

## A Characterization of the Small Suzuki Groups by the Number of the Same Element Order

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Received: 23 October 2014 / Revised: 3 January 2015 / Accepted: 17 May 2015

### Abstract

Suppose that  $G$  is a finite group. Then the set of all prime divisors of  $|G|$  is denoted by  $\pi(G)$  and the set of element orders of  $G$  is denoted by  $\pi_e(G)$ . Suppose that  $k \in \pi_e(G)$ . Then the number of elements of order  $k$  in  $G$  is denoted by  $m_k$  and the sizes of the set of elements with the same order is denoted by  $nse(G)$ ; that is,  $nse(G) = \{m_k : k \in \pi_e(G)\}$ . In this paper, we prove that if  $G$  is a group such that  $nse(G) = nse(Sz(n))$ , where  $n \in \{32, 128\}$ , then  $G \cong Sz(n)$ . Here  $Sz(n)$  denotes the family of Suzuki simple groups,  $n = 2^{2k+1}$ ,  $k \in \mathbb{N}$ . This proves that the second and third member of the family of Suzuki simple groups are characterizable by the set of the number of the same element order.

**Keywords:** Element order; Sylow subgroup; Simple  $K_n$ -group; Suzuki group.

### Introduction

Suppose that  $G$  is a finite simple group and  $|\pi(G)| = n$ , where  $|\pi(G)|$  denotes the number of prime numbers dividing the order of  $G$ . Then  $G$  is called a simple  $K_n$ -group. Suppose that  $G$  is a finite group. Then a Sylow  $q$ -subgroup of  $G$  is denoted by  $P_q$  and the number of Sylow  $q$ -subgroups of  $G$  is denoted by  $n_q$  and the greatest order of elements in  $P_q$  is denoted by  $\exp(P_q)$ . The Euler totient function is

denoted by  $\varphi(n)$ . The set of sizes of conjugacy classes has an essential role in determining the structure of a finite group. So one might ask whether the set of sizes of elements with the same order has an essential role in determining the structure of a finite group. In [9], it is proved that all simple  $K_4$ -groups can be uniquely determined by  $nse(G)$  and  $|G|$ . But in [1,6,10], it is proved that the groups  $A_4$ ,  $A_5$ ,  $A_6$ ,  $Sz(8)$  and the groups  $L_2(q)$ , for  $q \in \{7, 8, 11, 13\}$  are uniquely determined only by  $nse(G)$ . In this paper, we prove

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that if  $G$  is a group such that  $nse(G) = nse(Sz(n))$ , where  $n \in \{32, 128\}$ , then  $G \cong Sz(n)$ .

**Preliminary and Notations**

In this section, we bring some lemmas that is need in the proof of main theorem.

**Lemma 1.1 [5]** If  $G$  is a simple  $K_3$ -group, then  $G$  is isomorphic to one of the following groups:

- $A_5, A_6, L_2(7), L_2(8), L_2(17), L_3(3), U_3(3), U_4(2).$

**Lemma 1.2 [8]** If  $G$  is a simple  $K_4$ -group, then  $G$  is isomorphic to one of the following groups:

- (1)  $A_7, A_8, A_9, A_{10}.$
- (2)  $M_{11}, M_{12}, J_2.$
- (3) (a)  $L_2(r)$ , where  $r$  is a prime and satisfies  $r^2 - 1 = 2^a \times 3^b \times v^c$  with  $a \geq 1, b \geq 1, c \geq 1, v > 3, v$  is a prime.
- (b)  $L_2(2^m)$ , where  $m$  satisfies

$$\begin{cases} 2^m - 1 = u \\ 2^m + 1 = 3t^b \end{cases}$$

with  $m \geq 2, u, t$  are primes,  $t > 3, b \geq 1.$

(c)  $L_2(3^m)$ , where  $m$  satisfies

$$\begin{cases} 3^m + 1 = 4t \\ 3^m - 1 = 2u^c \end{cases} \text{ or } \begin{cases} 3^m + 1 = 4t^b \\ 3^m - 1 = 2u \end{cases}$$

with  $m \geq 2, u, t$  are odd primes,  $b \geq 1, c \geq 1.$

- (d)  $L_2(16), L_2(25), L_2(49), L_2(81), L_3(4), L_3(5), L_3(7), L_3(8), L_3(17), L_4(3), S_4(4), S_4(5), S_4(7), S_4(9), S_6(2), O_8^+(2), G_2(3), U_3(4), U_3(5), U_3(7), U_3(8), U_3(9), U_4(3), U_5(2), Sz(8), Sz(32), {}^3D_4(2), {}^2F_4(2)'$

