have been used to identify and measure the entanglement of two-dimensional bipartite systems [6-8]. Recent investigations have revealed that systems with higher dimensions may have advantages as regards channel capacities, security of quantum cryptography, quantum gate superiority and more efficient quantum information protocols [9-11]. This has prompted several investigations regarding qutrits (3-dimmensional systems) and their realization and manipulations [12-19]. On the other hand, it is already known that entangled coherent states [20-25] have applications in the domain of decoherence [26], quantum computation [27], quantum teleportation [28,29], interferometeric studies [30] and the test of the quantum non-locality [31, 32].

The above developments provide us the motivation to study the entanglement properties of superposition of two qutrits described in their relative coherent states, as a function of their parameters. Entanglement of two qubit-coherent states has also been studied by Berrada et al. [33]. The organization of the rest of this paper is as follows. The coherent state of a qutrit is introduced first. Next, generalized concurrence (I-concurrence) and its properties are considered. Finally superposition of two qutrit coherent states, its entanglement properties and results and discussion are presented in the last section.

Materials and Methods

Qutrit Coherent States

The Radcliffe spin coherent states are given by [34]

$$\left|\alpha, j\right\rangle = \frac{1}{(1+\left|\alpha\right|^2)^j} \sum_{m=-j}^{j} \binom{2j}{m+j}^{\frac{1}{2}} \alpha^{j+m} \left| j, m\right\rangle, \qquad (1)$$

where $\left| j, +m\right\rangle$ are the eigenvectors of the angular

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momentum operators $\hat{J}^2$ and $\hat{J}_z$ with eigenvalues equal to $j(j+1)$ and m, respectively. For $j=1$ we obtain

$$|\alpha,1\rangle = \frac{1}{(1+|\alpha|^2)}[(\frac{2!}{0!(2-0)!})^{\frac{1}{2}}|1,-1\rangle + (\frac{2!}{1!(2-1)!})^{\frac{1}{2}}\alpha|1,0\rangle + (\frac{2!}{2!(2-2)!})^{\frac{1}{2}}\alpha^2|1,1\rangle] \quad (2)$$

$$= \frac{1}{(1+|\alpha|^2)}[|-1\rangle + \sqrt{2}\alpha|0\rangle + \alpha^2|1\rangle],$$

where

$$|1,0\rangle = |0\rangle \ , \ |1,1\rangle = |1\rangle \ , \ |1,-1\rangle = |-1\rangle. \quad (3)$$

and

$$\alpha = e^{i\phi}\tan(\frac{\theta}{2}), \quad (4)$$

have been defined.

An unnormalized entangled pure state of two coherent states may be given by

$$|\psi\rangle' = \cos\theta(|\alpha\rangle\otimes|\beta\rangle) + e^{i\phi}\sin\theta(|\alpha'\rangle\otimes|\beta'\rangle)\ ; \quad (5)$$

defining $(|1\rangle\otimes|1\rangle = |11\rangle)$ and similar definitions for the other pair states and also substituting (2) in (5) we obtain

$$|\psi\rangle = \frac{1}{\sqrt{N}}[a|-1(-1)\rangle + b|-10\rangle + c|-1(1)\rangle + d|0(-1)\rangle + e|00\rangle + f|01\rangle + g|1(-1)\rangle + h|10\rangle + I|11\rangle], \quad (6)$$

where we have introduced

$$\lambda = \frac{\cos\theta}{(1+|\alpha|^2)(1+|\beta|^2)}\ ;\ \gamma = \frac{e^{i\varphi}\sin\theta}{(1+|\alpha'|^2)(1+|\beta'|^2)},$$

