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On Special Generalized Douglas-Weyl Metrics

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Abstract

In this paper, we study a special class of generalized Douglas-Weyl metrics whose Douglas curvature is constant along any Finslerian geodesic. We prove that for every Landsberg metric in this class of Finsler metrics, $\bar{\mathrm{E}} = 0$ if and only if $\mathrm{H} = 0$. Then we show that every Finsler metric of non-zero isotropic flag curvature in this class of metrics is a Riemannian if and only if $\bar{\mathrm{E}} = 0$.

Keywords: Douglas space; Landsberg metric; The non-Riemannian quantity H

Introduction

For a Finsler metric $F = F(x, y)$ on a manifold M, its geodesics curves are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}^i) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and given by following

$$G^i := \frac{1}{4} g^{il} \left\{ \frac{\partial^2 \left[F^2\right]}{\partial x^k \partial y^l} y^k - \frac{\partial \left[F^2\right]}{\partial x^l} \right\}, \quad y \in T_x M.$$

Thus F induced a spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ which determines the geodesics [9,15].

Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if any geodesic of the first is also geodesic for the second and the other way around. Hereby, there is a scalar function $P = P(x, y)$ defined on TM_0 such that

$$G^i = \bar{G}^i + P y^i,$$

where G^i and $\bar{G}^i$ are the geodesic spray coefficients of F and $\bar{F}$, respectively and P is positively y-homogeneous of degree one [6,8].

Let

$$D_j{}^i{}_{kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left[G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right].$$

It is easy to verify that, $D := D_j{}^i{}_{kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle TM_0. We call D the Douglas tensor. The Douglas tensor D is a non-Riemannian projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x, y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is same as that of $\bar{F}$ [8]. Finsler metrics with vanishing Douglas tensor are called Douglas metrics. The notion of Douglas curvature was proposed by Bácsó-Matsumoto as a generalization of Berwald curvature [3]. There is another projective invariant in Finsler geometry, namely $D_j{}^i{}_{kl|m} y^m = T_{jkl} y^i$, that is hold for some tensor T_{jkl}, where $D_j{}^i{}_{kl|m}$ denotes the

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horizontal covariant derivatives of $D^i_{j\ kl}$ whit respect to the Berwald connection of Finsler metric F. This equation implies that the rate of change of the Douglas curvature along a geodesic is tangent to the geodesic [6].

In this paper, we study on aclass of Finsler metrics whose Douglas curvature satisfies

$$D^i_{j\ kl|s}y^s = 0 \qquad (1)$$

The geometric mining of this equation is that on these new spaces, the Douglas tensor is constant along a geodesics.

Other than Douglas curvature, there are several important non-Riemannian quantities: the Cartan torsoin **C**, the Berwald curvature **B**, the mean Berwald curvature **E**, and the Landsberg curvature **L**, etc. [12-15]. The study show that the above mentioned non-Riemannian quantities are closely related to the Douglas metrics, namely Bácsó-Matsumoto proved that every Douglas metric with vanishing Landsberg curvature is a Berwald metric [1,2]. Is there any other interesting non-Riemannian quantity with such property? In [10], Shen find a new non-Riemannian quantity for Finsler metrics that is closely related to the *E*-curvature and call it $\bar{E}$ – curvature. Recall $\bar{E}$ is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics.

In this paper, we prove that for every Landsberg metric satisfies (1), $\bar{\mathrm{E}} = 0$ if and only if $\mathrm{H} = 0$. More precisely, we prove the following.

Theorem 1. *Let (M, F) be a Finsler space satisfies (1). Suppose that F is a Landsberg metric. Then* $\bar{\mathrm{E}} = 0$ if and only if $\mathrm{H} = 0$.

