Journal of Sciences, Islamic Republic of Iran 23(2): 179-184 (2012) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

On Special Generalized Douglas-Weyl Metrics

A. Tayebi[1] and E. Peyghan[2,*]

[1]*Department of Mathematics, Faculty of Science, University of Qom, Qom, Islamic Republic of Iran*
[2]*Department of Mathematics, Faculty of Science, Arak University, Arak, Islamic Republic of Iran*

Received: 16 August 2011 / Revised: 8 January 2012 / Accepted: 22 January 2012

Abstract

In this paper, we study a special class of generalized Douglas-Weyl metrics whose Douglas curvature is constant along any Finslerian geodesic. We prove that for every Landsberg metric in this class of Finsler metrics, $\bar{E} = 0$ if and only if $H = 0$. Then we show that every Finsler metric of non-zero isotropic flag curvature in this class of metrics is a Riemannian if and only if $\bar{E} = 0$.

Keywords: Douglas space; Landsberg metric; The non-Riemannian quantity H

Introduction

For a Finsler metric $F = F(x, y)$ on a manifold M, its geodesics curves are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}^i) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and given by following

$$G^i := \frac{1}{4} g^{il} \left\{ \frac{\partial^2 \left[F^2\right]}{\partial x^k \partial y^l} y^k - \frac{\partial \left[F^2\right]}{\partial x^l} \right\}, \quad y \in T_x M.$$

Thus F induced a spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ which determines the geodesics [9,15].

Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if any geodesic of the first is also geodesic for the second and the other way around. Hereby, there is a scalar function $P = P(x, y)$ defined on TM_0 such that

$$G^i = \bar{G}^i + P\, y^i,$$

where G^i and $\bar{G}^i$ are the geodesic spray coefficients of F and $\bar{F}$, respectively and P is positively y-homogeneous of degree one [6,8].

Let

$$D^i_{j\ kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left[G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right].$$

It is easy to verify that, $D := D^i_{j\ kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle TM_0. We call D the Douglas tensor. The Douglas tensor D is a non-Riemannian projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x, y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is same as that of $\bar{F}$ [8]. Finsler metrics with vanishing Douglas tensor are called Douglas metrics. The notion of Douglas curvature was proposed by Bácsó-Matsumoto as a generalization of Berwald curvature [3]. There is another projective invariant in Finsler geometry, namely $D^i_{j\ kl|m} y^m = T_{jkl} y^i$, that is hold for some tensor T_{jkl}, where $D^i_{j\ kl|m}$ denotes the

* Corresponding author, Tel.: +98-9122770859, Fax: +98(861)4173406, E-mail: epeyghan@gmail.com

horizontal covariant derivatives of $D_{j\ kl}^{i}$ whit respect to the Berwald connection of Finsler metric F. This equation implies that the rate of change of the Douglas curvature along a geodesic is tangent to the geodesic [6].

In this paper, we study on aclass of Finsler metrics whose Douglas curvature satisfies

$$D_{j\ kl|s}^{i} y^{s} = 0 \qquad (1)$$

The geometric mining of this equation is that on these new spaces, the Douglas tensor is constant along a geodesics.

Other than Douglas curvature, there are several important non-Riemannian quantities: the Cartan torsoin **C**, the Berwald curvature **B**, the mean Berwald curvature **E**, and the Landsberg curvature **L**, etc. [12-15]. The study show that the above mentioned non-Riemannian quantities are closely related to the Douglas metrics, namely Bácsó-Matsumoto proved that every Douglas metric with vanishing Landsberg curvature is a Berwald metric [1,2]. Is there any other interesting non-Riemannian quantity with such property? In [10], Shen find a new non-Riemannian quantity for Finsler metrics that is closely related to the *E*-curvature and call it $\bar{E}$ – curvature. Recall $\bar{E}$ is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics.

In this paper, we prove that for every Landsberg metric satisfies (1), $\bar{\mathrm{E}} = 0$ if and only if $\mathrm{H} = 0$. More precisely, we prove the following.

Theorem 1. *Let (M,F) be a Finsler space satisfies (1). Suppose that F is a Landsberg metric. Then* $\bar{\mathrm{E}} = 0$ if and only if $\mathrm{H} = 0$.

