Journal of Sciences, Islamic Republic of Iran 23(1): 59-65 (2012) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

B-Spline Solution of Boundary Value Problems of Fractional Order Based on Optimal Control Strategy

H. Azizi,[1] G.B. Loghmani,[1,*] M.R. Hooshmand Asl,[2] and A. Dehghan Nezhad[3]

[1]*Department of Applied Mathematics, Faculty of Mathematics, Yazd University, Yazd, Islamic Republic of Iran*
[2]*Department of Computer Science, Faculty of Mathematics, Yazd University, Yazd, Islamic Republic of Iran*
[3]*Department of Pure Mathematics, Faculty of Mathematics, Yazd University, Yazd, Islamic Republic of Iran*

Received: 10 April 2011 / Revised: 22 November 2011 / Accepted: 8 April 2012

Abstract

In this paper, boundary value problems of fractional order are converted into an optimal control problems. Then an approximate solution is constructed from translations and dilations of a B-spline function such that the exact boundary conditions are satisfied. The fractional differential operators are taken in the Riemann-Liouville and Caputo sense. Several example are given and the optimal errors are obtained for the sake of comparison. The obtained results are shown that the technique introduced here is accurate and easy to apply.

Keywords: B-spline functions; Fractional derivative; Fractional integral; Optimal control problem; Boundary value problems of fractional order

Introduction

Fractional calculus and fractional differential equations are considered as a part of classical calculus(refer to [15] or [17] a historical survey). They have been successfully applied to many fields, such as viscoelastic material, signal processing, control, quantum mechanics, meteorology, finance, life sciences [6, 9, 13, 15, 17, 18]. Many paper have focused on the analytical or the numerical study of fractional initial value problems [4, 5, 16, 18, 22]. Comparatively, little attention has been paid to the fractional boundary value problems. In this context, the existence of solution of the Strum-Liouville problem for an fractional differential equation and the Dirichlet-type fractional boundary value problems have been consider by Aleroev [1]. A class of fractional boundary value problems with Riemann-Liouville fractional derivatives, some kind of fractional boundary value problems with Caputo's derivatives, and a couple system of nonlinear fractional differential equations have been consider by Kilbas and Trujillo [11], Zhang [23], Bai and Lu [2], and Su [20] respectively; The least squares finite-element technique is employed by Roop and his coworkers to solve some kind of fractional boundary value problems [6, 7]; The Adomian decomposition method is used by Jafari and Daftardar-Gejji to find approximation and positive solution for a kind of fractional boundary value problems with Caputo's fractional derivative [10]. Moreover spline collocation method is used by Li and his coworkers for solving two-point boundary value problems of fractional differential equations [21].

In this paper we used least square method and B-spline functions for solving fractional boundary value problems that used by Loghmani for arbitrary-order problems with separated boundary conditions [12].

* Corresponding author, Tel.: +98(351)8210695, Fax: +98(351)8210695, E-mail: loghmani@yazduni.ac.ir

1. Preliminaries and Notations

In order to proceed, we need the following definitions of fractional derivatives and integrals. We first introduce the Riemann-Liouville definition of fractional derivative operator J_a^α.

Definition 2.1. Let $\alpha \in R^+$. The operator J_a^α, defined on the usual Lebesgue space $L_1[a,b]$ by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad (1)$$

$$J_a^0 f(x) = f(x),$$

for $a \le x \le b$, it is called the Riemann-Liouville fractional integral operator of order α.

Properties of the operator J_a^α can be found in [18]. For $f \in L_1[a,b]$, $\alpha, \beta \ge 0$ and $\gamma > -1$, we mention only the following:

(1) $J_a^\alpha f(x)$ *exists for almost every* $x \in [a,b]$,

(2) $J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x)$,

(3) $J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x)$,

(4) $J_a^\alpha (x-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a)^{\alpha+\gamma}$.