For a non-zero vector $y \in T_xM_0$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by $R_y(u) := R^i_{\ k}(y)u^k \frac{\partial}{\partial x^i}$, where $R^i_{\ k}(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$. The family $R := \{R_y\}_{y \in TM_0}$ is called the Riemann curvature [5]. Suppose $P \subset T_xM$ (flag) is an arbitrary plane and $y \in P$ (flag pole). The flag curvature $K(P, y)$ is defined by

$$K(P,y) = \frac{g_y(R_y(u),v)}{g_y(y,y)g_y(v,v) - g_y(v,y)g_y(v,y)}$$

where v is an arbitrary vector in P such that $P = span\{y, v\}$. A Finsler metric F is said to be of isotopic flag curvature if $K = K(x)$. In this paper, we show that every metrics in this class of Finsler metrics with non-zero isotropic flag curvature is a Riemannian metric if and only if $\bar{\mathrm{E}} = 0$.

Theorem 2. Let F be a Finsler metric satisfies (1) of non-zero isotropic flag curvature K = K(x). Then F is a Riemannian metric if and only if $\bar{\mathrm{E}} = 0$.

There are many connections in Finsler geometry [11]. In this paper we set the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

Preliminaries

Let M be a n-dimensional C^∞ manifold. Dnote by T_xM the tangent space at $x \in M$ by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, (iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive-definite,

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]\Big|_{s,t=0}, u,v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x define $C_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ *by*

$$\mathrm{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\left[\mathrm{g}_{y+tw}(u,v)\right]\Big|_{t=0}, u,v,w \in T_xM.$$

The family $\mathrm{C} := \{\mathrm{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $L_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $L_y(u,v,w) := L_{ijk}(y)u^i v^j w^k$, where $L_{ijk} := C_{ijk|s}y^s$. The family $L := \{L_y\}_{y \in TM_0}$ is called the

Landsberg curvature. A Finsler metric F is called a Landsberg metric if **L**=0 [4].

Given a Finsler manifold (M,F), then a global vector field G is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where G^i are local function on TM given by

$$G^i := \frac{1}{4} g^{il}\left\{\frac{\partial^2\left[F^2\right]}{\partial x^k \partial y^l} y^k - \frac{\partial\left[F^2\right]}{\partial x^l}\right\}, \quad y \in T_x M$$

G is called the associated spray to (M,F). The projection of an integral curve of G is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0.$

For a non-zero vector $y \in T_x M_0$, we can define $\mathrm{B}_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$ and $\mathrm{E}_y : T_x M \otimes T_x M \to \mathbb{R}$ by $\mathrm{B}_y(u,v,w) := B^i_{j\,kl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $\mathrm{E}_y(u,v) := E_{jk}(y)u^j v^k$ where

$$B^i_{j\,kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \qquad E_{jk} := \frac{1}{2} B^m_{j\,km}$$

$u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. The B and E are called the Berwald curvature and mean Berwald curvature respectively. A Finsler metric is called a Berwald metric and weakly Berwald metric if $\mathrm{B} = 0$ and $\mathrm{E} = 0$, respectively [11].

The quantity $\mathrm{H}_y = H_{ij} dx^i \otimes dx^j$ is defined as the covariant derivative of E along geodesics [7]. More precisely $H_{ij} := E_{ij|m} y^m$.

For a flag $P = span\{y,u\} \subset T_x M$ flagpole y, the flag curvature $K = K(P,y)$ is defined by

$$K(P,y) := \frac{g_y(u, R_y(u))}{g_y(y,y)g_y(u,u) - g_y(y,u)^2},$$

We say that a Finsler metric F is of scalar curvature if for any $y \in T_x M$, the flag curvature $K = K(x,y)$ is a scaler function on the slit tangent bundle TM_0.

By means of E-curvature, we can define $\bar{E}_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ *by*

$$\bar{E}_y(u,v,w) := \bar{E}_{jkl}(y) u^i v^j w^k,$$

where $\bar{E}_{ijk} := E_{ij|k}$. We call it $\bar{E}$-curvature. From a Bianchi identity, we have

$$B^i_{j\;ml|k} - B^i_{j\;km|l} = R^i_{j\;kl,m}$$

where R^i_{jkl} is the Riemannian curvature of Berwald connection [11]. This implies that $\bar{E}_{jlk} - \bar{E}_{jkl} = 2R^m_{j\;kl,m}$. Then $\bar{E}_{ijk}$ is not totally symmetric in all three of its indices.