**Lemma 1.3 [3]** Let  $G$  be a finite solvable group and  $|G| = mn$ , where  $m = p_1^{\alpha_1} \dots p_r^{\alpha_r}, (m, n) = 1.$  Let  $\pi = \{p_1, \dots, p_r\}$  and  $h_m$  be the number of Hall  $\pi$ -subgroups of  $G.$  Then  $h_m = q_1^{\beta_1} \dots q_s^{\beta_s}$  satisfies the following conditions for all  $i \in \{1, \dots, s\}:$

- (1)  $q_i^{\beta_i} \equiv 1 \pmod{p_j},$  for some  $p_j.$
- (2) The order of some chief factor of  $G$  is divisible by  $q_i^{\beta_i}.$

**Lemma 1.4 [2]** Let  $G$  be a finite group and  $m$  be a positive integer dividing  $|G|.$  If  $L_m(G) = \{g \in G : g^m = 1\},$  then  $m \mid |L_m(G)|.$

**Lemma 1.5 [10]** Let  $G$  be a group containing more than two elements. Let  $k \in \pi_e(G)$  and  $m_k$  be the number of elements of order  $k$  in  $G.$  If  $s = \sup \{m_k : k \in \pi_e(G)\}$  is finite, then  $G$  is finite and  $|G| \leq s(s^2 - 1).$

**Lemma 1.6 [7]** Let  $G$  be a finite group and  $p \in \pi(G)$  be odd. Suppose that  $P$  is a Sylow  $p$ -subgroup of  $G$  and  $n = p^s m,$  where  $(p, m) = 1.$  If  $P$  is not cyclic and  $s > 1,$  then the number of elements of order  $n$  is always a multiple of  $p^s.$

**Lemma 1.7 [4]** Let  $G$  be a solvable group and  $\pi$  be any set of primes. Then

- (1)  $G$  has a Hall  $\pi$ -subgroup.
- (2) If  $H$  is a Hall  $\pi$ -subgroup of  $G$  and  $V$  is any  $\pi$ -subgroup of  $G,$  then  $V \leq H^g$  for some  $g \in G.$  In particular, the Hall  $\pi$ -subgroups of  $G$  form a single conjugacy class of subgroups of  $G.$

**Lemma 1.8** Let  $G$  be a finite group which is not solvable. Then there is a normal series  $1 \leq N \leq M \leq G$  such that  $N$  is a maximal solvable normal subgroup of  $G$  and  $M/N$  is a non-abelian simple group or the direct product of isomorphic non-abelian simple groups.

**Proof.** Since  $G$  is a finite group, there is chief series  $1 = M_0 \leq M_1 \leq \dots \leq M_{n-1} \leq M_n = G.$  Since  $G$  is not solvable, there is a maximal  $i$  such that  $M_{i-1}$  is solvable and  $M_i/M_{i-1}$  is not solvable. On the other hand, we know that every chief factors is a simple group or the direct product of isomorphic simple groups. Therefore  $M_{i-1}$  is a maximal solvable normal

subgroup of  $G$  and  $M_i/M_{i-1}$  is a non-abelian simple group or the direct product of isomorphic non-abelian simple groups.

**Lemma 1.9** Let  $G$  be a group such that  $nse(G) = nse(Sz(n))$ , where  $n \in \{32, 128\}$ . Then  $G$  is finite and for every  $i \in \pi_e(G)$ ,

$$\begin{cases} \varphi(i) \mid m_i \\ i \mid \sum_{d \mid i} m_d \end{cases}$$

and if  $i > 2$ , then  $m_i$  is even.

**Proof.** By Lemma 1.5,  $G$  is a finite group. By Lemma 1.4,  $i \mid \sum_{d \mid i} m_d$ . We know that the number of elements of order  $i$  in a cyclic group of order  $i$  is equal with  $\varphi(i)$ . Hence  $m_i = \varphi(i)k$ , where  $k$  is the number of cyclic subgroups of order  $i$  in  $G$ . Thus  $\varphi(i) \mid m_i$ . We know that if  $i > 2$ , then  $\varphi(i)$  is even and since  $\varphi(i) \mid m_i$ , we conclude that  $m_i$  is even.  $\square$

### Results

In this section, we prove two theorems as the main results of our paper. The first theorem is the following theorem:

**Theorem 2.1** Suppose that  $G$  is a group such that  $nse(G) = nse(Sz(32))$ . Then  $G \cong Sz(32)$ .

**Proof.** By a program written in the *GAP*, we have in  $nse(G) = nse(Sz(32)) = \{1, 31775, 1016800, 1301504, 6507520, 7936000, 15744000\}$ .