$$a = \lambda + \gamma,$$

$$b = \sqrt{2}(\beta\lambda + \beta'\gamma),$$

$$c = \beta^2\lambda + \beta'^2\gamma,$$

$$d = \sqrt{2}(\alpha\lambda + \alpha'\gamma)$$

$$e = 2(\alpha\beta\lambda + \alpha'\beta'\gamma)$$

$$f = \sqrt{2}(\alpha\beta^2\lambda + \alpha'\beta'^2\gamma),$$

$$g = \alpha^2\lambda + \alpha'^2\gamma,$$

$$h = \sqrt{2}(\alpha^2\beta\lambda + \alpha'^2\beta'\gamma),$$

$$I = \alpha^2\beta^2\lambda + \alpha'^2\beta'^2\gamma. \quad (7)$$

$$N = \langle\psi|\psi\rangle. \quad (8)$$

I-Concurrence for a Pure Two-Qutrit System

While concurrence has been extensively used as an entanglement measure in the case of bipartite qubit systems, I-concurrence has been introduced as an appropriate measure of entanglement in the case of higher dimensional systems including qutrits [35, 36]. For a pure state it is defined by

$$C(\psi) = \sqrt{2(1 - tr(\rho_r)^2)}, \quad (9)$$

where, ρ_r represents the density matrix of one subsystem (*A* or *B*), derived from the bipartite density matrix ρ by tracing out the other

$$\rho^A = tr_B\,\rho\ ;\ \rho^B = tr_A\,\rho.$$

Concurrence and I-concurrence match for the two level systems. The minimum for both measures is equal to zero, but the maximum value for the former is unity, while the latter can obtain a maximum value equal to $\sqrt{\frac{2(d-1)}{d}}$ for d-dimensional systems.

Results and Discussion

Entanglement Properties of Two Qutrits

The density matrix of our pure bipartite system may be expressed by

$$\rho = |\psi\rangle\langle\psi|\ ;$$

thus

$$\rho^A = tr_B\,\rho = \sum_{j=1}^{3}(I_A\otimes\langle\phi_{Bj}|)\rho(I_A\otimes|\phi_{Bj}\rangle), \quad (10)$$

where, I_A is the unit operator in the A-subspace and $\{\phi_{Bj}\}$ are the orthogonal bases in the B-subspace, whose matrix representations are given by

$$\left|-1\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};\ \left|0\right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};\ \left|1\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{11}$$

Using (10) we obtain

$$\rho^A = \frac{1}{N}\begin{pmatrix} A & B & C \\ B^* & D & E \\ C^* & E^* & F \end{pmatrix}, \tag{12}$$

where we have defined

$$A = aa^* + bb^* + cc^*\ ;\ B = ad^* + be^* + cf^*,$$
$$C = ag^* + bh^* + cI^*\ ;\ D = dd^* + ee^* + ff^*,$$
$$E = dg^* + eh^* + fI^*\ ;\ F = gg^* + hh^* + II^*. \tag{13}$$

We also find

$$tr(\rho^A)^2 = \frac{1}{N^2}[A^2 + D^2 + F^2 + 2|B|^2 + 2|C|^2 + 2|E|^2]. \tag{14}$$

To simplify our calculations we assume that the coherent state parameters are real and also satisfy the following relations

$$\alpha' = -\alpha\ ;\ \beta' = -\beta. \tag{15}$$

Now, substituting for (14) from (7), (13) and (15) we find

$$tr(\rho^A)^2 = \frac{1}{N^2}[|\lambda+\gamma|^4(1+\beta^8+2\beta^4+20\alpha^4\beta^4$$
$$+\alpha^8+\alpha^8\beta^8+2\alpha^8\beta^4+2\alpha^4+2\alpha^4\beta^8)$$
$$+|\lambda-\gamma|^4(4\beta^4+4\alpha^4+4\alpha^4\beta^8+16\alpha^4\beta^4+4\alpha^8\beta^4)$$
$$+\alpha^8\beta^8+2\alpha^8\beta^4+2\alpha^4+2\alpha^4\beta^8)$$
$$+|\lambda-\gamma|^4(4\beta^4+4\alpha^4+4\alpha^4\beta^8+16\alpha^4\beta^4+4\alpha^8\beta^4)$$
$$+|\lambda+\gamma|^2|\lambda-\gamma|^2(4\beta^2+4\beta^6+24\alpha^4\beta^2+24\alpha^4\beta^6$$
$$+4\alpha^8\beta^2+4\alpha^8\beta^6+4\alpha^2+24\alpha^2\beta^4+4\alpha^2\beta^8+24\alpha^6\beta^4$$
$$+4\alpha^6\beta^8+4\alpha^6)+\{(\lambda+\gamma)^2(\lambda^*-\gamma^*)^2+(\lambda-\gamma)^2$$
$$\times(\lambda^*+\gamma^*)^2(8\alpha^2\beta^2+8\alpha^2\beta^6+8\alpha^6\beta^2+8\alpha^6\beta^6)]. \tag{16}$$