For a non-zero vector $y \in T_xM_0$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by $R_y(u) := R^i{}_k(y)u^k \frac{\partial}{\partial x^i}$, where $R^i{}_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$. The family $R := \{R_y\}_{y\in TM_0}$ is called the Riemann curvature [5]. Suppose $P \subset T_xM$ (flag) is an arbitrary plane and $y \in P$ (flag pole). The flag curvature $K(P,y)$ is defined by

$$K(P,y) = \frac{g_y(R_y(u),v)}{g_y(y,y)g_y(v,v) - g_y(v,y)g_y(v,y)}$$

where v is an arbitrary vector in P such that $P = span\{y,v\}$. A Finsler metric F is said to be of isotopic flag curvature if $K = K(x)$. In this paper, we show that every metrics in this class of Finsler metrics with non-zero isotropic flag curvature is a Riemannian metric if and only if $\bar{\mathrm{E}} = 0$.

Theorem 2. Let F be a Finsler metric satisfies (1) of non-zero isotropic flag curvature K = K(x). Then F is a Riemannian metric if and only if $\bar{\mathrm{E}} = 0$.

There are many connections in Finsler geometry [11]. In this paper we set the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

Preliminaries

Let M be a n-dimensional C^∞ manifold. Dnote by T_xM the tangent space at $x \in M$ by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0,\infty)$ which has the following properties:

(i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, (iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive-definite,

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y+su+tv)\right]\Big|_{s,t=0}, u,v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x define $C_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathrm{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\left[\mathrm{g}_{y+tw}(u,v)\right]\Big|_{t=0}, u,v,w \in T_xM.$$

The family $\mathrm{C} := \{\mathrm{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $L_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $L_y(u,v,w) := L_{ijk}(y)u^iv^jw^k$, where $L_{ijk} := C_{ijk|s}y^s$. The family $L := \{L_y\}_{y\in TM_0}$ is called the

Landsberg curvature. A Finsler metric F is called a Landsberg metric if **L**=0 [4].

Given a Finsler manifold (M,F), then a global vector field G is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where G^i are local function on TM given by

$$G^i := \frac{1}{4} g^{il}\left\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial[F^2]}{\partial x^l}\right\}, \quad y \in T_x M$$

G is called the associated spray to (M,F). The projection of an integral curve of G is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a non-zero vector $y \in T_x M_0$, we can define $\mathrm{B}_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$ and $\mathrm{E}_y : T_x M \otimes T_x M \to \mathbb{R}$ by $\mathrm{B}_y(u,v,w) := B^i_{j\ kl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $\mathrm{E}_y(u,v) := E_{jk}(y) u^j v^k$ where

$$B^i_{j\ kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \qquad E_{jk} := \frac{1}{2} B^m_{j\ km}$$

$u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. The B and E are called the Berwald curvature and mean Berwald curvature respectively. A Finsler metric is called a Berwald metric and weakly Berwald metric if $\mathrm{B} = 0$ and $\mathrm{E} = 0$, respectively [11].

The quantity $\mathrm{H}_y = H_{ij} dx^i \otimes dx^j$ is defined as the covariant derivative of E along geodesics [7]. More precisely $H_{ij} := E_{ij|m} y^m$.

For a flag $P = span\{y,u\} \subset T_x M$ flagpole y, the flag curvature $K = K(P,y)$ is defined by

$$K(P,y) := \frac{g_y(u, R_y(u))}{g_y(y,y) g_y(u,u) - g_y(y,u)^2},$$

We say that a Finsler metric F is of scalar curvature if for any $y \in T_x M$, the flag curvature $K = K(x,y)$ is a scaler function on the slit tangent bundle TM_0.

By means of E-curvature, we can define $\bar{E}_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ *by*

$$\bar{E}_y(u,v,w) := \bar{E}_{jkl}(y) u^i v^j w^k,$$

where $\bar{E}_{ijk} := E_{ij|k}$. We call it $\bar{E}$-curvature. From a Bianchi identity, we have

$$B^i_{j\ ml|k} - B^i_{j\ km|l} = R^i_{j\ kl,m}$$

where R^i_{jkl} is the Riemannian curvature of Berwald connection [11]. This implies that $\bar{E}_{jlk} - \bar{E}_{jkl} = 2R^m_{j\ kl,m}$. Then $\bar{E}_{ijk}$ is not totally symmetric in all three of its indices.

Results and Discusion

Sakaguchi Theorem

In this section, we are going to prove the well-known theorem of Sakaguchi. Our method is different from the Sakaguchi.

Theorem 3. Every Finsler metric of scalar flag curvature is a generalized Douglas-Weyl metric.

