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Transforming Geocentric Cartesian Coordinates to Geodetic Coordinates by a New Initial Value Calculation Paradigm

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Abstract

Transforming geocentric Cartesian coordinates (X, Y, Z) to geodetic curvilinear coordinates (φ, λ, h) on a biaxial ellipsoid is one of the problems used in satellite positioning, coordinates conversion between reference systems, astronomy and geodetic calculations. For this purpose, various methods including Closed-form, Vector method and Fixed-point method have been developed. In this paper, a new paradigm for calculation of initial values is presented. According to the new initial values, two state of the art iterative methods are modified to calculate the geodetic height and the geodetic latitude accurately and without iteration. The results show that for those points with height values between -10 to 1,000,000 km (30-fold more than the altitude of GPS satellites), the maximum error of the calculated height and geodetic latitude is less than 1.5×10^{-8} m and 1×10^{-14} rad (error lower than 0.001 mm in horizontal), respectively.

Keywords: Geodetic coordinate transformation, Cartesian geocentric coordinate, Curvilinear geodetic coordinate.

1. Introduction

Transforming Cartesian geocentric coordinates (X, Y, Z) to curvilinear geodetic coordinates (φ, λ, h) on biaxial ellipsoid is one of the conversions used in several applications such as navigation and satellite positioning, coordinates conversion from one datum to another, satellite orbit determination and positioning astronomy. According to Figure 1 and Equation (1), the main purpose is the calculation of geodetic coordinates (φ, λ, h) from Cartesian geocentric coordinates (X, Y, Z). In this case, geodetic longitude could be easily calculated from Equation (4) (Vaniček and Krakiwsky, 1986). However, for calculating geodetic latitude and geodetic height, a mathematical problem occurs. For solving this problem, various methods including Closed-form, Vector method and Fixed-point have been developed (Kumi-Boateng and Ziggah, 2016). Current Cartesian to geodetic coordinate transformation methods in the literature focus on the convergence speed, number of iteration, accuracy and innovation in their methods. These methods are summarized in Featherstone and Claessens (2008).

$$\begin{pmatrix} X_G \\ Y_G \\ Z_G \end{pmatrix} = \begin{pmatrix} (N+h)\cos\lambda\cos\varphi \\ (N+h)\sin\lambda\cos\varphi \\ \left(N\left(1-e^2\right)+h\right)\sin\varphi \end{pmatrix} \quad (1)$$

$$N = \frac{a}{\sqrt{1-e^2\sin^2\varphi}} \quad (2)$$

$$e = \sqrt{\frac{a^2-b^2}{a^2}} \quad (3)$$

$$\lambda = 2\tan^{-1}\left(\frac{Y_G}{X_G+\sqrt{X_G^2+Y_G^2}}\right) \quad (4)$$

where N is the radius of curvature in the prime vertical and e is Eccentricity of biaxial ellipsoid.

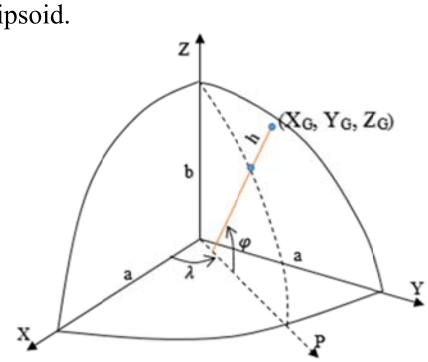

Figure 1. The position of a point with respect to biaxial ellipsoid.

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The fixed-point method is a numerical method that works by iteration (Heiskanen and Moritz, 1967). However, in the Closed-form method, the equations of the Fixed-point method are represented as third or fourth degree equations, and by solving these equations, geodetic latitude is calculated (Featherstone and Claessens, 2008; Vaniček and Krakiwsky, 1986; Vermeille, 2002, 2004; Vermeille, 2011).

In a vector method, the Cartesian coordinates are projected onto the ellipsoid along the normal vector. Then, the geodetic latitude and height are easily calculated from the transformed Cartesian coordinates (Feltens, 2008; Ligas and Banasik, 2011). In a study by Zhang et al. (2005), a new solution is introduced in which the relation between a point inside/outside the ellipsoid with its conjugated on the ellipsoid (along the normal vector) is established by a quadratic equation of the Lagrange parameter. To solve the quadratic equation of the Lagrange parameter, an algebraic algorithm (Zhang et al., 2005), a fast method (Shu and Li, 2010) and a numerical method (Turner, 2009) have been developed. This problem can also be solved by using differential search algorithm (Civicioglu, 2012).