Considering equations (7), it is clear that (16) is a function of the parameters α, β, θ and ϕ; maximizing (16) with respect to the parameters θ and ϕ we obtain the sets $[\theta = \frac{\pi}{4}, \phi = 0]$ and $[\theta = -\frac{\pi}{4}, \phi = 0]$, independent of the parameters α and β. For $[\theta = -\frac{\pi}{4}, \phi = 0]$ we find

$$tr(\rho^A)^2 = \frac{(\beta^4+\alpha^4+\alpha^4\beta^8+4\alpha^4\beta^4+\alpha^8\beta^4)}{(\beta^2+\alpha^2+\alpha^2\beta^4+\alpha^4\beta^2)^2},$$

and

$$C(\psi) = [2(1 - \frac{(\beta^4+\alpha^4+\alpha^4\beta^8+4\alpha^4\beta^4+\alpha^8\beta^4)}{(\beta^2+\alpha^2+\alpha^2\beta^4+\alpha^4\beta^2)^2})]^{\frac{1}{2}}. \tag{17}$$

Maximizing (17) with respect to α as a running parameter, we obtain

$$\alpha = \beta;\ \alpha = -\beta;\ \alpha = \frac{1}{\beta};\ \alpha = -\frac{1}{\beta}, \tag{18}$$

which correspond to the following states

$$|\psi_1\rangle = \frac{\sqrt{2}}{2}(|\alpha\rangle \otimes |\alpha\rangle) - \frac{\sqrt{2}}{2}(|-\alpha\rangle \otimes |-\alpha\rangle)$$

$$|\psi_2\rangle = \frac{\sqrt{2}}{2}(|\alpha\rangle \otimes |-\alpha\rangle) - \frac{\sqrt{2}}{2}(|-\alpha\rangle \otimes |\alpha\rangle); \tag{19a}$$

$$|\psi_3\rangle = \frac{\sqrt{2}}{2}(|\alpha\rangle \otimes |\alpha^{-1}\rangle) - \frac{\sqrt{2}}{2}(|-\alpha\rangle \otimes |-(\alpha^{-1})\rangle)$$

$$|\psi_4\rangle = \frac{\sqrt{2}}{2}(|\alpha\rangle \otimes |-(\alpha^{-1})\rangle) - \frac{\sqrt{2}}{2}(|-\alpha\rangle \otimes |\alpha^{-1}\rangle) \tag{19b}$$

and the maximum I-concurrence, $C_M(\psi_i) = 1$, for $i = 1, 2, 3, 4$ is obtained.

We have presented a three-dimensional plot of $C(\psi)$ as a function of the coherent state parameters α and β in Figure 1, using Mathematica. We have also displayed α versus β plots for the four equations in (18), which correspond to the maximum value of entanglement, in Figure 2. It is also observed that $C(\psi) = 0$ for $\alpha = 0$ and an arbitrary value of β, and vice versa; implying no entanglement at all.

For the set $[\theta = \frac{\pi}{4}, \phi = 0]$ we find

$$tr(\rho^A)^2 = \frac{(1+\beta^8)(1+\alpha^4)^2 + 2\beta^4(1+10\alpha^4+\alpha^8)}{(1+\beta^4+\alpha^4+\alpha^4\beta^4+4\alpha^2\beta^2)^2},$$

and

$$C(\psi)=[2(1-\frac{(1+\beta^8)(1+\alpha^4)^2+2\beta^4(1+10\alpha^4+\alpha^8)}{(1+\beta^4+\alpha^4+\alpha^4\beta^4+4\alpha^2\beta^2)^2})]^{\frac{1}{2}}. \quad (20)$$