Proof. Let F be a Finsler metric of scalar flag curvature K. The following holds

$$\begin{aligned} B^i_{jml|k} y^k = {} & 2K C_{jlm} y^i - \frac{1}{3} K_{.j.m} F^2 h^i_l \\ & - \frac{1}{3} K_{.j.l} F^2 h^i_m - \frac{1}{3} K_{.l.m} F^2 h^i_j \\ & - \frac{1}{3} K_{.l} \{F F_{.j} \delta^i_m + F F_{.m} \delta^i_j - 2g_{jm} y^i\} \\ & - \frac{1}{3} K_{.m} \{F F_{.j} \delta^i_m + F F_{.l} \delta^i_j - 2g_{jl} y^i\} \\ & - \frac{1}{3} K_{.j} \{F F_{.l} \delta^i_m + F F_{.m} \delta^i_l - 2g_{lm} y^i\} \end{aligned} \qquad (2)$$

It follows from (2) that

$$H_{jl} = -\frac{n+1}{6}\{y_l K_{.j} + y_j K_{.l} + K_{.j.l} F^2\}. \qquad (3)$$

We obtain

$$D^i{}_{jkl|m}\, y^m = 2K\, C_{jkl}\, y^i - \frac{2}{3}\{K_{.j}\, g_{kl} + K_{.l}\, g_{jk} + K_{.k}\, g_{jl}\} y^i - \frac{1}{3}\{K_{.j.l}\, y_k + K_{.j.k}\, y_l + K_{.k.l}\, y_j\} y^i - \frac{2}{n+1} E_{jk.l|m}\, y^m y^i \tag{4}$$

Thus, we can conclude that every Finsler metric of scalar flag curvature a generalized Douglas-Weyl metric. □

Proof of Theorem 1

To prove the Theorem 1, we need the following.

Lemma 2. Let (M, F) be a Finsler manifold. Then the following holds

$$B_j{}^i{}_{kl|m}\, y^m = \frac{2}{n+1}\big\{H_{jk}\,\delta^i_l + H_{kl}\,\delta^i_j + H_{lj}\,\delta^i_k + H_{jk,l}\, y^i - \bar{E}_{jkl}\, y^i\big\}. \tag{5}$$

Proof. By definition, we have

$$D^i_{jkl} = B^i_{jkl} - \frac{2}{n+1}\big\{E_{jk}\,\delta^i_l + E_{kl}\,\delta^i_j + E_{lj}\,\delta^i_k + E_{jk,l}\, y^i\big\}. \tag{6}$$

Thus

$$D_j{}^i{}_{kl|m}\, y^m = B_j{}^i{}_{kl|m}\, y^m - \frac{2}{n+1}\big\{E_{jk|m}\, y^m \delta^i_l + E_{kl|m}\, y^m \delta^i_j + E_{lj|m}\, y^m \delta^i_k\big\} - \frac{2}{n+1} E_{jk,l|m}\, y^m y^i. \tag{7}$$

On the other hand, the following Ricci identity for E_{ij} hold

$$E_{jk,l|k} - E_{ij|k,l} = E_{pj}\, B_i{}^p{}_{kl} + E_{ip}\, B_j{}^p{}_{kl}. \tag{8}$$

It follows from (5) that

$$E_{jk,l|m}\, y^m = E_{jk|m,l}\, y^m = \big[E_{jk|m}\, y^m\big]_{,l} - E_{jkl}, \tag{9}$$

This yields that

$$E_{jk,l|m}\, y^m = E_{jk,l|m}\, y^m = H_{jk,l} - \bar{E}_{jkl}. \tag{10}$$

By (7) and (10), we get (5). □

Lemma 2. Let (*M*,*F*) be a Finsler manifold. Then the following hold

$$R_j{}^i{}_{kl|m} + R_j{}^i{}_{lm|k} + R_j{}^i{}_{mk|l} = B_j{}^i{}_{ku} R^u{}_{lm} + B_j{}^i{}_{lu} R^u{}_{mk} + B_j{}^i{}_{mu} R^u{}_{kl} \tag{11}$$

$$B_j{}^i{}_{kl|m} - B_j{}^i{}_{ml|k} = R_j{}^i{}_{ml,k} \tag{12}$$

$$B_j{}^i{}_{kl,m} = B_j{}^i{}_{km,l} \tag{13}$$

Proof. The curvature form of Berwald connection is

$$\Omega^i{}_j = d\,\omega^i_j - \omega^k_j \wedge \omega^i_k = \frac{1}{2} R^i_{jkl}\,\omega^k \wedge \omega^l - B^h_{jkl}\,\omega^k \wedge \omega^{n+1}. \tag{14}$$