Definition 2.2. The fractional derivative of $f(x)$ in the Riemann-Liouville sense is defined as

$$D_a^\alpha f(x) = D^m J_a^{m-\alpha} f(x) = \frac{d^m}{dx^m}\frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1} f(t)dt, \quad (2)$$

where $m \in N$ and satisfies the relation $m-1 < \alpha \le m$, and $f \in L_1[a,b]$.

Properties of the operator D_a^α can be found in ([18], [19]). For $m-1 < \alpha, \beta \le m$, $x > a$ and $\gamma > -1$ we mention only the following:

(1) $D_a^\alpha (x-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(x-a)^{\gamma-\alpha}$,

(2) $D_a^\alpha J_a^\alpha f(x) = f(x)$.

The Riemann-Liouville derivatives have some certain disadvantages when we try to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by Caputo in his work on the theory of viscoelasticity [3].

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \quad (3)$$

$$D_*^\alpha J^\alpha f(x) = f(x),$$

for $m-1 < \alpha \le m$, $m \in N$, $x > 0$.

2. B-Spline Solution

The present work is to solve the more general form of fractional differential equations with boundary conditions. Consider the boundary fractional value problem

$$D_a^\alpha y(t) + Ly(t) + Ny(t) = f(t), \quad m-1 < \alpha \le m, \quad (4)$$

$$y(a) = \beta_1, \quad y(b) = \beta_2, \quad a \le t \le b, \quad (5)$$

where β_1 *and* β_2 are constants. The term $D_a^\alpha(t)$ denotes a linear fractional differential operator, $Ly(t)$ is linear ordinary differential operator, $Ny(t)$ is a nonlinear operator and $f(t)$ is a given function.

We define

$$G(y(t)) = D_a^\alpha y(t) + Ly(t) + Ny(t) - f(t) = 0,$$

with boundary condition (5).

We convert the problem to an optimal control problem

$$\min_y \|G(y(t))\|_{L^2[a,b]}^2 = \min_y \int_a^b (G(y(t)))^2 dt, \quad (6)$$

with boundary condition (5).

The actual solution of (4) and (5) is a function v such that

$$\begin{cases} \|G(v(t))\|_{L^2[a,b]}^2 = 0, \\ v(a) = \beta_1, \quad v(b) = \beta_2. \end{cases} \quad (7)$$

The sketch of the approximate solution is delineated as follow:

Consider B-spline function of order *p*, which due to iteration form

$$B_{i,0}(t) = \begin{cases} 1 & t_i < t \le t_{i+1} \\ 0 & 0.w \end{cases} \quad (8)$$

and if $l > 1$

$$B_{i,l}(t)=(\frac{t-t_i}{t_{i+l-1}-t_i})B_{i,l-1}(t)+(\frac{t_{i+l}-t}{t_{i+l}-t_{i+1}})B_{i+1,l-1}(t), \quad (9)$$

where $t_0,t_1,...,t_{n+1}$ is a non-decreasing sequence of knots and l is the order of the curve. Theses functions difficult to calculate directly for a general knot sequence. However, if the knot sequence is uniform (0,1,2,...n), it is quite straight forward to calculate these functions and they have some surprising properties.

For a fixed $k \in N$, consider an equal partition

$$a<a+h<a+2h<...<a+(p+1).2^{k-1}h=b$$

on $[a,b]$ where $h=\frac{b-a}{(p+1).2^{k-1}}$. Define

$$B_{ki}(t)=B(\frac{(p+1).2^{k-1}}{b-a}(t-a)-i), \quad i=-p,...,0,...,(p+1).2^{k-1}-1, \quad (10)$$

where B is a scaling function and $B_{ki}\,(k\in N, i=-p,...,...,(p+1).2^{k-1}-1)$ are translation and dilation of B as prescribe in [12].