We prove this theorem in five steps.

**Step 1.**  $\pi(G) = \{2, 5, 31, 41\}$ .

Since 31775 is odd, Lemma 1.9 implies that  $2 \in \pi(G)$  and  $m_2 = 31775$ . Assume that  $q \in \pi(G)$  and  $q \neq 2$ , by Lemma 1.9,  $q \mid (1 + m_q)$  and  $(q - 1) = \varphi(q) \mid m_q$ , which imply that

$q \in \{3, 5, 7, 13, 31, 41, 6507521\}$ . If  $6507521 \in \pi(G)$ , then by Lemma 1.9,  $m_{6507521} = 6507520$ . On the other hand, if  $13015042 = 2 \times 6507521 \in \pi_e(G)$ , then by Lemma 1.9,  $\varphi(13015042) \mid m_{13015042}$  and  $13015042 \mid (1 + m_2 + m_{6507521} + m_{13015042})$ , which is a contradiction. Hence  $2 \times 6507521 \notin \pi_e(G)$ . Thus  $P_{6507521}$  acts fixed point freely on the set of elements of order 2 by conjugation. Therefore  $|P_{6507521}| \mid m_2$ , which is a contradiction. So  $6507521 \notin \pi(G)$ . If  $13 \in \pi(G)$ , then by Lemma 1.9,  $m_{13} = 15744000$ . On the other hand, if  $26 = 2 \times 13 \in \pi_e(G)$ , then by Lemma 1.9,  $\varphi(26) \mid m_{26}$  and  $26 \mid (1 + m_2 + m_{13} + m_{26})$ , which is a contradiction. Hence  $2 \times 13 \notin \pi_e(G)$ . Thus  $P_{13}$  acts fixed point freely on the set of elements of order 2 by conjugation. Therefore  $|P_{13}| \mid m_2$ , which is a contradiction. So  $13 \notin \pi(G)$ . If  $7 \in \pi(G)$ , then by Lemma 1.9,  $m_7 = 15744000$ . On the other hand, if  $14 = 2 \times 7 \in \pi_e(G)$ , then by Lemma 1.9,  $\varphi(14) \mid m_{14}$  and  $14 \mid (1 + m_2 + m_7 + m_{14})$ , which is a contradiction. Hence  $14 = 2 \times 7 \notin \pi_e(G)$ . Thus  $P_7$  acts fixed point freely on the set of elements of order 2 by conjugation. Therefore  $|P_7| \mid m_2$ , which is a contradiction. So  $7 \notin \pi(G)$ . Therefore we conclude that  $\pi(G) \subseteq \{2, 3, 5, 31, 41\}$ .

If  $\{2, 3, 5, 31, 41\} \subseteq \pi(G)$ , then by Lemma 1.9,  $m_2 = 31775$ ,  $m_3 = 1301504$ ,  $m_5 = 1301504$ ,  $m_{31} = 15744000$ ,  $m_{41} = 7936000$  and  $2^{13}, 3^3, 5^3, 31^2, 41^2, 2 \times 31, 3 \times 41, 31 \times 41 \notin \pi_e(G)$ .

Since  $2^{13} \notin \pi_e(G)$ , we conclude that  $\exp(P_2) \in \{2, \dots, 2^{12}\}$ . If  $\exp(P_2) = 2^2$ , then by Lemma 1.4 and considering  $m = |P_2|$ , we conclude that  $|P_2| \mid 2^{20}$  otherwise  $|P_2| \mid 2^{19}$ .

Since  $3^3 \notin \pi_e(G)$ , we conclude that  $\exp(P_3) = 3$  or  $3^2$ . There are two cases:

**Case 1.** If  $\exp(P_3) = 3$ , then by Lemma 1.4 and considering  $m = |P_3|$ , we conclude that  $|P_3| = 3$ .

Hence  $P_3$  is cyclic and  $n_3 = \frac{m_3}{\varphi(3)} = 2^9 \times 31 \times 41$ .