Results and Discusion

Sakaguchi Theorem

In this section, we are going to prove the well-known theorem of Sakaguchi. Our method is different from the Sakaguchi.

Theorem 3. Every Finsler metric of scalar flag curvature is a generalized Douglas-Weyl metric.

Proof. Let F be a Finsler metric of scalar flag curvature K. The following holds

$$\begin{aligned} B^i{}_{jml|k}\, y^k = {} & 2K\,C_{jlm}\, y^i - \frac{1}{3}K_{.j.m}\, F^2 h^i{}_l \\ & - \frac{1}{3}K_{.j.l}\, F^2 h^i{}_m - \frac{1}{3}K_{.l.m}\, F^2 h^i{}_j \\ & - \frac{1}{3}K_{.l}\,\{F F_{.j}\delta^i_m + F F_{.m}\delta^i_j - 2g_{jm} y^i\} \\ & - \frac{1}{3}K_{.m}\,\{F F_{.j}\delta^i_m + F F_{.l}\delta^i_j - 2g_{jl} y^i\} \\ & - \frac{1}{3}K_{.j}\,\{F F_{.l}\delta^i_m + F F_{.m}\delta^i_l - 2g_{lm} y^i\} \end{aligned} \qquad (2)$$

It follows from (2) that

$$H_{jl} = -\frac{n+1}{6}\left\{y_l K_{.j} + y_j K_{.l} + K_{.j.l} F^2\right\}. \qquad (3)$$

We obtain

$$D^{i}{}_{jkl|m}\, y^{m} = 2K\, C_{jkl}\, y^{i} - \frac{2}{3}\{K_{.j}\, g_{kl} + K_{.l}\, g_{jk} + K_{.k}\, g_{jl}\} y^{i} - \frac{1}{3}\{K_{.j.l}\, y_{k} + K_{.j.k}\, y_{l} + K_{.k.l}\, y_{j}\} y^{i} - \frac{2}{n+1} E_{jk.l|m}\, y^{m} y^{i} \tag{4}$$

Thus, we can conclude that every Finsler metric of scalar flag curvature a generalized Douglas-Weyl metric. □

Proof of Theorem 1

To prove the Theorem 1, we need the following.

Lemma 2. Let (M, F) be a Finsler manifold. Then the following holds

$$B_{j\ kl|m}^{\,i}\, y^{m} = \frac{2}{n+1}\left\{H_{jk}\delta^{i}_{l} + H_{kl}\delta^{i}_{j} + H_{lj}\delta^{i}_{k} + H_{jk,l}\, y^{i} - \bar{E}_{jkl}\, y^{i}\right\}. \tag{5}$$

Proof. By definition, we have

$$D^{i}_{jkl} = B^{i}_{jkl} - \frac{2}{n+1}\left\{E_{jk}\delta^{i}_{l} + E_{kl}\delta^{i}_{j} + E_{lj}\delta^{i}_{k} + E_{jk,l}\, y^{i}\right\}. \tag{6}$$

Thus

$$D_{j\ kl|m}^{\,i}\, y^{m} = B_{j\ kl|m}^{\,i}\, y^{m} - \frac{2}{n+1}\left\{E_{jk|m}\, y^{m}\delta^{i}_{l} + E_{kl|m}\, y^{m}\delta^{i}_{j} + E_{lj|m}\, y^{m}\delta^{i}_{k}\right\} - \frac{2}{n+1} E_{jk,l|m}\, y^{m} y^{i}. \tag{7}$$

On the other hand, the following Ricci identity for E_{ij} hold

$$E_{jk,l|k} - E_{ij|k,l} = E_{pj} B_{i\ kl}^{\,p} + E_{ip} B_{j\ kl}^{\,p}. \tag{8}$$