We approximation the solution of boundary value problems of fractional order (4) and (5) by combination of the translation and dilation vertion of a B-spline function as follow:

$$v_k(t)=\sum_{i=-p}^{(p+1).2^{k-1}-1} c_i B_{ki}(t), \quad (11)$$

where the coefficients $\{c_i\}$ are determined from the condition $v_k(a)=\beta_1, v_k(b)=\beta_2$ and the following least square problem

$$\min_{c_i}\|G(v_k(t))\|^2_{L^2[a,b]}. \quad (12)$$

The minimization problem is equivalent to the following system which is called normal equation:

$$\begin{cases}\frac{\partial}{\partial c_i}\|G(v_k(t))\|^2_{L^2[a,b]}=0 \quad i=-p,...,0,...,(p+1).2^{k-1}-1,\\ v_k(a)=\beta_1, v_k(b)=\beta.\end{cases} \quad (13)$$

3. The Algorithm

In this section, we present the detailed steps of the our method:

step 1)

Choose a degree k and construct $B_{ki}(t)'s$, $i=-p,\cdots,0,\cdots,(p+1).2^{k-1}-1$, on interval $[a,b]$ then express the solution $v_k(t)$ as described in equation (11), section 3.

step 2)

Substitute the approximate solution $v_k(t)$ into the differential equation (4), to obtain the function $G(v_k(t))$.

step 3)

Substitute the approximate solution $v_k(t)$ into boundary conditions (5), to obtain the following equations:

$$\begin{cases}g_1(c_{-p},\cdots,c_0,\cdots,c_{(p+1).2^{k-1}-1})=v_k(a)-\beta_1=0,\\ g_2(c_{-p},\cdots,c_0,\cdots,c_{(p+1).2^{k-1}-1})=v_k(b)-\beta_2=0.\end{cases}$$

step 4)

Construct the error function G over $[a,b]$:

$$G(c_{-p},\cdots,c_0,\cdots,c_{(p+1).2^{k-1}-1})=\int_a^b\left(G(v_k(t))\right)^2dt.$$

step 5)

Find $c_i, i=-p,\cdots,0,\cdots,(p+1).2^{k-1}-1$, by solving the following optimization problem:

$$\min_{c_{-p},\cdots,c_0,\cdots,c_{(p+1).2^{k-1}-1}} G(c_{-p},\cdots,c_0,\cdots,c_{(p+1).2^{k-1}-1}),$$

with boundary conditions (13). This minimization problem can be solved by finding the solution of the following normal equations .

$$\begin{cases}\frac{\partial}{\partial c_i}G(c_{-p},\cdots,c_0,\cdots,c_{(p+1).2^{k-1}-1})=0,\\ g_1(c_{-p},\cdots,c_0,\cdots,c_{(p+1).2^{k-1}-1})=v_k(a)-\beta_1=0,\\ g_2(c_{-p},\cdots,c_0,\cdots,c_{(p+1).2^{k-1}-1})=v_k(b)-\beta_2=0.\end{cases}$$

step 6)

Form the approximate solution $v_k(t)=\sum_{i=-p}^{(p+1).2^{k-1}-1} c_i B_{ki}(t)$.

4. Analysis of the Method

For discussion about analysis of the method we denote that in equation (4) if put $\alpha=m$ then we obtain an ordinary diffferential equation. In [12] least square method has been applied for arbitrary-order problems with separated boundary conditions. In that paper by selection of $B_{k,i}(t)$ in the form (10) the numerical method find a sequence of functions $\{v_k\}$ of B-spline functions such that the exact boundary conditions are satisfied. Also, up to an error ε_k, the functions v_k satisfies the differential equation where $\varepsilon_k \to 0$ as $k \to +\infty$.

Therefore, convergence analysis of the method when $\alpha = m$ was described in [12] and for fractional differential equation is an open problem.

5. Numerical Examples

In this section, six problems will be tested using the mentioned method. The results of minimization problem have been obtained by MAPLE12. The least square errors ($LSE \;\; i.e. \int_a^b (y(t) - v_k(t))^2 dt$) in the analytical solution for test problems were calculated and are depicted in Tables 1-4. In example 6.4 our method have been compared with Haar wavlet method in [16].