Currently, a huge number of equipment use the satellite positioning systems. Among these equipment, most of them use low-cost receivers (such as mobile phones, unmanned vehicles, and cars). Hence, low-cost GPS receivers require fast and simple mathematical models due to their limitation in computational resources. Thus, the speed of convergence to the solution in transforming Cartesian geocentric coordinates to curvilinear geodetic coordinates on biaxial ellipsoid is always one of the most important criteria in developing and introducing various methods. Fukushima (2006) by solving the equation introduced in (Bowring, 1976), shows that the speed of convergence of his method is faster than the previous works such as: Borkowski, 1989; Fukushima, 1999; Heiskanen and Moritz, 1967; Jones, 2002; Laskowski, 1991; Lin and Wang, 1995; Pollard, 2002; Vermeille, 2002; however, the disadvantage of this method is the singularities at the poles.

The geodetic coordinate transformation was found to be promising in earlier approaches but has not yet been sufficiently examined and still holds potential for further improvements. In this research, a new initial value calculation paradigm is introduced that could transform Cartesian geocentric coordinates (X, Y, Z) to geodetic coordinates (φ, λ, h) on biaxial ellipsoid with no singularities at the poles and without iteration. In the following, the detail of proposed method is explained. Then, in the next section, the method is numerically evaluated and the paper ends with conclusion.

2. Methods

This paper proposes a new paradigm to calculate initial values for two state-of-the-art methods. Hence, at first, the geodetic height and tangent of the parametric latitude (Section 2.1 Equation (13)) are calculated approximately. Then, the geodetic height and geodetic latitude are calculated by modified fixed point and Fukushima's methods. Besides, it is assumed that the biaxial ellipsoid is a geocentric ellipsoid.

2-1. Proposed initial value computation paradigm

In this section, the initial value computation parading is proposed. Hence, the initial values of the geodetic height and the tangent of the parametric latitude T are calculated. For this purpose, a new geocentric ellipsoid with the same eccentricity is scaled to intersect the station point. According to Figure 2, the parameters of the scaled ellipsoid are defined as follows:

$$P_G = \sqrt{X_G^2 + Y_G^2} \tag{5}$$

$$\frac{P_G^2}{A^2} + \frac{Z_G^2}{B^2} = 1 \tag{6}$$

$$A = ka \tag{7}$$

$$B = kb \tag{8}$$

$$k = \sqrt{\frac{P_G^2}{a^2} + \frac{Z_G^2}{b^2}} \tag{9}$$

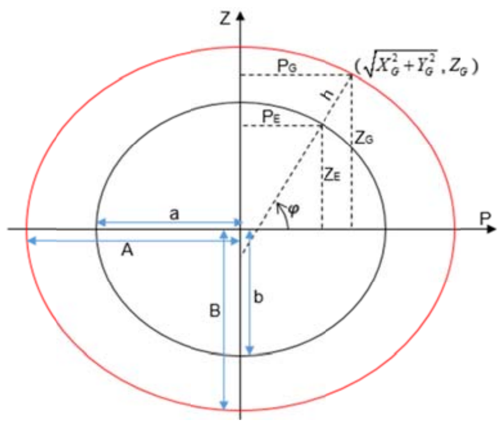

Figure 2. Position of the station point with respect to meridian plane; red: scaled ellipsoid and black: Reference ellipsoid.

In order to calculate the initial value of geodetic height, at first, the following approximations are considered.

$$P_E \cong \frac{P_G}{k}, Z_E \cong \frac{Z_G}{k} \tag{10}$$

According to Figure 2 and Equation (10), the initial value of height is calculated by Equation (12).

