

Direct Limit of Krasner (m, n) -Hyperrings

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Abstract

The purpose of this paper is the study of direct limits in category of Krasner (m, n) -hyperrings. In this regards we introduce and study direct limit of a direct system in category (m, n) -hyperrings. Also, we consider fundamental relation Γ^* , as the smallest equivalence relation on an (m, n) -hyperring R such that the quotient space R/Γ^* is an (m, n) -ring, to introduce the fundamental functor from category of Krasner (m, n) -hyperrings to the category of (m, n) -rings. Finally, we study the relationship between fundamental functor and direct limit on Krasner (m, n) -hyperrings. In particular, we prove that the fundamental functor is exact and obtain some its basic properties.

Keywords: Krasner (m, n) -hyperrings; Direct system; Fundamental functor; Direct limit.

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Introduction

An n -ary hyperoperation is a mapping

$$f : \underbrace{H \times \cdots \times H}_n \rightarrow P^*(H), \quad \text{where } P^*(H) \text{ is}$$

the set of all nonempty subsets of H . In this case, (H, f) is said to be an n -ary hypergroupoid. This is a generalization of a hypergroupoid (when $n = 2$), that was defined by Marty in [1] as founder of hyperstructure theory (for more details refer to [2], [3], [4], [5], [6], [7], [8] and [9]). An n -ary hyperoperation initiated an n -ary hyperstructure. Nowadays, n -ary hyperstructures is a well-known field of researches on hyperstructures theory (for more see [10], [11], [12], [13], [14], [15], [16] and [17]). Also, recently some researchers studied direct systems

and direct limit on (fuzzy) hyperstructures (for instance see [18], [19], [20], [21], [22] and [23]).

In this paper, we consider category of Krasner (m, n) -hyperrings ([11]), introduce, and study direct limit of a direct system in this category. In this regards, we introduce the fundamental functor from category of (m, n) -hyperrings into category of (m, n) -rings, via the fundamental relation. In particular, we prove that this functor preserves all direct limits.

Preliminaries

In this section, we present some basic concepts of n -ary hyperstructures which we need to development our paper.

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In dealing with n -ary hyperstructures, for abbreviation, we will show the sequence x_i, x_{i+1}, \dots, x_j by x_i^j , and we put $x_i^j = \emptyset$ for $j < i$. Hence $f(x_1, \dots, x_i, y_{i+1}, \dots, y_j, z_{j+1}, \dots, z_n)$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. Also, if $y_{i+1} = \dots = y_j = y$, then it is shown by $f(x_1^i, y^{(j-i)}, z_{j+1}^n)$. Moreover, if f is an n -ary hyperoperation and $t = l(n-1)+1$, for some $l \geq 0$, then t -ary hyperoperation $f_{(l)}$ is given by $f_{(l)}(x_1^{l(n-1)+1}) = \underbrace{f(f(\dots f(f(x_1^n))$

$, x_{n+1}^{2n-1}), \dots), x_{(l-1)(n-1)+1}^{l(n-1)+1})$. For nonempty subsets A_1, \dots, A_n of H we define $f(A_1^n) = \bigcup \{f(x_i^n) \mid x_i \in A_i, i = 1, \dots, n\}$. We say (H, f) is an n -ary semihypergroup if f is associative, that is, $f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$, holds, for every $1 \leq i < j \leq n$ and all $x_1, x_2, \dots, x_{2n-1} \in H$. If (H, f) is an n -ary semihypergroup and $f(x_1^{i-1}, H, x_{i+1}^n) = H$ for all $x_1^n \in H$ and $1 \leq i \leq n$, then (H, f) is called an n -ary hypergroup. (H, f) is said to be commutative, if for all $\sigma \in S_n$ and for every $a_i^n \in H$, we have $f(a_i^n) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)})$. moreover, a nonempty subset B of an n -ary hypergroup (H, f) is called an n -ary subhypergroup of H , if (B, f) is an n -ary hypergroup.

Let (H, f) be a commutative n -ary hypergroup. (H, f) is called a canonical n -ary hypergroup ([12]), if

- i) there exists a unique $e \in H$ such that for every $x \in H, f(x, e^{n-1}) = \{x\}$;
- ii) for all $x \in H$ there exists a unique $x^{-1} \in H$ such that $e \in f(x, x^{-1}, e^{n-2})$;
- iii) if $x \in f(x_1^n)$ then, for all $1 \leq i \leq n$ we have $x_i \in f(x, x^{-1}, \dots, x_{i-1}^{-1}, x_{i+1}^{-1}, \dots, x_n^{-1})$.

Definition [16] (R, f, g) is said to be an (m, n) -hyperring, if:

- i) (R, f) is an m -ary hypergroup.
- ii) (R, g) is an n -ary semihypergroup.