Riemann-Liouville Fractional Derivative

Example 6.1. [21] Let us consider the linear boundary value problem of Riemann-Liouville fractional order

$$y''(t) + \sin t\, D^{0.5} y(t) + t y(t) = f(t), \quad 0 < t < 1, \quad (14)$$

with the boundary conditions:

$$y(0) = y(1) = 0,$$

where

$$f(t) = t^9 - t^8 + 56t^6 - 42t^5 + \sin t\left(\frac{32768}{6435\sqrt{\pi}} t^{7.5} - \frac{2048}{429\sqrt{\pi}} t^{6.5}\right), \quad (15)$$

and the exact solution is $y(t) = t^8 - t^7$.

We use the mentioned method and the following result has been obtained. Approximate solution for k=2 and p=3 is

$$y_2(t) = 0.000037 B_{2,-3}(t) + 0.000006 B_{2,-2}(t)$$
$$-0.000061 B_{2,-1}(t) - 0.000074 B_{2,0}(t) - 0.000505 B_{2,1}(t)$$
$$-0.003200 B_{2,2}(t) - 0.012373 B_{2,3}(t) - 0.034505 B_{2,4}(t)$$
$$-0.057381 B_{2,5}(t) - 0.032491 B_{2,6}(t) + 0.187344 B_{2,7}(t).$$

In Figure1 the approximate solution and the exact solution of Eq.(14) have been plotted for (k=2, p=3).

Example 6.2. [19] Consider the following linear boundary value problem of fractional order

Table 1. Least square error(LSE) for test problems

Example	k	p	LSE
1	2	1	0.177200e − 4
	2	3	0.306464e − 6
2	2	2	0.849746e − 8
	2	3	0.742175e − 12
3	2	3	0.205210e − 4
	3	3	0.301583*e* − 5

Table 2. Comparison of Haar wavlet [16] and our method with $k = 2, p = 2$ for $\alpha = 1.4$

x	Harr wavlet method with M=32	Our method for K=2, p=2
0.1	1.45713×10^{-6}	8.23871×10^{-5}
0.2	6.98702×10^{-9}	2.69316×10^{-5}
0.3	3.38714×10^{-8}	0.47382×10^{-5}
0.4	3.13234×10^{-7}	1.09416×10^{-4}
0.5	7.84210×10^{-7}	3.44962×10^{-4}
0.6	1.58534×10^{-6}	1.29711×10^{-5}
0.7	4.81312×10^{-7}	4.76136×10^{-5}
0.8	7.98561×10^{-7}	0.69128×10^{-5}
0.9	1.17157×10^{-6}	5.81512×10^{-5}

Table 3. Comparison of Haar wavlet [16] and our method with $k = 2, p = 3$ for $\alpha = 1.8$

x	Harr wavlet method with M=32	Our method for K=2, p=3
0.1	1.49112×10^{-7}	2.74437×10^{-11}
0.2	8.90900×10^{-7}	1.08624×10^{-11}
0.3	1.14669×10^{-7}	4.39901×10^{-11}
0.4	1.86018×10^{-7}	6.21045×10^{-11}
0.5	2.66526×10^{-7}	0.29815×10^{-10}
0.6	6.05455×10^{-7}	1.69136×10^{-11}
0.7	1.05672×10^{-6}	1.72410×10^{-11}
0.8	1.85259×10^{-7}	2.41326×10^{-11}
0.9	3.54806×10^{-7}	3.91543×10^{-12}

Table 4. Least square error(LSE) for different values of k, p for Example 5.6

k	p	Least squar error
1	1	0.28419e − 4
1	2	0.35964e − 5
2	2	0.34786e − 6
2	3	0.76747e − 6
3	3	0.15693e − 7

$$D^{0.5}y(t)=\frac{1}{\sqrt{\pi t}}+e^{t}\ erf\ (\sqrt{t}),\quad 0<t<1, \qquad (16)$$

with the boundary conditions:

$$y(0)=1,\ y(1)=e,$$

one can easily check that $y(t)=e^{t}$ is the exact solution.