$$h = \sqrt{(P_G - P_E)^2 + (Z_G - Z_E)^2} \Rightarrow$$
$$h \cong \sqrt{(P_G - \frac{P_G}{k})^2 + (Z_G - \frac{Z_G}{k})^2} \tag{11}$$

$$h_0 = \frac{(k-1)}{k}\sqrt{P_G^2 + Z_G^2} \tag{12}$$

The tangent of the parametric latitude T is a variable that was defined by (Bowring, 1976) as:

$$T = e_c \tan\varphi$$
$$e_c = \sqrt{1-e^2} \tag{13}$$

According to the proposed method by Heiskanen and Moritz (1967), the relationship between the geodetic latitude and the tangent of the parametric latitude T can be expressed as follow:

$$T = e_c \frac{(N+h)Z_G}{(N(1-e^2)+h)P_G} \tag{14}$$

Hence, the tangent of the parametric latitude T is initially calculated by considering initial values for h and N. According to Equation (12), the initial value of h is calculated easily. However, the initial value of N is approximately calculated using a new relationship. According to Figure 2, the initial value of height could also be calculated by Equation (16).

$$h_0 = \begin{cases} (k-1)a & if \quad \varphi = 0 \\ (k-1)b & if \quad \varphi = 90 \end{cases} \tag{15}$$

So,

$$h_0 = \sqrt{(k-1)^2 a^2 \cos^2\varphi + (k-1)^2 b^2 \sin^2\varphi}$$
$$\Rightarrow h_0 = (k-1)a\sqrt{1-e^2\sin^2\varphi} \tag{16}$$

Therefore, from Equation (2), the initial value of N is equal to:

$$N_0 = \frac{(k-1)a^2}{h_0} \tag{17}$$

By substituting Eqs. (16) and (17) in Equation (14), the initial tangent of the parametric latitude T could be calculated as follows:

$$T_0 = \frac{e_c\left(k^2a^2 + (k-1)\left(P_G{}^2 + Z_G{}^2\right)\right)Z_G}{\left(k^2b^2 + (k-1)\left(P_G{}^2 + Z_G{}^2\right)\right)P_G + \varepsilon} \tag{18}$$

In order to avoid the singularities at the poles, ε is added to Equation (18), which has a value of 10^{-6}. Compared to ε, the large amount of $\left(k^2b^2 + (k-1)\left(P_G{}^2 + Z_G{}^2\right)\right)P_G$ has no significant effect on the final results. For example, around the North Pole (0.1 millimeter), if P_G, Z_G, and K were 1×10^{-4} m, 6.4×10^6 m, and 1 respectively, $\left(k^2b^2 + (k-1)\left(P_G{}^2 + Z_G{}^2\right)\right)P_G$ will be 4×10^9 that compared with 10^{-6} is neglected.

2-2. Modified fixed point method

In this section, the geodetic latitude and height are calculated using modified fixed point method. The geodetic height is calculated from the proposed method by Fukushima (2006).

$$h = \left(e_c P_G + Z_G T - b\sqrt{1+T^2}\right)C \tag{19}$$

where

$$C = \frac{1}{\sqrt{e_c{}^2 + T^2}} \tag{20}$$

By replacing the initial tangent of the parametric latitude T_0 in Equations (19) and (20), the geodetic height is calculated accurately.

Moreover, to calculate the geodetic latitude, the equations of the fixed-point method (Equation (14)) which was presented by Heiskanen and Moritz (1967) are used. Hence, the radius of prime vertical (N) must be calculated.

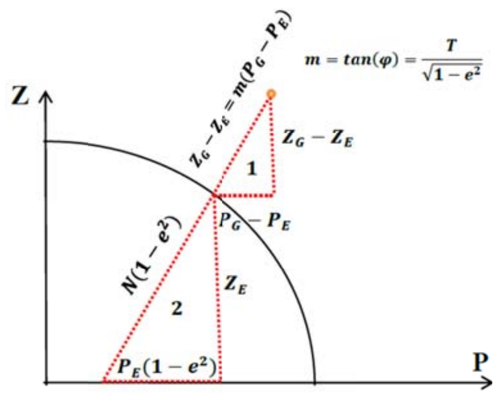

Figure 3. The geometric relationships between geodetic latitude, height and radius of prime vertical.

