Journal of Sciences, Islamic Republic of Iran 22(4): 355-361 (2011) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Frames and Homogeneous Spaces

A. Ghaani Farashahi[1,*] and R.A. Kamyabi-Gol[2]

[1]*Department of Mathematics, Faculty of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Islamic Republic of Iran*
[2]*Department of Mathematics, Faculty of Pure Mathematics, Ferdowsi University of Mashhad, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, Mashhad, Islamic Republic of Iran*

Received: 14 May 2011 / Revised: 23 November 2011 / Accepted: 15 December 2011

Abstract

Let G be a locally compact non-abelian group and H be a compact subgroup of G also let μ be a G-invariant measure on the homogeneous space G/H. In this article, we extend the linear operator $T_H:\mathcal{C}_c(G)\rightarrow\mathcal{C}_c(G/H)$ as a bounded surjective linear operator for all L^p-spaces with $p\geq 1$. As an application of this extension, we show that each frame for $L^2(G)$ determines a frame for $L^2(G/H,\mu)$ and each frame for $L^2(G/H,\mu)$ arises from a frame in $L^2(G)$ via the linear operator T_H.

Keywords: Homogeneous spaces; *G*-invariant measure; Bessel sequence; Frame; Voice (Wavelet) transform

Introduction

Essentially a homogeneous space X of a locally compact group G is a transitive G-space X. Due to Proposition 2.44 of [7], if G is σ-compact then each transitive G-space can be considered as a quotient space G/H for some closed subgroup H of G. Although G/H is not a group whenever H is not normal, but the classical harmonic analysis on G carries over on homogeneous spaces. Theory of classical harmonic analysis on coset space G/H was studied by several authors [7,14]. Many homogeneous spaces in mathematical physics such as the n-dimensional unit sphere, can be considered as a homogeneous space of the form G/H, where H is a compact subgroup of G.

Approximation theory in the function spaces of homogeneous spaces such as the Hilbert space $L^2(G/H,\mu)$ plays an important role in physics and also engineering. A useful functional analysis approach of the approximation theory is frame theory. In [6] Duffin and Schaeffer studied theory of frames for abstract Hilbert spaces. Frame theory can be considered as an advanced tool in wavelet theory, image and signal processing and approximation theory. For more on these applications we refer the readers to [2-4,9]. It is worthwhile to mentioned that the main technique of frame theory is to present any elements of the Hilbert

Subject Classification: Primary 43A85, 43A15.
* Corresponding author, Tel.: +98-9153599411, Fax: +98(511)8828606, E-mail: ghaanifarashahi@hotmail.com

space as a infinite linear combination of the frame elements. Theory of multiresolution analysis (MRA) for abstract Hilbert spaces has many connections with the frame theory of Hilbert spaces. Theory of multireslution analysis for $L^2(G)$ where G is a non-abelian, type I and unimodular group was studied in [15], also in the special case of LCA groups see [10].

In this article, as a main result we show that the linear operator $T_H : \mathcal{C}_c(G) \to \mathcal{C}_c(G/H)$ can be extended to a bounded surjective operator for all L^p-spaces with $p \geq 1$. It is also shown that, in this case if μ is a G-invariant measure on G/H the linear operator $T_H : L^2(G) \to L^2(G/H, \mu)$ is a partially isometric operator. As an application, we will show that via the linear operator T_H each frame for $L^2(G)$ determines a frame for $L^2(G/H,\mu)$ and conversely each frame for $L^2(G/H,\mu)$ arises from a frame in $L^2(G)$ via the linear operator T_H also as an another application of this extension, we find an interesting relation about the admissibility conditions associated to the voice transforms related to the left regular representation of G on $L^2(G)$ and the left regular representation of G on $L^2(G/H,\mu)$.