It follows from (5) that

$$E_{jk,l|m}\, y^{m} = E_{jk|m,l}\, y^{m} = \left[E_{jk|m}\, y^{m}\right]_{,l} - E_{jkl}, \tag{9}$$

This yields that

$$E_{jk,l|m}\, y^{m} = E_{jk,l|m}\, y^{m} = H_{jk,l} - \bar{E}_{jkl}. \tag{10}$$

By (7) and (10), we get (5). □

Lemma 2. Let (M,F) be a Finsler manifold. Then the following hold

$$R_{j\ kl|m}^{\,i} + R_{j\ lm|k}^{\,i} + R_{j\ mk|l}^{\,i} = B_{j\ ku}^{\,i} R^{u}{}_{lm} + B_{j\ lu}^{\,i} R^{u}{}_{mk} + B_{j\ mu}^{\,i} R^{u}{}_{kl} \tag{11}$$

$$B_{j\ kl|m}^{\,i} - B_{j\ ml|k}^{\,i} = R_{j\ ml,k}^{\,i} \tag{12}$$

$$B_{j\ kl,m}^{\,i} = B_{j\ km,l}^{\,i} \tag{13}$$

Proof. The curvature form of Berwald connection is

$$\Omega^{i}{}_{j} = d\,\omega^{i}_{j} - \omega^{k}_{j}\wedge\omega^{i}_{k} = \frac{1}{2} R^{\,i}_{jkl}\,\omega^{k}\wedge\omega^{l} - B^{\,h}_{jkl}\,\omega^{k}\wedge\omega^{n+1}. \tag{14}$$

For the Berwald connection, we have the following structure equation

$$dg_{ij} - g_{jk}\Omega^{k}_{i} - g_{ik}\Omega^{k}_{j} = -2L_{ijk}\,\omega^{k} + 2C_{ijk}\,\omega^{n+1}. \tag{15}$$

Differentiating (15) yields the following Ricci identity

$$g_{pj}\Omega^{p}_{i} - g_{pi}\Omega^{p}_{j} = -2L_{ijk}\,\omega^{k}\wedge\omega^{l} - 2L_{ijk}\,\omega^{k}\wedge\omega^{n+1} - 2C_{ijl|k}\,\omega^{k}\wedge\omega^{n+1} - 2C_{ijl,k}\,\omega^{n+k}\wedge\omega^{n+1} - 2C_{ijp}\Omega^{p}_{l}\, y^{l}. \tag{16}$$

Differentiating of (14) yields

$$d\,\Omega^{j}_{i} - \omega^{k}_{i}\wedge\Omega^{k}_{j} + \omega^{j}_{k}\wedge\Omega^{k}_{i} = 0. \tag{17}$$

Define $B^{\,i}_{jkl|m}$ and $B^{\,i}_{jkl,m}$ by

$$dB^{\,i}_{jkl} - B^{\,i}_{mkl}\,\omega^{m}_{i} - B^{\,i}_{jml}\,\omega^{m}_{k} - B^{\,i}_{jkm}\,\omega^{m}_{l} + B^{\,i}_{jkl}\,\omega^{i}_{m} = B^{\,i}_{jkl|m}\,\omega^{m} + B^{\,i}_{jkl,m}\,\omega^{n+m}. \tag{18}$$

Similary, we define $R^{\,i}_{jkl|m}$ and $R^{\,i}_{jkl,m}$ by

$$dR^{\,i}_{jkl} - R^{\,i}_{mkl}\,\omega^{m}_{i} - B^{\,i}_{jml}\,\omega^{m}_{k} - R^{\,i}_{jkm}\,\omega^{m}_{l} + R^{\,i}_{jkl}\,\omega^{i}_{m} = R^{\,i}_{jkl|m}\,\omega^{m} + R^{\,i}_{jkl,m}\,\omega^{n+m}. \tag{19}$$

From (16), (17), (18) and (19), we get the proof. □

Proof of Theorem 1: From (16), it follows that

$$C_{ijl|k} - L_{ijk.l} = \frac{1}{2} g_{pj} B^p{}_{ikl} + \frac{1}{2} g_{ip} B^p{}_{jkl} \qquad (20)$$