For the Berwald connection, we have the following structure equation

$$dg_{ij} - g_{jk}\,\Omega^k_i - g_{ik}\,\Omega^k_j = -2L_{ijk}\,\omega^k + 2C_{ijk}\,\omega^{n+1}. \tag{15}$$

Differentiating (15) yields the following Ricci identity

$$g_{pj}\,\Omega^p_i - g_{pi}\,\Omega^p_j = -2L_{ijk}\,\omega^k \wedge \omega^l - 2L_{ijk}\,\omega^k \wedge \omega^{n+1} - 2C_{ijl|k}\,\omega^k \wedge \omega^{n+1} - 2C_{ijl,k}\,\omega^{n+k} \wedge \omega^{n+1} - 2C_{ijp}\,\Omega^p_l\, y^l. \tag{16}$$

Differentiating of (14) yields

$$d\,\Omega^j_i - \omega^k_i \wedge \Omega^k_j + \omega^j_k \wedge \Omega^k_i = 0. \tag{17}$$

Define $B^i_{jkl|m}$ and $B^i_{jkl,m}$ by

$$dB^i_{jkl} - B^i_{mkl}\,\omega^m_i - B^i_{jml}\,\omega^m_k - B^i_{jkm}\,\omega^m_l + B^i_{jkl}\,\omega^i_m = B^i_{jkl|m}\,\omega^m + B^i_{jkl,m}\,\omega^{n+m}. \tag{18}$$

Similary, we define $R^i_{jkl|m}$ and $R^i_{jkl,m}$ by

$$dR^i_{jkl} - R^i_{mkl}\,\omega^m_i - B^i_{jml}\,\omega^m_k - R^i_{jkm}\,\omega^m_l + R^i_{jkl}\,\omega^i_m = R^i_{jkl|m}\,\omega^m + R^i_{jkl,m}\,\omega^{n+m}. \tag{19}$$

From (16), (17), (18) and (19), we get the proof. □

Proof of Theorem 1: From (16), it follows that

$$C_{ijl|k} - L_{ijk.l} = \frac{1}{2} g_{pj} B^p{}_{ikl} + \frac{1}{2} g_{ip} B^p{}_{jkl} \tag{20}$$

Contracting (20) with y^j and using $y^i{}_{.j} = \delta^i_j$ and $y_{i.j} = 0$ yields

$$L_{jkl} = -\frac{1}{2} g_{im} y^m B^i{}_{jkl}. \tag{21}$$

By assumption, we have

$$B^i_{j\,kl|m} y^m = \frac{2}{n+1}\left\{H_{jk}\delta^i_l + H_{kl}\delta^i_j + H_{lj}\delta^i_k + H_{jk,l} y^i - \bar{E}_{jkl} y^i\right\}. \tag{22}$$

Multiplying (22) with y_i and using (21), we get

$$\bar{E}_{jkl} = \left\{H_{jk} y_l + H_{kl} y_j + H_{lj} y_k\right\} F^{-2} + H_{jk.l}. \tag{23}$$

By (23), we get the proof. □

Proof of Theorem 2: Let $R^i{}_{kl} := y^j R^i_{j\,kl}$. Then we have

$$R^i_{j\,kl} = \frac{1}{3}\left\{\frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k}\right\}. \tag{24}$$

Here, we assume that a Finsler metric F is of scalar curvature $K = K(x, y)$. In local coordinates,

$$R^i{}_k = K F^2 h^i_k. \tag{25}$$

Plugging (25) into (24) gives

$$\begin{aligned} R^i_{j\,kl} &= \frac{K_{.j.l}}{3} F^2 h^i{}_k - \frac{K_{.j.k}}{3} F^2 h^i{}_l \\ &+ K_{.j}\{FF_{.l} h^i{}_k - FF_{.k} h^i{}_l\} \\ &+ \frac{1}{3} K_{.k}\{2F F_{.j}\delta^i_l - g_{jl} y^i - F F_{.l}\delta^i{}_j\} \\ &+ K\{g_{jl}\delta^i{}_k - g_{jk}\delta^i{}_l\} \\ &+ \frac{1}{3} K_{.l}\{2F F_{.j}\delta^i_k - g_{jk} y^i - F F_{.k}\delta^i{}_j\} \end{aligned} \tag{26}$$