Let X be a locally compact Hausdorff space, $\mathcal{C}_c(X)$ be the space of all continuous complex valued functions on X with compact supports and also when μ is a positive Radon measure on X, for each $1 \leq p < \infty$ the Banach space of all equivalence classes of μ-measurable complex valued functions $f : X \to \mathbb{C}$ such that

$$\|f\|_p^p := \int_X |f(x)|^p \, dx < \infty,$$

is denoted by $L^p(X,\mu)$ which contains $\|.\|_p$-dense subspace $\mathcal{C}_c(X)$.

Let G be a locally compact group with identity e and left Haar measure dx, H a closed subgroup of G with the left Haar measure dh and also let Δ_G and Δ_H be the modular functions of G and H respectively. For $p \geq 1$ the notation $L^p(G)$ stands for $L^p(G,dx)$. For $x \in G$ and also a function $f : G \to \mathbb{C}$, the left translation $L_x f$ of f by x is defined by $L_x f(y) = f(x^{-1}y)$ and also the right translation $R_x f$ of f by x is defined via $R_x f(y) = f(xy)$ for $y \in G$. Also, the left coset space G/H is considered as a homogeneous space that G acts on it from the left and $q : G \to G/H$ is the surjective canonical mapping. More precisely we consider G/H as the left coset space of the closed subgroup G. Proposition 2.48 of [7] implies that $\mathcal{C}_c(G/H)$ consists of all functions $T_H(f)$, where $f \in \mathcal{C}_c(G)$ and

$$T_H(f)(xH) = \int_H f(xh)\,dh \tag{1}$$

Let μ be a Radon measure on G/H and $x \in G$. The translation μ_x of μ is defined by $\mu_x(E) = \mu(xE)$, for all Borel subsets E of G/H. The measure μ is called G-invariant if $\mu_x = \mu$, for all $x \in G$. It is well known that, the homogeneous space G/H admits a G-invariant measure μ if and only if $\Delta_G|_H = \Delta_H$, which satisfies the following generalized Mackey-Bruhat formula,

$$\int_{G/H} T_H(f)(xH)\,d\mu(xH) = \int_G f(x)\,dx. \tag{2}$$

The formula (2) is also known as the Weil's formula (see [7]). If μ is a G-invariant measure on G/H, then the surjective linear erator $T_H : \mathcal{C}_c(G) \to \mathcal{C}_c(G/H)$ is bounded in L^1-norms that is for all $f \in \mathcal{C}_c(G)$ we have $\|T_H(f)\|_1 \leq \|f\|_1$. Due to the boundedness, it can be extended to a bounded surjective linear operator from $L^1(G)$ onto $L^1(G/H,\mu)$.

Results

Throughout this article, we assume that H is a compact subgroup of a non-abelian locally compact group G with a normalized Haar measure dh. First we find a generalized notation of the linear operator T_H for other L^p-function spaces, with $p > 1$. It should be mentioned that when H is a compact subgroup of a locally compact group G, automatically we have $\Delta_G|_H = \Delta_H = 1$ and therefore, existence of a G-invariant measure on G/H is guaranteed.

Proposition 2.1. Let H be a compact subgroup of a locally compact group G and also let μ be a G-invariant measure on G/H. The linear operator

$T_H : \mathcal{C}_c(G) \to \mathcal{C}_c(G/H)$ has an extension to a bounded linear operator from $L^p(G)$ onto $L^p(G/H,\mu)$, for each $p \geq 1$.

Proof. We show that each $f \in \mathcal{C}_c(G)$ we have $\|T_H(f)\|_p \leq \|f\|_p$. Using compactness of H and also the Weil's formula (2) we get

$$\begin{aligned}
\|T_H(f)\|_p^p &= \int_{G/H} |T_H(f)(xH)|^p \, d\mu(xH) \\
&= \int_{G/H} \left| \int_H f(xh)dh \right|^p d\mu(xH) \\
&\leq \int_{G/H} \left(\int_H |f(xh)| dh \right)^p d\mu(xH) \\
&\leq \int_{G/H} \left(\int_H |f(xh)|^p \, dh \right) d\mu(xH) \\
&= \int_{G/H} \left(\int_H |f|^p (xh) dh \right) d\mu(xH) \\
&= \int_{G/H} T_H(|f|^p)(xH) d\mu(xH) \\
&= \int_G |f(x)|^p \, dx = \|f\|_p^p .
\end{aligned}$$