Presented method has been applied for the above example. Approximate solution for k=2 and p=3 is

$$y_2(t)=0.880201B_{2,-3}(t)+0.997399B_{2,-2}(t)$$
$$+1.130203B_{2,-1}(t)+1.280686B_{2,0}(t)+0.1.451207B_{2,1}(t)$$
$$+1.644434B_{2,2}(t)+1.863390B_{2,3}(t)+2.111495B_{2,4}(t)$$
$$+2.392635B_{2,5}(t)+2.711214B_{2,6}(t)+3.072199B_{2,7}(t).$$

In Figure 2 the approximate solution and the exact solution of Eq.(16) have been plotted for (k=2, p=3).

Example 6.3. [21] Consider the following linear boundary value problem of fractional order

$$y''(t)+D^{0.5}y(t)=f(t),\ 0<t<1, \qquad (17)$$

with the boundary conditions:

$$y(0)=y(1)=0,$$

where

$$f(t)=\frac{256}{63\sqrt{\pi}}t^{4.5}-\frac{128}{35\sqrt{\pi}}t^{3.5}+20t^{3}-12t^{2}, \qquad (18)$$

one can easily check $y(t)=t^{4}(t-1)$ is the analytical solution.

We use the mentioned method and the following result has been obtained. Approximate solution for k=3 and p=3 is

$$y_3(t)=-0.001701B_{3,-3}(t)-0.000130B_{3,-2}(t)$$
$$-0.002222B_{3,-1}(t)-0.002321B_{3,0}(t)-0.004609B_{3,1}(t)$$
$$-0.004561B_{3,2}(t)-0.050314B_{3,3}(t)-0.078343B_{3,4}(t)$$
$$-0.081544B_{3,5}(t)-0.020926B_{3,6}(t)-0.016524B_{3,7}(t)$$
$$-0.044217B_{3,8}(t)-0.026497B_{3,9}(t)-0.104850B_{3,10}(t)$$
$$-0.197241B_{3,11}(t)-0.460017B_{3,12}(t)+0.176244B_{3,13}(t)$$
$$+0.820379B_{3,14}(t)+0.389004B_{3,15}(t).$$

In Figure 3 the approximate solution and the exact solution of Eq.(17) have been plotted for (k=3, p=3).

Examples 1-3 show that when k and p are changed the least square errors are different. Least square errors for above examples are listed in the following Table for different values of k and p.

Example 6.4. [16] Consider the boundary value problem for inhomogeneous linear fractional differential equation

$$D^{\alpha}y(t)+ay(x)=g(t),\quad t\in[0,1], \qquad (19)$$

$$y(0)=0,\ y(1)=\frac{1}{\Gamma(\alpha+2)},$$

where $1<\alpha\le 2$, $a\in R$. For $g(x)=t+\frac{at^{\alpha+1}}{\Gamma(\alpha+2)}$ the exact solution of boundary value problem is $y(t)=\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$. For comparison of presented method and Haar wavlet method in [16] the absolute error is given in the Tables 2 and 3. In the Tables 2 and 3 we compare our method and Haar wavlet method for $\alpha=1.4, a=\frac{3}{57}, k=2, p=2$ and $\alpha=1.8, a=\frac{3}{57}, k=2$ and $p=3$, respectively.

Caputo Fractional Derivative

Example 6.5. [8] Let us consider the nonlinear boundary value problem of Caputo fractional order

$$D_{*}^{2.4}y(t)-y^{3}(t)=f(t), \qquad (20)$$

with the boundary conditions:

$$y(0)=0,\ y(1)=2,$$

where

$$f(t)=\frac{10}{\Gamma(\frac{3}{5})}t^{\frac{3}{5}}+t^{6}+3t^{7}+t^{9}, \qquad (21)$$

and the exact solution is $y(t)=t^{2}+t^{3}$.

We use the mentioned method and the following result has been obtained. Approximate solution for k=2 and p=2 is

$$y_2(t)=0.008334B_{2,-2}(t)+0.002346B_{2,-1}(t)$$
$$+0.021948B_{2,0}(t)+0.041673B_{2,1}(t)+0.032124B_{2,2}(t)$$
$$+0.347687B_{2,2}(t)+0.528754B_{2,4}(t)+0.612854B_{2,5}(t).$$

In Figure 4 the approximate solution and the exact solution of Eq.(20) have been plotted for k=2, p=2.