**Case 2.** If  $\exp(P_3) = 3^2$ , then by Lemma 1.4 and considering  $m = |P_3|$ , we conclude that  $|P_3| \mid 3^3$ . If  $|P_3| = 3^3$ , then  $P_3$  is not cyclic. Hence by Lemma 1.6,  $9 \mid m_9 = 15744000$ , which is a contradiction.

Therefore  $|P_3| = 3^2$  and  $n_3 = \frac{m_3^2}{\varphi(3^2)} = 2^9 \times 5^3 \times 41$ .

Since  $5^3 \notin \pi_e(G)$ , we conclude that  $\exp(P_5) = 5$  or  $5^2$ . If  $\exp(P_5) = 5$ , then by Lemma 1.4 and by considering  $m = |P_5|$ , we conclude that  $|P_5| = 5$  and  $n_5 = \frac{m_5}{\varphi(5)} = 2^8 \times 31 \times 41$ . If  $\exp(P_5) = 5^2$ , then

by Lemma 1.4 and considering  $m = |P_5|$ , we conclude that  $|P_5| = 5^2$  and  $n_5 = \frac{m_5^2}{\varphi(5^2)} = 2^8 \times 31 \times 41$ .

Since  $31^2 \notin \pi_e(G)$ , by Lemma 1.4 and considering  $m = |P_{31}|$ , we conclude that  $|P_{31}| = 31$  and  $n_{31} = \frac{m_{31}}{\varphi(31)} = 2^9 \times 5^2 \times 41$ .

Since  $41^2 \notin \pi_e(G)$ , by Lemma 1.4 and considering  $m = |P_{41}|$ , we conclude that  $|P_{41}| \mid 41^2$ .

Now we show that  $3 \notin \pi(G)$ .

If  $3 \in \pi(G)$ , then by the above discussion,  $n_3 = 2^9 \times 31 \times 41$  or  $2^9 \times 5^3 \times 41$ . Hence  $41 \mid |G|$ . Since  $3 \times 41 \notin \pi_e(G)$ , we conclude that  $P_3$  acts fixed point freely on the set of elements of order 41 by conjugation. Hence  $|P_3| \mid m_{41}$ , which is a contradiction. So  $3 \notin \pi(G)$ . Therefore

$$\pi(G) \subseteq \{2, 5, 31, 41\}.$$

If  $\pi(G) = \{2\}$ , then we know that  $|nse(G)| = 7$ .

Thus  $\exp(P_2) > 4$ . Hence  $|G| = |P_2| \mid 2^{19}$ . So  $1 \leq m_4 \leq 2^{19}$ , but

$m_4 \in \{1016800, 1301504, 6507520, 7936000, 15744000\}$ , which is a contradiction.

If  $\pi(G) = \{2, 41\}$ , then we know that  $2^{13}, 41^2 \notin \pi_e(G)$  and  $|P_2| \mid 2^{20}, |P_{41}| \mid 41^2$ . Hence  $\pi_e(G) \subseteq \{1, 2, \dots, 2^{12}\} \cup \{41, 41 \times 2, \dots, 41 \times 2^{12}\}$ .

Therefore,

$$|G| = 2^1 \times 41^k = 32537600 + 1016800k_1 + 1301504k_2 + 6507520k_3 + 7936000k_4 + 15744000k_5$$

where  $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 19, l \leq 20, k \leq 2$ .

It is easy to check that this equation has no solution.

If  $5 \in \pi(G)$ , then  $n_5 = 2^8 \times 31 \times 41$ . We know that  $n_5 \mid |G|$ . Hence  $31 \mid |G|$ .

Therefore in any cases we can assume that  $31 \in \pi(G)$ .

Now we prove that  $\pi(G) = \{2, 5, 31, 41\}$ . Since  $31 \in \pi(G)$ , we conclude that  $|P_{31}| = 31$  and

$n_{31} = \frac{m_{31}}{\varphi(31)} = 2^9 \times 5^2 \times 41$ . We know that  $n_{31} \mid |G|$ ,

hence  $2^9 \times 5^2 \times 41 \mid |G|$ . It follows that

$$\pi(G) = \{2, 5, 31, 41\}.$$

**Step 2.**  $|G| = 2^k \times 5^l \times 31 \times 41$ , where  $k \leq 10, l \leq 2$ .

By the above discussion  $|P_{31}| = 31, |P_5| \mid 5^2$ .