Now since T_H is bounded in L^p-norms and also it maps $\mathcal{C}_c(G)$ onto $\mathcal{C}_c(G/H)$, we can extend T_H into a bounded linear operator from $L^p(G)$ onto $L^p(G/H,\mu)$. □

Remark 2.2. Throughout this article, we still denote the extended linear operator in Proposition 3.1 by T_H. From now on, for all $p \geq 1$ by $T_H : L^p(G) \to L^p(G/H,\mu)$ we mean the mentioned extension of the bounded linear operator $T_H : \mathcal{C}_c(G) \to \mathcal{C}_c(G/H)$ according to Proposition 2.1. Thus, for all $p \geq 1$ we can fix the notation $\mathcal{J}^p(G,H)$ as follows

$$\mathcal{J}^p(G,H) := \{ f \in L^p(G) : T_H(f) = 0 \} \tag{3}$$

which is a closed subspace of $L^p(G)$. When $p = 2$, $\mathcal{J}^2(G,H)^\perp$ stands for the standard orthogonal completion of $\mathcal{J}^2(G,H)$ in $L^2(G)$. Since the linear operator T_H commutes with the left action of G we deduce that $\mathcal{J}^p(G,H)$ and also $\mathcal{J}^p(G,H)^\perp$ are invariant under left translation by elements of G.

In the next theorem we illustrate a worthwhile property of the linear operator T_H when $p = 2$. We recall that, when $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, a bounded linear operator $T : \mathcal{H} \to \mathcal{K}$ is called a partially isometric operator if and only if $\|Tx\|_{\mathcal{K}} = \|x\|_{\mathcal{H}}$ for all $x \in \ker(T)^\perp$.

Theorem 2.3. *Let H be a compact subgroup of a locally compact group G and also let μ be a G-invariant measure on G/H. The adjoint operator $T_H^* : L^2(G/H,\mu) \to L^2(G)$ of the bounded linear operator $T_H : L^2(G) \to L^2(G/H,\mu)$ is given via $\psi \mapsto T_H^*(\psi) := \psi^q$, where*

$$\psi^q(x) := \psi \circ q(x) = \psi(xH),$$

for all $x \in G$ and also T_H is a partially isometric operator.

Proof. Let μ be a G-invariant measure on G/H and $\psi \in L^p(G/H,\mu)$ also let $\psi^q := \psi \circ q$. Then, ψ^q belongs to $L^2(G)$. Because, due to the Weil's formula and also compactness of H we have

$$\begin{aligned}
\|\psi^q\|_2 &= \int_G |\psi^q(x)|^2 \, dx \\
&= \int_G |\psi^q|^2 (x) dx \\
&= \int_{G/H} T_H(|\psi^q|^2)(xH) d\mu(xH) \\
&= \int_{G/H} \int_H |\psi^q|^2 (xh) d\mu(xH) \\
&= \int_{G/H} \int_H |\psi^q(xh)|^2 \, d\mu(xH) \\
&= \int_{G/H} \int_H |\psi \circ q(xh)|^2 \, d\mu(xH)
\end{aligned}$$