According to triangle (2) in Figure 3, the following equation is hold:

$$N^2\left(1-e^2\right)^2 = Z_E{}^2 + P_E{}^2\left(1-e^2\right)^2 \Rightarrow$$
$$N^2\left(1-e^2\right) = \frac{a^2 Z_E{}^2}{b^2} + P_E{}^2 - e^2 P_E{}^2 \quad (21)$$

In addition, considering the definition of an ellipsoid:

$$\frac{Z_E{}^2}{b^2} + \frac{P_E{}^2}{a^2} = 1 \Rightarrow \frac{a^2 Z_E{}^2}{b^2} + P_E{}^2 = a^2 \quad (22)$$

By substituting Equation (22) in (21), N is obtained as follows:

$$N = \sqrt{\frac{a^2 - e^2 P_E{}^2}{(1-e^2)}} \quad (23)$$

On the other hand, according to triangle (1) in Figure 3, P_E is equal to:

$$P_E = P_G - \frac{h}{\sqrt{1+\tan^2\varphi}} \Rightarrow$$
$$P_E = P_G - \frac{h\sqrt{1-e^2}}{\sqrt{1-e^2+T^2}} \Rightarrow \quad (24)$$
$$P_E = P_G - e_c hC$$

By substituting T from Equation (18), h from Equation (19) and P_E from Equation (24) in Equation (23), the initial value of N is obtained as follows:

$$N = \frac{\sqrt{a^2 - e^2\left(P_G - e_c hC\right)^2}}{e_C} \quad (25)$$

After obtaining N from Equation (25) and h from Equation (19) and substituting them in Equation (14), the geodetic latitude is calculated as follows:

$$\varphi = \tan^{-1}\frac{\left(N+h\right)Z_G}{\left(Ne_C{}^2 + h\right)P_G + \varepsilon} \quad (26)$$

In order to avoid singularities at the poles, ε is also added to Equation (26), which has a value of 10^{-6}.

2-3. Modified Fukushima's method

The geodetic latitude and height could also be calculated using modified Fukushima's method. The Fukushima's method solves the tangent of the parametric latitude by Halley's method. The tangent of the parametric latitude is expressed as follows:

$$g(T) = PT - Z - \frac{ET}{\sqrt{1+T^2}}$$
$$P = \frac{P_G}{a}, Z = \frac{|Z_G| e_c}{a}, E = e^2 \quad (27)$$

The Halley's method could be simplified as follows:

$$T = T_0 - \frac{2D^2 g_1 g}{2g_1{}^2 - 3ET_0 g} \quad (28)$$

where,

$$T_0 = \frac{e_c\left(k^2a^2 + (k-1)\left(P_G{}^2 + Z_G{}^2\right)\right)|Z_G|}{\left(k^2b^2 + (k-1)\left(P_G{}^2 + Z_G{}^2\right)\right)P_G + \varepsilon}$$
$$g_1 = PD^3 - E \quad (29)$$
$$g = DPT_0 - DZ - ET_0$$
$$D = \sqrt{1+T_0^2}$$

Finally, the geodetic latitude and height are calculated as follows:

$$\varphi = sign(z_G)\arctan\left(\frac{T}{e_c}\right) \quad (30)$$

$$h = \frac{e_c P_G + |Z_G| T - Db}{\sqrt{e_c^{\,2} + T^2}} \qquad (31)$$

3. Numerical results

This section presents a description of data and the results of numerical assessment and comparison of the modified methods. The World Geodetic System 1984 (WGS84) is used as the reference system. The parameters of this biaxial ellipsoid are:

a = 6378137.0 m

e = 0.081819191310869

The indices and factors that lead to the priority of one method to another have no singularities at the poles speed of convergence and low error in calculations.

In order to compare the methods, among available methods, the methods of Fukushima (2006), Vermeille (2011), and Zhang et al. (2005) are selected. All programs and tests have been coded in MATLAB, on Z500 Lenovo notebook with five-core 2.6 GHz processor. In Tables 1-3, dh and $d\varphi$ are geodetic height error and geodetic latitude error, according to meter and radian, respectively.

In this section, at first, some points with known geodetic coordinates are considered, and then these points are converted to Cartesian coordinates using Equation (1). To numerically evaluate the available methods, the Cartesian coordinates of these points are converted to geodetic coordinates, then the converted geodetic coordinates are compared with the original geodetic coordinates.