Since  $62 \notin \pi_e(G)$ , we conclude that  $P_2$  acts fixed point freely on the set of elements of order 31 by conjugation. Therefore  $|P_2| \mid m_{31}$ . Hence  $|P_2| \mid 2^{10}$ .

Since  $1271 \notin \pi_e(G)$ , we conclude that  $P_{41}$  acts fixed point freely on the set of elements of order 31 by conjugation. Therefore  $|P_{41}| \mid m_{31}$ . Hence  $|P_{41}| = 41$ .

**Step 3.**  $G$  is not solvable.

If  $G$  is solvable, then by Lemma 1.7,  $G$  has a Hall

$\pi$ -subgroup  $H$ , where  $\pi = \{5, 31, 41\}$  and all the Hall  $\pi$ -subgroups of  $G$  are conjugate and the number of Hall  $\pi$ -subgroups of  $G$  is  $|G : N_G(H)| \mid 2^{10}$ . Since  $G$  is solvable, we conclude that  $H$  is solvable. Hence by Lemma 1.3, there are non negative integers  $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s$  such that  $n_{31}(H) = 5^{\alpha_1 + \dots + \alpha_r} \times 41^{\beta_1 + \dots + \beta_s}$ ,  $5^{\alpha_i} \equiv 1 \pmod{31}$ ,  $41^{\beta_i} \equiv 1 \pmod{31}$ . Since  $|G| = 2^k \times 5^l \times 31 \times 41$ , where  $k \leq 10$ ,  $l \leq 2$ , we conclude that  $\alpha_1 + \dots + \alpha_r \leq 2$ ,  $\beta_1 + \dots + \beta_s \leq 1$ . Therefore  $n_{31}(H) = 1$ . So  $30 \leq m_{31}(G) \leq (2^{10} \times 30) = 30720$ , but we have  $m_{31}(G) = 15744000$ , which is a contradiction.

**Step 4.**  $|G| = 2^{10} \times 5^2 \times 31 \times 41$ .

Since  $G$  is a finite group which is not solvable, there is a normal series  $1 \subset N \subset M \subset G$  such that  $N$  is a maximal solvable normal subgroup of  $G$  and  $M/N$  is a non-abelian simple group or the direct product of isomorphic non-abelian simple groups, by Lemma 1.8. Let  $M/N \cong S_1 \times \dots \times S_r$ , where  $S_1$  is a non-abelian simple group and  $S_i \cong \dots \cong S_r$ . Since  $1 \subset N \subset M \subset G$  and  $|G| = 2^k \times 5^l \times 31 \times 41$ , where  $k \leq 10$ ,  $l \leq 2$ , we conclude that  $r = 1$  and  $M/N$  is a simple  $K_3$ -group or a simple  $K_4$ -group.

If  $M/N$  is a simple  $K_3$ -group, then by Lemma 1.1 and  $|G| = 2^k \times 5^l \times 31 \times 41$ , where  $k \leq 10$ ,  $l \leq 2$ , we conclude a contradiction.

If  $M/N$  is a simple  $K_4$ -group, then by Lemma 1.2 and  $|G| = 2^k \times 5^l \times 31 \times 41$ , where  $k \leq 10$ ,  $l \leq 2$ , we conclude that  $M/N \cong Sz(32)$ . Hence  $2^{10} \times 5^2 \times 31 \times 41 = |M/N| \mid |G| \mid 2^{10} \times 5^2 \times 31 \times 41$ . So  $|G| = |Sz(32)|$ .

**Step 5.**  $G \cong Sz(32)$ .

Since  $1 \subset N \subset M \subset G$ ,  $M/N \cong Sz(32)$  and  $|G| = |Sz(32)|$ , we can conclude  $N = 1$ ,  $G = M \cong Sz(32)$  and the proof is completed.  $\square$

The second theorem as the main result is the following theorem:

**Theorem 2.2** Suppose that  $G$  is a group such that  $nse(G) = nse(Sz(128))$ . Then  $G \cong Sz(128)$ .

**Proof.** By a program written in the *GAP*, we have  $nse(G) = nse(Sz(128)) =$

$\{1, 2080895, 266354560, 235126784, 1645887488, 6583549952, 8447918080, 16912465920\}$ .

We prove this theorem in four steps.