The n -ary hyperoperation g is distributive with respect to the m -ary hyperoperation f , i.e., for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$ $g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \dots, g(a_1^{i-1}, x_m, a_{i+1}^n))$. A nonempty subset S of R is called an (m, n) -subhyperring, if (R, f, g) is an (m, n) -hyperring. Let $i \in \{1, \dots, n\}$. An i -hyperideal I of R is an (m, n) -subhyperring of R such that $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$ for every $x_1^n \in R$. I is called a hyperideal, if I is a i -hyperideal, for all $1 \leq i \leq n$.

An (m, n) -hyperring (R, f, g) is said to be Krasner if (R, f) is a canonical n -ary hypergroup and (R, g) is an n -ary semigroup such that 0 is a zero element (absorbing element) of the n -ary operation g , i.e. for all $x_2^n \in R$ we have $g(0, x_2^n) = g(x_2, 0, x_3^n) = \dots = g(x_2^n, 0)$.

Example [16] Suppose that (L, \vee, \wedge) is a relatively complemented distributive lattice. Define f and g on L as follows: $f(a_1, a_2) = \{c \in L \mid a_1 \wedge c = a_2 \wedge c = a_1 \wedge a_2\}$, and $g(a_1^n) = \bigvee_{i=1}^n a_i, \forall a_i^n \in L$.

It is easy to verify that (L, f, g) is a Krasner $(2, n)$ -hyperring.

Definition [24] A category consists of the following data:

- Objects A, B, C
- Arrows: f, g, h
- For each arrow f , there are given objects $dom(f), cod(f)$ called the domain and codomain of f . We write $f: A \rightarrow B$ to indicate that $A = dom(f)$ and $B = cod(f)$.
- Given arrows $f: A \rightarrow B$ and $g: B \rightarrow C$, that is, with $cod(f) = dom(g)$ there is given an

arrow $g \circ f : A \rightarrow C$ called the composite of f and g .

- For each object A , there is given an arrow $1_A : A \rightarrow A$ called the identity arrow of A . These data are required to satisfy the following laws:
 - Associatively: $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$.
 - Unit: $f \circ 1_A = f = 1_B \circ f$ for all $f : A \rightarrow B$.

Example [24] group and group homomorphisms is a category.

Definition [24] In any category C , an object

- 0 is initial if for any object c there is a unique morphism $0 \rightarrow c$
- 1 is terminal if for any object c there is a unique morphism $c \rightarrow 1$.

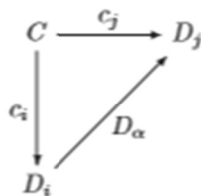
Definition [24] A functor $F : C \rightarrow D$ between categories C and D is a mapping of objects and arrows to arrows, in such a way that

- $F(f : A \rightarrow B) =$
 - $F(f) : F(A) \rightarrow F(B)$,
 - $F(1_A) = 1_{F(A)}$,
 - $F(g \circ f) = F(g) \circ F(f)$.

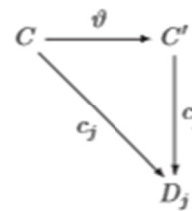
Definition [24] Let J and C be categories. A diagram of type J in C is a functor $F : J \rightarrow C$.

We write the objects in the index category J lower case, i, j, \dots and the values of the functor $D : J \rightarrow C$ in the form D_i, D_j , etc.

A cone to a diagram D consists of an object c in C and a family of arrows in C , $c_j : C \rightarrow D_j$ one for each object $j \in J$, such that for each arrow $\alpha : i \rightarrow j$ in J , the following triangle commutes:



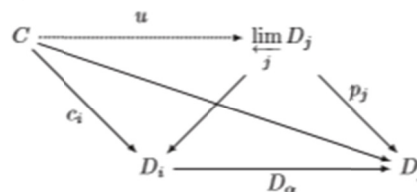
A morphism of cones $\vartheta : (C, c_j) \rightarrow (C', c'_j)$ is an arrow ϑ in C making each triangle,



commute. Thus we have evident category $Cone(D)$ of cones to D .

Definition [24] A limit for a diagram $D : J \rightarrow C$ is a terminal object in category $Cone(D)$.

We often denote a limit in the form $p_i : \varinjlim D_j \rightarrow D_i$.



In general, a colimit for a diagram $D : J \rightarrow C$ is of course, an initial object in the category of cocones. We write such a colimit in the form $\varinjlim D_j$.

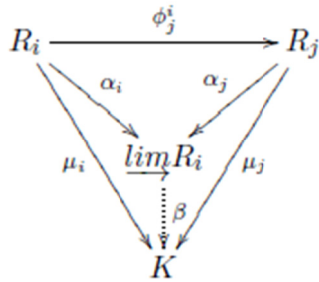
Results

Direct System (Limit) of Krasner (m, n)-Hyperrings

We say a partially ordered set I is a directed set if for each $(i, j) \in I \times I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Let I be a directed set and (m, n) -KH_r the category of Krasner (m, n) -hyperrings with strong homomorphisms. Let $(R_i, \mathcal{f}_i, \mathcal{g}_i)_{i \in I}$ be a family of Krasner (m, n) -hyperrings indexed by I . For each pair $i, j \in I$ such that $i \leq j$, let $\phi_j^i : R_i \rightarrow R_j$ be a strong homomorphism where ϕ_i^i is the identity homomorphism for all $i \in I$ and $\phi_k^i = \phi_k^j \circ \phi_j^i$ for $i \leq j \leq k$. Then $R = (R_i, \phi_j^i)$ is said to be a direct system over the direct set I .