$$= \int_{G/H}\int_{H} |\psi(xhH)|^2 \, d\mu(xH)$$

$$= \int_{G/H}\int_{H} |\psi(xH)|^2 \, d\mu(xH)$$

$$= \int_{G/H} |\psi(xH)|^2 \left(\int_H dh\right) d\mu(xH) = \|\psi\|_2 .$$

Now again using the Weil's formula, for each $f \in \mathcal{C}_c(G)$ we have

$$T_H^*(\psi), f_{L^2(G)} = \psi, T_H(f)_{L^2(G/H,\mu)}$$

$$= \int_{G/H} \psi(xH)\overline{T_H(f)(xH)} d\mu(xH)$$

$$= \int_{G/H} \psi(xH) T_H(\bar{f})(xH) d\mu(xH)$$

$$= \int_{G/H} \psi^q(x) T_H(\bar{f})(xH) d\mu(xH)$$

$$= \int_{G/H} T_H(\psi^q . \bar{f})(xH) d\mu(xH)$$

$$= \int_G \psi^q(x)\overline{f(x)} dx = \psi^q, f_{L^2(G)}.$$

Thus, by continuity we achieve $T_H^*(\psi), f_{L^2(G)} = \psi^q, f_{L^2(G)}$ for all $f \in L^2(G)$. Hence, that $T_H^*(\psi) = \psi^q$. A straightforward calculation implies $T_H T_H^*(\psi) = \psi$ for all $\psi \in L^2(G/H,\mu)$, which guarantees that $T_H = T_H T_H^* T_H$. By Theorem 2.3.3 of [12], T_H is a partial isometric operator. □

As an immediate consequence of Theorem 2.3 we have the following corollary.

Corollary 2.4. Let H be a compact subgroup of a locally compact group G and also let μ be a G-invariant measure on G/H. The following statements hold.

(1) $\mathcal{J}^p(G,H)^{\perp} = \{\psi^q : \psi \in L^2(G/H,\mu)\}$.

(2) *For all* $f \in \mathcal{J}^2(G,H)^{\perp}$ *and also each* $h \in H$ *we have* $R_h f = f$.

(3) *For all* $\psi \in L^2(G/H,\mu)$ *we have* $\|\psi^q\|_2 = \|\psi\|_2$.

(4) *For all* $f, g \in \mathcal{J}^2(G,H)^{\perp}$ *we have* $T_H(f), T_H(g)_{L^2(G/H,\mu)} = f, g_{L^2(G)}$.

We can also identify the structure of orthogonal projections onto $\mathcal{J}^2(G,H)$ and $\mathcal{J}^2(G,H)^{\perp}$ as follows.

Corollary 2.5. Let H be a compact subgroup of a locally compact group G and also let μ be a G-invariant measure on G/H also let $P_{\mathcal{J}^2(G,H)}$ and $P_{\mathcal{J}^2(G,H)^{\perp}}$ be the orthogonal projections onto the closed subspaces $\mathcal{J}^2(G,H)$ and $\mathcal{J}^2(G,H)^{\perp}$, respectively. Then, for all $f \in L^2(G)$ and also for a.e. $x \in G$ we have

$$P_{\mathcal{J}^2(G,H)^{\perp}}(f)(x) = T_H(f)(xH), \quad P_{\mathcal{J}^2(G,H)}(f)(x) = f(x) - T_H(f)(xH). \tag{4}$$

Proof. Let $f \in L^2(G)$. Obviously we have $f = T_H(f)^q + f - T_H(f)^q$. Due to Corollary 3.4 we have $T_H(f)^q \in \mathcal{J}^2(G,H)^{\perp}$ and also

$$T_H\left(f - T_H(f)^q\right) = T_H(f) - T_H(T_H(f)^q)$$

$$= T_H(f) - T_H(f) = 0.$$

Now, since decomposition of each $f \in L^2(G)$ as a sum of two elements in $\mathcal{J}^2(G,H)$ and $\mathcal{J}^2(G,H)^{\perp}$ is unique, we get $P_{\mathcal{J}^2(G,H)^{\perp}}(f) = T_H(f)^q$ and $P_{\mathcal{J}^2(G,H)}(f) = f - T_H(f)^q$. □

Remark 2.6. Let $\widetilde{T_H}$ be the restriction of the linear operator T_H to the closed subspace $\mathcal{J}^2(G,H)^{\perp}$. Then, $\widetilde{T_H} : \mathcal{J}^2(G,H)^{\perp} \to L^2(G/H,\mu)$ is a bijective bounded linear operator and so there exists a bounded operator $T_H^{\dagger} : L^2(G/H,\mu) \to L^2(G)$ such that $T_H T_H^{\dagger}(\psi) = \psi$, for all $\psi \in L^2(G/H,\mu)$ and also $T_H^{\dagger} T_H(f) = f$ for all $f \in \mathcal{J}^2(G,H)^{\perp}$.