In the first scenario, the error rate and the speed of convergence are evaluated. 20,000 points in the height range of -10 to 30,000 km and in the geodetic latitude range of -90 to +90 degrees and the longitude geodetic of 45 degrees, with respect to elliptical GRS80, are randomly selected. The performance (speed of convergence and the maximum error of the geodetic height and latitude calculation) of the modified fixed point and Fukushima's methods, Zhang et al., and Vermeille and original Fukushima's methods is presented in Table 1. In this evaluation, processing times are normalized with respect to the original Fukushima's method, the unit of the geodetic latitude and height are radian and meter respectively. In this scenario, all methods implemented without any iteration. As presented in Table 1, the accuracy of original Fukushima's method is low in the case of non-iterative implementation. Although the numerical results of this evaluation demonstrate the universality of modified methods, however, to examine all potential of the proposed initial values, another numerical assessment has been considered on a real dataset. To this end, the transformation between geodetic and Cartesian coordinates between the positions of GRACE satellite around its orbit have been evaluated. According to scenario 1, 17226 positions in the height of 461 km around the whole earth are considered. Table 1 also presents the speed of convergence and geodetic height and latitude errors for GRACE satellite tracing. These results also demonstrate the advantage of the proposed initial value computation paradigm.

Table 1. Comparison of the methods with respect to the time consuming and achieved precision for 20,000 points in the height range of -10 to 30,000 km and in the geodetic latitude range of -90 to +90 degrees and 17226 positions of GRACE satellite.

method	20000 random points			Real satellite tracing		
	dh_{max} (m)	$d\varphi_{max}$ (rad)	**Normalized Time**	dh_{max} (m)	$d\varphi_{max}$ (rad)	**Normalized Time**
Vermeille (2011)	1.1×10^{-8}	5×10^{-16}	1.17	5×10^{-9}	5×10^{-16}	1.20
Zhang et al. (2005)	1.1×10^{-8}	5×10^{-16}	1.13	5×10^{-9}	5×10^{-16}	1.20
Modified fixed point	1.5×10^{-8}	1×10^{-14}	0.9	4×10^{-9}	5×10^{-16}	0.91
Modified Fukushima's method (2006)	1.1×10^{-8}	5×10^{-16}	0.93	4×10^{-9}	5×10^{-16}	0.98
Original Fukushima's method (2006)	1.5×10^{-8}	1×10^{-11}	1	3.6×10^{-9}	8×10^{-14}	1

In the second scenario, the singularity and number of iteration for points located in the geodetic latitude 0, 45 and 90 with the height of 10, 1000, 30,000 and 1,000,000 km are presented in Table 2. Considering the height of a million kilometers (30 times than height of satellites GPS) is just to check universality of the modified methods.

As mentioned earlier, the aim of this paper is to introduce a new initial values paradigm and subsequently a fast way with no singularities at the poles, to convert geocentric coordinates to geodetic coordinates. The results show that the modified methods could successfully achieve what is desired. According to Table 1, it was observed that the speed of convergence to the modified methods are faster than Fukushima, Zhang et al., and Vermeille's methods.

As presented in Table 1, the computational error of Vermeille and Zhang et al. methods are less than of the modified fixed point method. However, this amount of error is negligible due to the effect of these errors that is less than 0.001 mm in horizontal. While in these circumstances, original Fukushima's method (in the non-iteration state) has the amount of error less than 1 mm, and in cases where higher accuracy is required, it must be solved with repetition.

One factor that leads to the priority of one method to another is the universality of that method. Hence, to check no singularities and the number of iteration, the modified fixed point method is compared with original Fukushima's method (Fukushima, 2006).

As presented in Tables 1 and 2, the modified methods calculate the geodetic height and latitude with maximum error of 1.5×10^{-8} meters and 1×10^{-14} radians (error lower than 0.001 millimeter in horizontal) respectively, which is more accurate than original Fukushima's method. In addition, unlike Fukushima's method, the modified methods do not have singularities at the poles and are also considered as universal methods.

Besides, the results of Table 2 showed that for points with a height of one million kilometers (30 times than height of satellites GPS), the modified methods are able to solve this problem without iteration and with the computational error less than 1 mm. While for points with this amount of height, original Fukushima's method needs repetition.

Table 2. Comparison of the modified fixed point method with original Fukushima's method in singularities and number of iteration.