Again maximizing with respect to α we find

$$\alpha=\pm(A'-B')^{\frac{1}{2}}\ ;\ \alpha=\pm(A'+B')^{\frac{1}{2}}, \quad (21)$$

where we have defined

$$A'=\frac{2\beta^2}{1+\beta^4}\ ;\ B'=\frac{\sqrt{2\beta^4-\beta^8-1}}{1+\beta^4}. \quad (22)$$

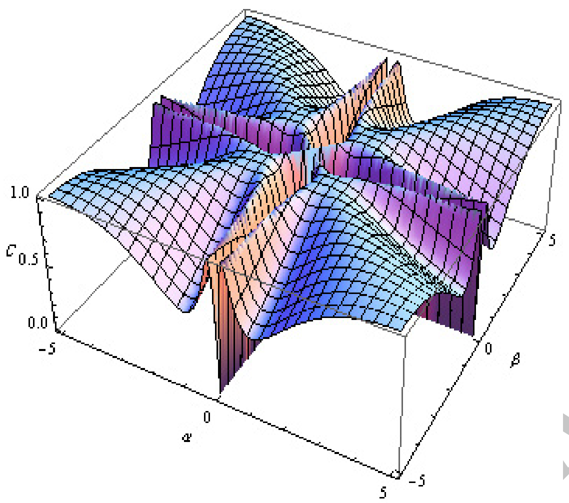

Figure 1. Plot of $C(\psi)$ as a function of the coherent state parameters α, β.

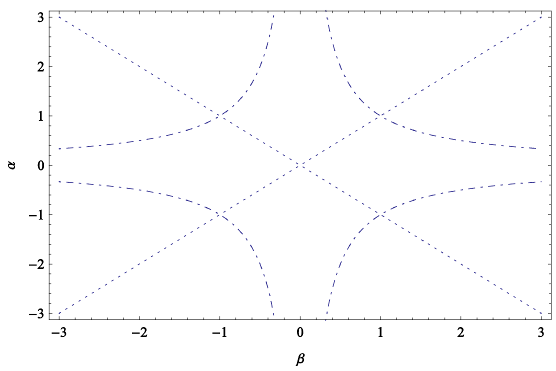

Figure 2. Plots of the equations $\alpha=\pm\frac{1}{\beta}$ (dashed-dotted line) and $\alpha=\pm\beta$ (dotted lines), which correspond to maximum entanglement.

Requiring real parameters, as we have assumed initially, the acceptable values for β are

$$\beta=\pm 1 \quad (23a)$$

leading to

$$\alpha=\pm 1. \quad (23b)$$

The corresponding entangled states are

$$|\psi_1'\rangle=\frac{\sqrt{2}}{2}(|1\rangle\otimes|1\rangle)+\frac{\sqrt{2}}{2}(|-1\rangle\otimes|-1\rangle),$$

$$|\psi_2'\rangle=\frac{\sqrt{2}}{2}(|-1\rangle\otimes|1\rangle)+\frac{\sqrt{2}}{2}(|1\rangle\otimes|-1\rangle)\ , \quad (24)$$

which also lead to the maximum concurrence for both states

$$C(\psi_1')=C(\psi_2')=1. \quad (25)$$

We note that equations (19a) at $\alpha=1$, along with equations (24), bring together a set of four Bell-like states; they not only display the maximum entanglement as we have shown in this work, but they also display the maximum violations of the Bell-CHSH inequalities, as has been demonstrated by Gerry et al. [37].

In summary, considering the superposition of qutrit spin coherent states, we have studied their entanglement properties as a function of their amplitudes, using I-concurrence as the measure of entanglement. It is observed that choosing appropriate parameter values, as expressed by equations (18) and (23), renders this system to attain maximum entanglement; while, for $\alpha=0$ and arbitrary β or $\beta=0$ and arbitrary α no entanglement is observed at all. Moreover, we introduced a set of four qutrit Bell-like states, as expressed by equations (19a) for $\alpha=1$ and (24) that not only display the maximum entanglement, but also show the maximum violations of the Bell-CHSH inequalities.

Our investigations in this work are limited to real and specific values of the coherence parameters and only three dimensional systems (qutrits) have been considered. Investigations along these lines, but with arbitrary values of the coherence parameters, including complex ones, and also considering higher dimensional systems could be pursued in the future.

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