In the sequel as an application, we will find a characterization for frames associate to the Hilbert space $L^2(G/H,\mu)$, where H is a compact subgroup of a locally compact group G and μ is a G-invariant measure on G/H. We recall that a sequence $\{f_n\}$ in a Hilbert space $\mathcal{H}$ is called a Bessel sequence with Bessel bound B for $\mathcal{H}$, if for each $f \in \mathcal{H}$ we have

$$\sum_n \left|\langle f_n, f\rangle_{\mathcal{H}}\right|^2 \leq B\|f\|_{\mathcal{H}}^2. \tag{5}$$

A Bessel sequence $\{f_n\}$ with the Bessel bound B is called a frame with frame pair bound (A,B) for $\mathcal{H}$, if each $f \in \mathcal{H}$ satisfies

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_n \left|\langle f_n, f\rangle_{\mathcal{H}}\right|^2 \leq B\|f\|_{\mathcal{H}}^2. \tag{6}$$

Using Corollary 5.3.2 of [5] we can deduce that, each frame for $L^2(G)$ determines a frame for $L^2(G/H,\mu)$ via the linear operator T_H.

Theorem 2.7. *Let H be a compact subgroup of a locally compact group G and also let μ be a G-invariant measure on G/H also let $\{f_n\}$ be a frame for $L^2(G)$ with frame pair bound (A,B). Then, $\{\psi_n := T_H(f_n)\}$ is a frame for $L^2(G/H,\mu)$ with frame pair bound (A,B).*

Proof. Due to Remark 2.6 and also Theorem 3.3 we have $\|T_H\| = \|T_H^\dagger\| = 1$. Now Corollary 5.3.2 of [5] implies that $\{T_H(f_n)\}$ is a frame for $L^2(G/H,\mu)$ with frame bounds (A,B). □

Next theorem can be considered as a converse of Theorem 2.7. But first we show that each Bessel sequence for $L^2(G/H,\mu)$ arises from a Bessel sequence in $L^2(G)$ via Theorem 2.7.

Proposition 2.8. *Let H be a compact subgroup of a locally compact group G and also let μ be a G-invariant measure on G/H. Every Bessel sequence for $L^2(G/H,\mu)$ arises from a Bessel sequence in $L^2(G)$ via Theorem 2.7.*

Proof. Let $\{\psi_n\}$ be a Bessel sequence for $L^2(G/H,\mu)$ with Bessel bound B. For each n let

$$f_n := T_H^*(\psi_n) = \psi_n^q.$$

Then, for each n we have $T_H(f_n) = \psi_n$ and also $f_n \in \mathcal{J}^2(G,H)^\perp$. Now, we show that $\{f_n\}$ is a Bessel sequence for $L^2(G)$ with Bessel bound B. Indeed, if $f \in L^2(G)$ is given, then

$$\begin{aligned}\sum_n \left|\langle f_n, f\rangle_{L^2(G)}\right|^2 &= \sum_n \left|\langle \psi_n^q, f\rangle_{L^2(G)}\right|^2 \\ &= \sum_n \left|\langle T_H^*(\psi_n), f\rangle_{L^2(G)}\right|^2 \\ &= \sum_n \left|\langle \psi_n, T_H(f)\rangle_{L^2(G/H,\mu)}\right|^2 \\ &\leq B\|T_H(f)\|_{L^2(G/H,\mu)}^2 \leq B\|f\|_{L^2(G)}^2.\end{aligned}$$

□

In the next theorem we prove that each frame for $L^2(G/H,\mu)$ arises from a frame for $\mathcal{J}^2(G,H)^\perp$.