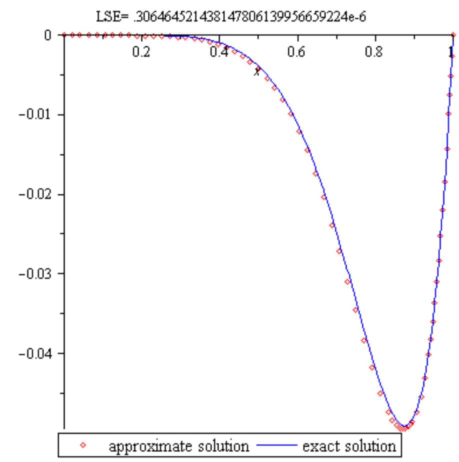

Figure 1. Approximate solution for Example 6.1.

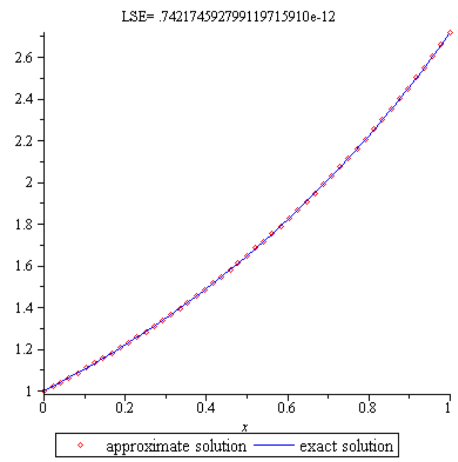

Figure 2. Approximate solution for Example 6.2.

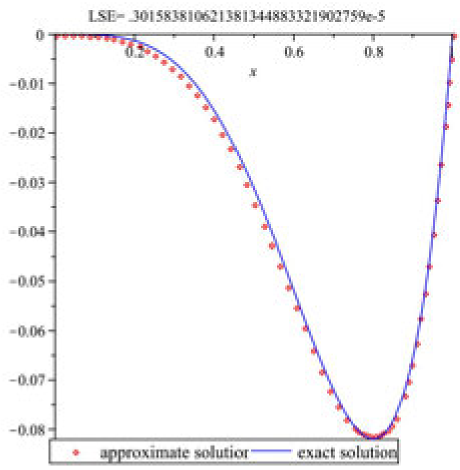

Figure 3. Approximate solution for Example 6.3.

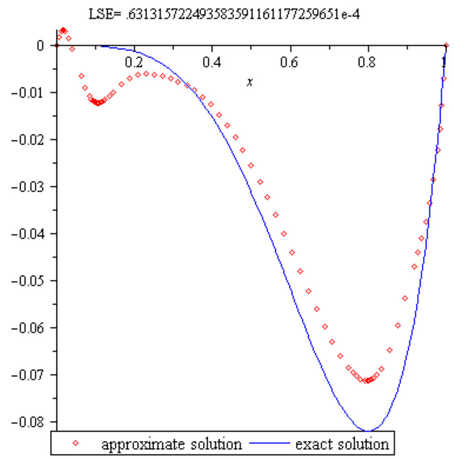

Figure 4. Approximate solution for Example 6.5.

Example 6.6. Consider the boundary value problem of fractional order

$$D_*^{1.5} y(t) + y'(x) = g(t), \ t \in [0,1], \tag{22}$$

with the boundary conditions:

$$y(0) = \frac{1}{2}, \ y(1) = \frac{1}{3}. \tag{23}$$

The exact solution of this problem is $y(t) = \dfrac{1}{2+t}$ and $g(t)$ determine by substitute the exact solution in above equation.

We apply the mentioned method and the least squar errors for different values of k,p is shown in the Table 4.