position	**dh (m)**		$d\varphi(rad)$		**Number of iteration**	
(φ, λ, h)	Modified fixed point method	Original Fukushima (2006)	Modified fixed point method	Original Fukushima (2006)	Modified fixed point method	Original Fukushima (2006)
$(\frac{\pi}{4},\frac{\pi}{4},10)$	9×10^{-10}	2×10^{-9}	1×10^{-16}	1×10^{-16}	0	0
$(\frac{\pi}{4},\frac{\pi}{4},1000)$	9×10^{-10}	9×10^{-10}	9×10^{-16}	3×10^{-13}	0	0
$(\frac{\pi}{4},\frac{\pi}{4},30000)$	4×10^{-9}	4×10^{-9}	1×10^{-14}	1×10^{-11}	0	0
$(\frac{\pi}{4},\frac{\pi}{4},1000000)$	0	0	1×10^{-15}	0	0	1
$(0,\frac{\pi}{4},10)$	3×10^{-10}	3×10^{-10}	0	0	0	0
$(0,\frac{\pi}{4},1000)$	1×10^{-9}	1×10^{-9}	0	0	0	0
$(0,\frac{\pi}{4},30000)$	0	0	0	0	0	0
$(0,\frac{\pi}{4},1000000)$	0	0	0	0	0	0
$(\frac{\pi}{2},\frac{\pi}{4},10)$	6×10^{-11}	singular	0	singular	0	0
$(\frac{\pi}{2},\frac{\pi}{4},1000)$	5×10^{-10}	singular	0	singular	0	0
$(\frac{\pi}{2},\frac{\pi}{4},30000)$	0	singular	0	singular	0	0
$(\frac{\pi}{2},\frac{\pi}{4},1000000)$	0	singular	0	singular	0	0

In another test for checking the number of iteration and universality of the modified methods, in this study, the most well-known global biaxial ellipsoids and a synthetic ellipsoid with different eccentricity are evaluated. In this evaluation, considering various ellipsoids, for points set in Table 2, the maximum geodetic latitude and height error of the modified methods are presented in Table 3. In this table, all results achieved without iteration.

Due to the eccentricity of less than 0.1 for known defined ellipsoids, the results in Table 3 show that the modified methods calculate the geodetic height and latitude without iteration, thus the modified methods are considered as the non-iterative method. Hence, the step-by-step of the modified methods are presented in Appendix 1and 2 without iteration.

5. Conclusion

The aim of this paper is to modify two state-of-the-art methods with high speed of convergence and no singularities at the poles, to convert Cartesian coordinates to geodetic coordinates. The results show that the speed of convergence to the modified methods are faster than original Fukushima (2006), Zhang et al. and Vermeille's methods and the error of geodetic height and latitude calculation are also less than mentioned methods. In addition, unlike Fukushima's method in the modified methods, the points located at poles do not have singularities.

Besides, according to the results, for ellipsoids with eccentricity less than 0.3, the calculations show that the modified methods (in the height range of -10 to 1 million kilometers away from the elliptical surface) can convert Cartesian coordinates to geodetic coordinates without iteration.

Table 3. Evaluation of the modified fixed point and modified Fukushima's methods for a number of well-known global ellipsoids.

Ellipsoid	**a**	**Eccentricity**	**Modified Fukushima's method**		**Modified fixed point method**	
			$dh_{max}\ (m)$	$d\varphi_{max}\ (rad)$	$dh_{max}\ (m)$	$\boldsymbol{d\varphi_{max}\ (rad)}$
Airy 1830	6377563.396	0.08167337	1.1×10^{-8}	6×10^{-16}	1.5×10^{-8}	1×10^{-14}
Bessel 1841	6377397.155	0.08169683	1.1×10^{-8}	6×10^{-16}	1.5×10^{-8}	1×10^{-14}
Clarke 1880	6378249.145	0.08248321	1.1×10^{-8}	6×10^{-16}	1.9×10^{-8}	1×10^{-14}
ED50	6378388.000	0.08199189	1.1×10^{-8}	6×10^{-16}	1.5×10^{-8}	1×10^{-14}
SAD69	6378160.000	0.08182018	1.1×10^{-8}	6×10^{-16}	1.9×10^{-8}	1×10^{-14}
GRS80	6378137.000	0.08181919	1.1×10^{-8}	6×10^{-16}	1.5×10^{-8}	1×10^{-14}
WGS84	6378137.000	0.08181919	1.1×10^{-8}	6×10^{-16}	1.5×10^{-8}	1×10^{-14}
Synthetic ellipsoid	6378137.000	0.05	2.2×10^{-8}	5×10^{-16}	2.2×10^{-8}	5×10^{-16}
Synthetic ellipsoid	6378137.000	0.06	1.8×10^{-8}	6×10^{-16}	1.8×10^{-8}	7×10^{-16}
Synthetic ellipsoid	6378137.000	0.07	1.8×10^{-8}	6×10^{-16}	2.2×10^{-8}	2×10^{-15}
Synthetic ellipsoid	6378137.000	0.08	1.8×10^{-8}	6×10^{-16}	1.8×10^{-8}	8×10^{-15}
Synthetic ellipsoid	6378137.000	0.09	1.8×10^{-8}	5×10^{-16}	1.8×10^{-8}	3×10^{-14}
Synthetic ellipsoid	6378137.000	0.1	1.4×10^{-8}	6×10^{-16}	1.8×10^{-8}	7×10^{-14}
Synthetic ellipsoid	6378137.000	0.15	$1.3\times10-7$	6×10^{-16}	1.3×10^{-7}	4.4×10^{-12}
Synthetic ellipsoid	6378137.000	0.2	$4.3\times10-6$	6×10^{-16}	4.3×10^{-6}	8×10^{-11}
Synthetic ellipsoid	6378137.000	0.3	6.5×10^{-4}	5×10^{-16}	6.5×10^{-4}	5×10^{-9}