Theorem 2.9. *Let H be a compact subgroup of a locally compact group G and also let μ be a G-invariant measure on G/H. Every frame for $L^2(G/H,\mu)$ arises from a frame for $\mathcal{J}^2(G,H)^\perp$ via Theorem 2.7.*

Proof. Let $\{\psi_n\}$ be a frame for $L^2(G/H,\mu)$ with frame bound pair (A,B). Proposition 2.8 guarantee that $\{\psi_n^q\}$ is a Bessel sequence for $L^2(G)$ with Bessel bound B and so that it is a Bessel sequence for $\mathcal{J}^2(G,H)^\perp$. Now we show that $\{\psi_n^q\}$ admits a lower frame bound and so that it is a frame for $\mathcal{J}^2(G,H)^\perp$. Using Corollary 2.4 for all $f \in L^2(G)$ we have

$$\begin{aligned}A\|P_{\mathcal{J}^2(G,H)^\perp}(f)\|_{L^2(G)}^2 &= A\left\|T_H\left(P_{\mathcal{J}^2(G,H)^\perp}(f)\right)\right\|_{L^2(G/H,\mu)}^2 \\ &= A\|T_H(f)\|_{L^2(G/H,\mu)}^2 \\ &\leq \sum_n \left|\langle \psi_n, T_H(f)\rangle_{L^2(G/H,\mu)}\right|^2 \\ &= \sum_n \left|\langle T_H^*(\psi_n), f\rangle_{L^2(G)}\right|^2 \\ &= \sum_n \left|\langle \psi_n^q, f\rangle_{L^2(G)}\right|^2 = \sum_n \left|\langle \psi_n^q, P_{\mathcal{J}^2(G,H)^\perp}(f)\rangle_{L^2(G)}\right|^2.\end{aligned}$$

□

In the following corollary, we show that if $\{\psi_n\}$ is a frame for $L^2(G/H,\mu)$, then $\{\psi_n^q\}$ is a frame for the Hilbert space $L^2(G)$ if and only if H is the identity group.

Corollary 2.10. *Let H be a compact subgroup of a locally compact group G and also let μ be a G-invariant measure on G/H. If $\{\psi_n\}$ is a frame for $L^2(G/H,\mu)$, then $\{\psi_n^q\}$ is a frame for the Hilbert space $L^2(G)$ if and only if H is the identity group.*

Proof. If H be the identity group, it is clear that the result holds. Now, let $\{\psi_n\}$ be a frame for $L^2(G/H,\mu)$ with frame bounds $0<A\le B<\infty$ such that $\{\psi_n^q\}$ be also a frame for $L^2(G)$ with frame bounds (C,D). Then, for each $f\in L^2(G)$ we have

$$CP_{\mathcal{J}^2(G,H)}(f)^2_{L^2(G)} \le \sum_n \left|\psi_n^q, P_{\mathcal{J}^2(G,H)}(f)_{L^2(G)}\right|^2 = 0$$

which implies that $\mathcal{J}^2(G,H)=\{0\}$ and equivalently H is the identity group. □

In spite of Corollary 2.10, we can still find a frame for $L^2(G)$ in which $\{\psi_n\}$ arises from it via Theorem 2.7.

Corollary 2.11. *Let H be a compact subgroup of a locally compact group G and also let μ be a G-invariant measure on G/H. Every frame for $L^2(G/H,\mu)$ arises from a frame for $L^2(G)$ via Theorem 2.7.*

Proof. Let $\{\psi_n\}$ be a frame for $L^2(G/H,\mu)$. Using Theorem 2.9, the sequence $\{\psi_n^q\}$ is a frame for $\mathcal{J}^2(G,H)^\perp$. Now let $\{g_n\}$ be an arbitrary frame for $\mathcal{J}^2(G,H)$, for instance it can be an ONB for $\mathcal{J}^2(G,H)$. According to Theorem 3.2 of [13], sequence $\{g_n\}\cup\{\psi_n^q\}$ is a frame for $L^2(G)$ with $T_H(\{g_n\}\cup\{\psi_n^q\})=\{\psi_n\}$. □