Results and Discussion

In this paper, the B-spline method has been successfully used for finding the solution of linear and nonlinear boundary value problems of Riemann-Liouville and Caputo fractional order. The method was used in a direct way without using linearization, discritization or perturbation assumptions. Our method can be used to solved the linear and nonlinear multi-term fractional (arbitrary) orders differential equation. Several examples are given to demonstrate the powerfulness of the proposed method.

Acknowledgements

The authors are grateful to the anonymous reviewers for their helpful comments, which greatly improves this paper.

References

1. Aleroev, T. S. The Sturm-Liouville problem for a second order ordinary differential equation with fractional derivatives in the lower terms, Differential'nye Uravneniya.**18**: 341-342 (1982).
2. Bai, Z. and Lu, H. Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. **311**: 495-505 (2005).
3. Caputo, M. Linear models of dissipation whose Q is almost frequency independent. Part II, J. Roy. Astr. Soc. **13**: 529-539 (1967).
4. Diethelm, K. and Ford, N. J. Numerical solution of the Bagley-Torvik equation, BIT. **42**: 490- 507 (2002).
5. Diethelm, K., Ford, N. J. and Freed, A. D. Detailed error analysis for a fractional Adams method, Numerical Algorithms. **36**: 31-52 (2004).
6. Ervin,V. J. and Roop, J. P. Variational formulation for a stationary fractional Advection Dispersion equation, Numer. Meth. PDE. **22**: 558-576 (2006).
7. Fix, G. J. and Roop, J. P. Least squares finite element solution of a fractional order two-point boundary value problem, Comput. Math. Appl. **48**: 1017-1033 (2004).
8. Ghorbani, A. Toward a new analytical method for solving nonlinear fractional differential Equations, Comput. Methods Appl. Mech. Engrg. **197**: 4173-4180 (2008).
9. Hilfer, R. Applications of fractional calculus in physics, World Scientific, Singapore (2000).
10. Jafari, H. and Daftardar-Gejji, V. Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method, Appl. Math. Comput. **180**: 700-706 (2006).
11. Kilbas, A. A. and Trujillo, J. J. Differential equations of fractional order: methods, results and problems II. Applicable Analysis, **81**: 435-493 (2002).
12. Loghmani, G. B. Application of least square method to arbitrary-order problems with separated boundary conditions, J. Comput. Appl. Math. **222**: 500-510 (2008).
13. Magin, R. Fractional calculus in Bioengineering, Part 1-3, Critical Reviews in Bioengineering. **32**: Part 1-3 (2004).
14. Miyakoda, T. Discretized fractional calclus with a series of chebyshev polynomial, Electronic Notes Theoretical computer Science. **225**: 239-244 (2009).
15. Miller, K. S. and Ross, B. An introduction to the fractional calculus and fractional differential Equations, John Wiley and Sons, New York (1993).
16. Mujeeb ur R. and Rahmat A. K. A numerical method for solving boundary value problems for fractional differential equations, Appl. Math. Modell. (2011), doi:10.1016/j.apm.2011.07.045.
17. Oldham, K. B. and Spanier, J. The fractional calculus, Academic Press, New York (1974).
18. Podlubny, I. fractional differential equations,in Mathematics in Science and Engineering, Academic Press vol 198, San Diego (1999).
19. Podlubny, I. Numerical solution of ordinary fractional differential equations by the fractional difference method. in: S. Elaydi, I. Gyori, G. Ladas, Advances in Difference Equations, Gordon and Breach, Amsterdam (1997).
20. X. Su. Boundary value problem for a coupled system of nonlinear fractional differential equations, Appl. Math. Lett. **22**: 64-69 (2008).
21. Vazqueze, L., Tanga, Y., Nie, N., Jimenez, S. and Li, M. Solving two-point boundary value problems of fractional differential equations by spline collocation methods, Available from: http://www.cc.ac.cn/2009 research _ report/0903.pdf (2009).
22. Weilbeer, M. Effcient numerical methods for fractional differential equations and their analytical background. Ph.D. *Thesis*, Technischen Universität, 218 p. (2006).
23. Zhang, S. Q. Existence of solution for a boundary value problem of fractional order, Acta Mathematica Scientia. **26**: 220-228 (2006).