Differentiating (26) with respect to y^m gives a formula for $R^i_{j\,kl.m}$ expressed in terms of K and its derivatives. Contracting (12) with y^k, we obtain

$$\begin{aligned} B^i{}_{jml|k} y^k &= 2K C_{jlm} y^i - \frac{1}{3} K_{.j.m} F^2 h^i{}_l \\ &- \frac{1}{3} K_{.j.l} F^2 h^i{}_m - \frac{1}{3} K_{.l.m} F^2 h^i{}_j \\ &- \frac{1}{3} K_{.l}\{F F_{.j}\delta^i{}_m + F F_{.m}\delta^i{}_j - 2g_{jm} y^i\} \\ &- \frac{1}{3} K_{.m}\{F F_{.j}\delta^i{}_m + F F_{.l}\delta^i{}_j - 2g_{jl} y^i\} \\ &- \frac{1}{3} K_{.j}\{F F_{.l}\delta^i{}_m + F F_{.m}\delta^i{}_l - 2g_{lm} y^i\} \end{aligned} \tag{27}$$

Since $K = K(x)$, then by (27) we get

$$B^i_{j\;ml|k}\, y^k = 2K C_{jlm} y^i \tag{28}$$

Since F be a weakly Douglas Finsler metric, then we have

$$B^i_{j\,kl|m} y^m = \frac{2}{n+1}\left\{H_{jk}\delta^i_l + H_{kl}\delta^i_j + H_{lj}\delta^i_k + H_{jk,l} y^i - \bar{E}_{jkl} y^i\right\}. \tag{29}$$

From the assumptions, one can obtains

$$B^i_{j\,kl|m} y^m = 0.$$

By (28), we can conclude that $C_{ijk} = 0$ and then F is Riemannian. □

Acknowledgements

The authors wish to thank referees and the editor for their useful comments and suggestions.

References

1. Bácsó, S. Ilosvay, F. and Kis, B. Landsberg spaces with common geodesics. Publ. Math. Debrecen. **42**: 139-144 (1993).
2. Bácsó, S. and Matsumoto, M. Reduction theorems of certain Landsberg spaces to Berwald spaces, Publ. Math. Debrecen. **48**: 357-366 (1996).
3. Bácsó, S. and Matsumoto, M. On Finsler spaces of Douglas type, A generalization of notion of Berwald space, Publ. Math. Debrecen. **51**: 385-406 (1997).
4. Hashiguchi, M. and Ichijyo, Y. On Some special (α,β)-

metrics, Rep. Fac. Sci., Kagoshima Univ. **8**: 39-46 (1975).
5. Matsumoto, M. On Finsler spaces with Randers metric and special forms of important tensors, J. Math. Kyoto Univ. **14**: 477-498 (1974).
6. Najafi, B. Shen, Z. and Tayebi, A. On a projective class of Finsler metrics, Publ. Math. Debrecen. **70**: 211-219 (2007).
7. Najafi, B. Shen, Z. and Tayebi, A. Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties, Geom. Dedicata. **131**: 87-97 (2008).
8. Najafi, B. and Tayebi, A. Finsler Metrics of scalar flag curvature and projective invariants, Balkan. J. Geom. Appl. **15**: 90-99 (2010).
9. Peyghan, E. and Tayebi, A., Generalized Berwald metrics, Turkish Journal Math, **35**: 1-10 (2011).
10. Shen, Z. Differential Geometry of Spray and Finsler Spaces, Kluwer Academic Publishers, Dordrecht, (2001).
11. Tayebi, A. Azizpour, E. and Esrafilian, E. On a family of connections in Finsler geometry, Publ. Math. Debrecen. **72**: 1-15 (2008).
12. Tayebi, A. and Najafi, B. On m-th root Finsler metrics, Journal Geometry and Physics. **61**: 1479-1484 (2011).
13. Tayebi, A. and Najafi, B. On m-th Root Metrics with special curvature properties, C. R. Acad. Sci. Paris, Ser. I, **349**: 691-693 (2011)
14. Tayebi, A. and Peyghan, E. On Ricci tensors of Randers metrics, Journal of Geometry and Physics. **60**: 1665-1670 (2010).
15. Tayebi, A. and Peyghan, E. Special Berwald Metrics, Symmetry, Integrability and Geometry: Methods and its Applications (SIGMA), **6**: 008 (2010).