Contracting (20) with y^j and using $y^i{}_{.j} = \delta^i_j$ and $y_{i.j} = 0$ yields

$$L_{jkl} = -\frac{1}{2} g_{im} y^m B^i{}_{jkl}. \qquad (21)$$

By assumption, we have

$$B^i_{j\ kl|m} y^m = \frac{2}{n+1}\left\{H_{jk}\delta^i_l + H_{kl}\delta^i_j + H_{lj}\delta^i_k + H_{jk,l} y^i - \bar{E}_{jkl} y^i\right\}. \qquad (22)$$

Multiplying (22) with y_i and using (21), we get

$$\bar{E}_{jkl} = \left\{H_{jk} y_l + H_{kl} y_j + H_{lj} y_k\right\} F^{-2} + H_{jk.l}. \qquad (23)$$

By (23), we get the proof. □

Proof of Theorem 2: Let $R^i{}_{kl} := y^j R^i_{j\ kl}$. Then we have

$$R^i_{j\ kl} = \frac{1}{3}\left\{\frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k}\right\}. \qquad (24)$$

Here, we assume that a Finsler metric F is of scalar curvature $K = K(x, y)$. In local coordinates,

$$R^i{}_k = K F^2 h^i_k. \qquad (25)$$

Plugging (25) into (24) gives

$$\begin{aligned} R^i_{j\ kl} = {} & \frac{K_{.j.l}}{3} F^2 h^i{}_k - \frac{K_{.j.k}}{3} F^2 h^i{}_l \\ & + K_{.j}\{FF_{.l} h^i{}_k - FF_{.k} h^i{}_l\} \\ & + \frac{1}{3} K_{.k}\{2F F_{.j}\delta^i{}_l - g_{jl} y^i - F F_{.l}\delta^i{}_j\} \qquad (26) \\ & + K\{g_{jl}\delta^i{}_k - g_{jk}\delta^i{}_l\} \\ & + \frac{1}{3} K_{.l}\{2F F_{.j}\delta^i{}_k - g_{jk} y^i - F F_{.k}\delta^i{}_j\} \end{aligned}$$

Differentiating (26) with respect to y^m gives a formula for $R^i_{j\ kl.m}$ expressed in terms of K and its derivatives. Contracting (12) with y^k, we obtain

$$\begin{aligned} B^i{}_{jml|k} y^k = {} & 2K C_{jlm} y^i - \frac{1}{3} K_{.j.m} F^2 h^i{}_l \\ & - \frac{1}{3} K_{.j.l} F^2 h^i{}_m - \frac{1}{3} K_{.l.m} F^2 h^i{}_j \\ & - \frac{1}{3} K_{.l}\{F F_{.j}\delta^i{}_m + F F_{.m}\delta^i{}_j - 2g_{jm} y^i\} \qquad (27) \\ & - \frac{1}{3} K_{.m}\{F F_{.j}\delta^i{}_m + F F_{.l}\delta^i{}_j - 2g_{jl} y^i\} \\ & - \frac{1}{3} K_{.j}\{F F_{.l}\delta^i{}_m + F F_{.m}\delta^i{}_l - 2g_{lm} y^i\} \end{aligned}$$

Since $K = K(x)$, then by (27) we get

$$B^i_{j\ ml|k} y^k = 2K C_{jlm} y^i \qquad (28)$$

Since F be a weakly Douglas Finsler metric, then we have

$$B^i_{j\ kl|m} y^m = \frac{2}{n+1}\left\{H_{jk}\delta^i_l + H_{kl}\delta^i_j + H_{lj}\delta^i_k + H_{jk,l} y^i - \bar{E}_{jkl} y^i\right\}. \qquad (29)$$

From the assumptions, one can obtains

$$B^i_{j\ kl|m} y^m = 0.$$

By (28), we can conclude that $C_{ijk} = 0$ and then F is Riemannian. □

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