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Appendix 1: The step-by-step of the modified fixed point method

The algorithm of the modified fixed point method to convert Cartesian geocentric coordinates to geodetic coordinates
(For ellipsoids with eccentricities less than 0.1).

1. Calculate PG from Equation (5)

$$P_G = \sqrt{X_G^{\ 2} + Y_G^{\ 2}}$$

2. Calculate k from Equation (9)

$$k = \sqrt{\frac{P_G^{\ 2}}{a^2} + \frac{Z_G^{\ 2}}{b^2}}$$

3. Calculate T_0 from Equation (18)

$$T_0 = \frac{e_c\left(k^2a^2 + (k-1)\left(P_G^{\ 2} + Z_G^{\ 2}\right)\right)Z_G}{\left(k^2b^2 + (k-1)\left(P_G^{\ 2} + Z_G^{\ 2}\right)\right)P_G + 10^{-6}}$$

$$e_c = \sqrt{1-e^2}$$

4. Calculate C from Equation (20)

$$C = \frac{1}{\sqrt{e_c^{\ 2} + T_0^{\ 2}}}$$

5. Calculate h from Equation (19)

$$h = \left(e_c P_G + Z_G T_0 - b\sqrt{1+T_0^{\ 2}}\right)C$$

6. Calculate N from Equation (25)

$$N = \frac{\sqrt{a^2 - e^2\left(P_G - e_c hC\right)^2}}{e_C}$$

7. Calculate φ from Equation (26)

$$\varphi = \tan^{-1}\frac{(N+h)Z_G}{\left(Ne_C^{\ 2} + h\right)P_G + 10^{-6}}$$

Appendix 2: The step-by-step of the modified Fukushima's (2006) method

The algorithm of modified Fukushima's method (2006) to convert Cartesian geocentric coordinates to geodetic coordinates

(For ellipsoids with eccentricities less than 0.1).

1. Calculate PG from Equation (5)

$$P_G = \sqrt{X_G^{\ 2} + Y_G^{\ 2}}$$

2. Calculate k from Equation (9)

$$k = \sqrt{\frac{P_G^{\ 2}}{a^2} + \frac{Z_G^{\ 2}}{b^2}}$$

3. Calculate the following parameters

$$P = \frac{P_G}{a}, Z = \frac{|Z_G| e_c}{a}, E = e^2$$

4. Calculate T_0 from the following equation

$$T_0 = \frac{e_c\left(k^2 a^2 + (k-1)\left(P_G^{\ 2} + Z_G^{\ 2}\right)\right)|Z_G|}{\left(k^2 b^2 + (k-1)\left(P_G^{\ 2} + Z_G^{\ 2}\right)\right)P_G + 10^{-6}}$$

$$e_c = \sqrt{1 - e^2}$$

5. Calculate the following parameters

$$D = \sqrt{1 + T_0^{\ 2}}$$

$$g_1 = PD^3 - E$$

$$g = DPT_0 - DZ - ET_0$$

6. Calculate T from Equation (28)

$$T = T_0 - \frac{2D^2 g_1 g}{2g_1^{\ 2} - 3ET_0 g}$$

7. Calculate h from Equation (31)

$$h = \frac{e_c P_G + |Z_G| T_0 - Db}{\sqrt{e_c^{\ 2} + T_0^{\ 2}}}$$

8. Calculate φ from Equation (30)

$$\varphi = sign(z_G) \tan^{-1}\left(\frac{T}{e_c}\right)$$