**Step 1.**  $\pi(G) = \{2, 5, 29, 113, 127\}$ .

Since 2080895 is odd, Lemma 1.9 implies that  $2 \in \pi(G)$  and  $m_2 = 2080895$ . Assume that  $q \in \pi(G)$  and  $q \neq 2$  by Lemma 1.9,  $q \mid (1 + m_q)$  and  $(q - 1) = \varphi(q) \mid m_q$  which imply that  $q \in \{3, 5, 11, 13, 29, 113, 127\}$ . If  $13 \in \pi(G)$ , then by Lemma 1.9,  $m_{13} = 16912465920$ . On the other hand, by Lemma 1.9,  $13^2 \notin \pi_e(G)$ . Thus  $|P_{13}| \mid (1 + m_{13})$ . Therefore  $|P_{13}| = 13$  and  $n_{13} = \frac{m_{13}}{\varphi(13)} = 1409372160$ . Since  $113 \mid n_{13}$ , we deduce that  $113 \in \pi(G)$ . Now by Lemma 1.9,  $13 \times 113 \notin \pi_e(G)$ . Thus  $P_{13}$  acts fixed point freely on the set of elements of order 113 by conjugation. Therefore  $|P_{13}| \mid m_{113} = 8447918080$ , which is a contradiction. So  $13 \notin \pi(G)$ . Similarly, we can prove that  $11 \notin \pi(G)$ .

If  $3 \in \pi(G)$ , then by Lemma 1.9,  $m_3 \in \{235126784, 1645887488, 6583549952\}$ . On the other hand, by Lemma 1.9,  $3^2 \notin \pi_e(G)$ . Thus  $|P_3| \mid (1 + m_3)$ . Therefore  $|P_3| = 3$  and  $n_3 = \frac{m_3}{\varphi(3)} \in \{117563392, 822943744, 3291774976\}$ . Since  $127 \mid n_3$ , we deduce that  $127 \in \pi(G)$ . Now by Lemma 1.9,  $127^2 \notin \pi_e(G)$ . Thus

$|P_{127}| \mid (1+m_{127}) = (1+16912465920)$ . Therefore  $|P_{127}| = 127$  and  $n_{127} = \frac{m_{127}}{\varphi(127)} = 134225920$ .

Since  $29 \mid n_{127}$ , we deduce that  $29 \in \pi(G)$ . Now by Lemma 1.9,  $3 \times 29 \notin \pi_e(G)$ . Thus  $P_3$  acts fixed point freely on the set of elements of order 29 by conjugation. Therefore  $|P_3| \mid m_{29} = 1645887488$ , which is a contradiction. So  $3 \notin \pi(G)$ . If  $\{2, 5, 29, 113, 127\} \subseteq \pi(G)$ , then by Lemma 1.9,  $m_2 = 2080895$ ,

$$\begin{aligned} m_5 &= 235126784, \\ m_{29} &= 1645887488, \\ m_{113} &= 8447918080, \\ m_{127} &= 16912465920 \end{aligned}$$

and  $2^{18}, 5^2, 29^2, 113^2, 127^2 \notin \pi_e(G)$ . Thus by Lemma 1.4 and considering  $m = |P_5|$ , we conclude that  $|P_5| = 5$ . Similarly,  $|P_{29}| = 29$ ,  $|P_{113}| = 113$  and  $|P_{127}| = 127$ .

If  $\pi(G) = \{2\}$ , then since  $2^{18} \notin \pi_e(G)$ , we conclude that  $\pi_e(G) \subseteq \{1, 2, 2^2, \dots, 2^{17}\}$ . Therefore

$$\begin{aligned} |G| = 2^k &= 34093383680 + 266354560k_1 \\ &\quad + 235126784k_2 + 1645887488 \\ &\quad + 6583549952k_4 + 8447918080k_5 \\ &\quad + 16912465920k_6 \end{aligned}$$

where  $k, k_1, k_2, k_3, k_4, k_5$  and  $k_6$  are non-negative integers and  $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 \leq 10$ . It is easy to check that this equation has no solution.

If  $5 \in \pi(G)$ , then  $|P_5| = 5$  and  $n_5 = \frac{m_5}{\varphi(5)} = \frac{235126784}{4} = 58781696$ . We know that  $n_5 \mid |G|$ . Hence  $127 \in \pi(G)$ . Similarly, we can prove that if  $29 \in \pi(G)$  or  $113 \in \pi(G)$ , then  $127 \in \pi(G)$ . So in any cases, we can assume that  $127 \in \pi(G)$ .