We can also use the preceeding results to justify admissibility condition for the left regular representation of G on the Hilbert space $L^2(G/H,\mu)$. A continuous unitary representation $(\pi,\mathcal{H}_\pi)$ of a locally compact group G is a homomorphism π from G into the group $\mathcal{U}(\mathcal{H}_\pi)$, the group of all unitary operators on the Hilbert space $\mathcal{H}_\pi$, which is continuous with respect to the strong (or weak) operator topology (see [7]). Let $(\pi,\mathcal{H}_\pi)$ be a continuous unitary representation of G and $\zeta\in\mathcal{H}_\pi$. The voice transform of $\xi\in\mathcal{H}_\pi$ generated by the representation π and also the parameter ζ is the complex valued function defined on G via

$$x \to V_\zeta\xi(x) = \xi,\pi(x)\zeta_{\mathcal{H}_\pi}.$$

The voice transform $V_\zeta:\mathcal{H}_\pi\to\mathcal{C}_b(G)$ is a bounded linear operator. But in general setting the voice transform is not square integrable. The continuous unitary representation $(\pi,\mathcal{H}_\pi)$ is called square integrable if for some non-zero vector $\zeta\in\mathcal{H}_\pi$ we have $V_\zeta\zeta\in L^2(G)$ and in this case the vector ζ is called admissible (see [15]). It is worthwhile to remember that many standard transformations in signal possessing can be deduced by the voice transform, for instance the affine wavelet transform or the Gabor transform are in fact special voice transforms [8, 14]. It should be noted that the term "Voice transform" in many references replaced by the "Continuous wavelet transform (CWT)"(see [1]).

We recall that the left regular representation $\varrho_G:G\to\mathcal{U}(L^2(G))$ of a locally compact group G is defined by

$$[\varrho_G(x)f](y)=[L_xf](y)=f(x^{-1}y). \qquad (7)$$

for all $f\in L^2(G)$. Also a function (vector) $f\in L^2(G)$ is called ϱ_G-admissible if and only if the function $x\mapsto L_xf,f_{L^2(G)}$ belongs to $L^2(G)$. If H is a closed subgroup of G and also μ is a G-invariant measure on G/H, the left regular representation of G on the Hilbert space $L^2(G/H,\mu)$ via $\varrho_{G/H}:G\to\mathcal{U}(L^2(G/H,\mu))$ is defined by

$$[\varrho_{G/H}(x)\psi](yH)=[L_x\psi](yH)=\psi(x^{-1}yH). \qquad (8)$$

for all $\psi\in L^2(G/H,\mu)$. A function (vector) $\varphi\in L^2(G/H,\mu)$ is called $\varrho_{G/H}$-admissible if and only if the function given by $x\mapsto L_x\varphi,\varphi_{L^2(G/H,\mu)}$ belongs to $L^2(G)$ (see [11]).

In the following theorem we show that for $f\in\mathcal{J}^2(G,H)^\perp$, ϱ_G-admissibility of f is equivalent to the $\varrho_{G/H}$-admissibility of $T_H(f)$.

Theorem 2.12. *Let H be a compact subgroup of a locally compact group G, μ be a G-invariant measure on G/H and $f \in \mathcal{J}^2(G,H)^\perp$. Then, $T_H(f)$ is $\varrho_{G/H}$-admissible if and only if f is ϱ_G-admissible.*

Proof. Let $f \in \mathcal{J}^2(G,H)^\perp$. Since $\mathcal{J}^2(G,H)^\perp$ is a left invariant subspace of $L^2(G)$ we get $L_x f \in \mathcal{J}^2(G,H)^\perp$ for all $x \in G$. Using Corollary 2.4 and the fact that left translations commute T_H, we have

$$\int_G \left| \langle L_x f, f \rangle_{L^2(G)} \right|^2 dx = \int_G \left| \langle T_H(L_x f), T_H(f) \rangle_{L^2(G/H,\mu)} \right|^2 dx$$
$$= \int_G \left| \langle L_x T_H(f), T_H(f) \rangle_{L^2(G/H,\mu)} \right|^2 dx,$$

which implies that $T_H(f)$ is $\varrho_{G/H}$-admissible if and only if f is ϱ_G-admissible. □

Corollary 2.13. *Let H be compact subgroup of a locally compact unimodular group G and also let μ be a G-invariant measure on G/H. Then, every $\varphi \in L^1(G/H,\mu) \cap L^1(G/H,\mu)$ is $\varrho_{G/H}$-admissible.*

Proof. Using unimodularity of G and due to Theorem 10.2 of [15], each f in $L^1(G) \cap L^2(G)$ is ϱ_G-admissible and so that each function in $L^1(G) \cap \mathcal{J}^2(G,H)^\perp$ is a ϱ_G-admissible vector. Now, if $\varphi \in L^1(G/H,\mu) \cap L^1(G/H,\mu)$ is arbitrary. Invoking Corollary 2.4, we achieve that $\varphi^q \in L^1(G) \cap \mathcal{J}^2(G,H)^\perp$. Then, Theorem 2.12 implies that $\varphi = T_H(\varphi^q)$ is a $\varrho_{G/H}$-admissible vector. □

As an immediate consequence we have the following corollary for compact groups.

Corollary 2.14. *Let H be compact subgroup of a compact group G and also let μ be a G-invariant measure on G/H. Then, every $\varphi \in L^1(G/H,\mu) \cap L^1(G/H,\mu)$ is $\varrho_{G/H}$-admissible.*

Acknowledgements

The authors would like to thank the referees for their valuable comments and remarks. We also would like to thank Prof. Feichtinger for stimulating discussions and pointing out various references to us.

References

1. Arefijamal, A. and Kamyabi-Gol, R. A Characterization of Square Integrable Representations Associated with CWT. J. Sci. Islam. Repub. Iran. **18**(2): 159-166 (2007).
2. Benedetto, J. and Powell, A. and Yilmaz, O. Sigm-Delta quantization and finite frames. IEEE Trans. Inform. Theory **52**: 1990-2005 (2006).
3. Bolcskel, H. and Hlawatsch, F. and Feichtinger, H. G. Frame-theoretic analysis of oversampled filter banks. IEEE Trans. Signal Process. **46**: 3256-3268 (1998).
4. Candas, E.J. and Donoho, D.L. New tight frames of curvelets and optimal representations of objects with piecewise C^2-singularities. Comm. Pure. Appl. Math. **56**: 216-266 (2004).
5. Christensen, Ole. Frames and Bases. An Introductory Course, Birkhäuser (2008).
6. Duffin, R.J. and Shaeffer, A.C. A class of non-harmonic Fourier series. Trans. Amer. Math. Soc. **72**: 341-366 (1952).
7. Folland, G.B. A Course in Abstract Harmonic Analysis, CRC press, (1995).
8. Fuhr, H. Abstract Harmonic Analysis of Continuous Wavelet Transforms (Lecture Notes in Mathematics), Springer (2005).
9. Heath, R. W. and Paulraj, A.J. Linear dispersion codes for MIMO systems based on frame theory. IEEE Trans. Signal Process. **50**: 2429-2441 (2002).
10. Kamyabi-Gol, R. A. and Raisi Tousi, R. Some equivalent multiresolution conditions on locally compact abelian groups, Proc. Indian Acad. Sci. Math. Sci. **120**(3): 317-331 (2010).
11. Kamyabi-Gol, R. and Tavallaei, N. Wavelet transforms via generalized quasi-regular representations. Appl. Comput. Harmon. Anal. **26**: 291–300 (2009).
12. Murphy, G.J. C*-Algebras and Operator theory. Academic Press, INC, (1990).
13. Pati, Y. C. Frames generated by subspace addition. Technical research report, (2006).
14. Reiter, H. and Stegeman, J. D. Classical Harmonic Analysis. 2nd Ed, Oxford University Press, New York, (2000).
15. Wong, M. W. Wavelet Transforms and Localization Operators. Operator Theory Advances and Applications, (2002).
16. Yang, Q. Multiresolution analysis on non-abelian locally compact groups. PhD Thesis, (1999).