Now we prove that  $\pi(G) = \{2, 5, 29, 113, 127\}$ . Since  $127 \in \pi(G)$ , we conclude that  $|P_{127}| = 127$

and  $n_{127} = \frac{m_{127}}{\varphi(127)} = 134225920$ . We know that  $n_{127} \mid |G|$ . Hence  $134225920 \mid |G|$ . It follows that  $\pi(G) = \{2, 5, 29, 113, 127\}$ .

**Step 2.**  $|G| = 2^k \times 5 \times 29 \times 113 \times 127$ , where  $13 \leq k \leq 14$ .

By the above discussion  $|P_5| = 5$ ,  $|P_{29}| = 29$ ,  $|P_{113}| = 113$  and  $|P_{127}| = 127$ .

By Lemma 1.9,  $2 \times 127 \notin \pi_e(G)$ . Thus  $P_2$  acts fixed point freely on the set of elements of order 127 by conjugation. Therefore  $|P_2| \mid m_{127}$ . Hence  $|P_2| \mid 2^{14}$ . On the other hand, since  $n_{127} \mid |G|$ , we deduce that  $2^{13} \mid |G|$ . Hence  $2^{13} \mid |P_2|$ .

**Step 3.**  $G$  is not solvable.

If  $G$  is solvable, then by Lemma 1.7,  $G$  has a Hall  $\pi$ -subgroup  $H$ , where  $\pi = \{5, 29, 113, 127\}$  and all the Hall  $\pi$ -subgroups of  $G$  are conjugate and the number of Hall  $\pi$ -subgroups of  $G$  is  $|G : N_G(H)| \mid 2^{14}$ . Since  $G$  is solvable, we conclude that  $H$  is solvable. Hence by Lemma 1.3, there are nonnegative integers  $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_t$  such that

$$\begin{aligned} n_5(H) &= 29^{\alpha_1 + \dots + \alpha_r} \times 113^{\beta_1 + \dots + \beta_s} \times 127^{\gamma_1 + \dots + \gamma_t}, \\ 29^{\alpha_i} &\equiv 1 \pmod{5}, & 113^{\beta_i} &\equiv 1 \pmod{5}, \\ 127^{\gamma_i} &\equiv 1 \pmod{5}. \end{aligned}$$

Since  $|G| = 2^k \times 5 \times 29 \times 113 \times 127$ , where  $13 \leq k \leq 14$ , we conclude that  $\alpha_1 + \dots + \alpha_r \leq 1$ ,  $\beta_1 + \dots + \beta_s \leq 1$ ,  $\gamma_1 + \dots + \gamma_t \leq 1$ . Therefore  $n_5(H) = 1$ . So  $4 \leq m_5(G) \leq (2^{14} \times 4) = 65536$ , but we have  $m_5(G) = 235126784$ , which is a contradiction.

**Step 4.**  $G \cong Sz(128)$ .

Since  $G$  is a finite group which is not solvable, there is a normal series  $1 \subset N \subset M \subset G$  such that  $N$  is a maximal solvable normal subgroup of  $G$  and  $M/N$  is a non-abelian simple group or the direct product of isomorphic non-abelian simple groups, by

Lemma 1.8. Let  $M/N \cong S_1 \times \dots \times S_r$ , where  $S_1$  is a non-abelian simple group and  $S_1 \cong \dots \cong S_r$ . Since  $1 \in N \subset M \subset G$  and  $|G| = 2^k \times 5 \times 29 \times 113 \times 127$ , where  $13 \leq k \leq 14$ , we conclude that  $r=1$  and  $M/N$  is a non-abelian simple group. Since  $3 \nmid |G|$ , we deduce that  $3 \nmid |M/N|$ . We know that the group  $Sz(q)$  is only non-abelian simple group such that  $3 \nmid |Sz(q)|$ . Hence  $M/N \cong Sz(128)$  and since  $|G| = 2^k \times 5 \times 29 \times 113 \times 127$ , where  $13 \leq k \leq 14$ , we deduce that  $|N| = 1$  and  $G = M \cong Sz(128)$ .

### Acknowledgement

The authors would like to thank the referees for a very careful reading of the paper and for all their insightful comments and valuable suggestions, which improve considerably the presentation of this paper. This work was partially supported by Center of Excellence of Algebraic Hyperstructures and its Applications of Tarbiat Modares University (CEAHA).

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