The direct limit of a direct system $R = (R_i, \phi_j^i)$ in (m, n) -KH denoted by $\varinjlim R_i$, is a Krasner (m, n) -hyperring and a family of strong homomorphisms

$\alpha_i : R_i \rightarrow \varinjlim R_i \mid \alpha_i = \alpha_j \phi_j^i; i \leq j$, which for every Krasner (m, n) -hyperring K and every family of strong homomorphisms $\{\mu_i : R_i \rightarrow K \mid \mu_i = \mu_j \phi_j^i; i \leq j\}$ there is a unique strong homomorphism $\beta : \varinjlim R_i \rightarrow K$ such that the following diagram is commutated:



Let $X = \bigcup_{i \in I} R_i$ and define the following equivalence relation on X for $a_i \in R_i$ and $a_j \in R_j$: $a_i \sim a_j \Leftrightarrow \phi_k^i a_i = \phi_k^j a_j$ for $k \geq i, j$. Suppose that $\bar{X} = \{[a_i] \mid a_i \in X\}$ where $[a_i]$ is the equivalent class of a_i . It is clear that $a_i \sim \phi_j^i a_j$ for $j \geq i$ in \bar{X} .

Theorem 1.1. (\bar{X}, F) is an m -ary canonical hypergroup where $\{[x] \mid x \in f(a_{1k}^{mk}); a_{lk} = \phi_k^l a_l\}$ and $a_{lk} = \phi_k^l a_l$ for $k \geq l, 1 \leq l \leq m$.

Proof. First we show that F is well-defined. Let $([a_1], \dots, [a_m]) = ([b_1], \dots, [b_m])$, then $[a_i] = [b_i]$ for all $1 \leq i \leq m$. Thus, for every $1 \leq i \leq m$ there exist $k_i \geq i$ such that $\phi_{k_i}^i a_i = \phi_{k_i}^i b_i$. Hence, we have $[x] \in F([a_1], \dots, [a_m]) \Leftrightarrow x \in f(\phi_{k_{(m+1)}}^1 a_1, \dots, \phi_{k_{(m+1)}}^m a_m)$ for $k_{(m+1)} \geq 1, \dots, m \Leftrightarrow x \in f(\phi_k^1 a_1, \dots, \phi_k^m a_m)$ for $k \geq k_1, \dots, k_m, k_{(m+1)} \Leftrightarrow x \in f(\phi_k^{k_1} \phi_{k_1}^1 a_1, \dots, \phi_k^{k_m} \phi_{k_m}^m a_m) \Leftrightarrow x \in f(\phi_k^{k_1} \phi_{k_1}^1 b_1, \dots, \phi_k^{k_m} \phi_{k_m}^m b_m) \Leftrightarrow x \in f(\phi_k^1 b_1, \dots, \phi_k^m b_m) \Leftrightarrow [x] \in F([b_1], \dots, [b_m])$. So, F is well-defined. Now, let $[a]_l^{2m-1} \in \bar{X}$ and

$k \geq 1, \dots, 2m-1$. Also, let $[x] \in F([a]_l^{i-1}, F([a]_i^{m+i-1}), [a]_{m+i}^{2m-1})$ for all $1 \leq i \leq m$. So, there exists $[y] \in F([a]_i^{m+i-1})$ such that $[x] \in F([a]_l^{i-1}, [y], [a]_{m+i}^{2m-1})$. Therefore $y \in f(a_{ik}, \dots, a_{(m+i-1)k})$ for some $k \geq i, \dots, m+i-1$ such that $\phi_k^i a_i = a_{ik}, \dots, \phi_k^{m+i-1} a_{m+i-1} = a_{(m+i-1)k}$. Also $x \in f(a_{1k}, \dots, a_{(i-1)k}, b_k, a_{(m+i)k}, \dots, a_{(2m-1)k})$ for some $k \geq 1, \dots, i-1, m+i, \dots, 2m-1$ such that $\phi_k^1 a_1 = a_{1k}, \dots, \phi_k^{i-1} a_{i-1} = a_{(i-1)k}, \dots, \phi_k^{m+i} a_{m+i} = a_{(m+i)k}, \dots, \phi_k^{2m-1} a_{2m-1} = a_{(2m-1)k}$. Since $y \in R_k$, $b_k = \phi_k^k y = y$. Thus by associativity of R_k we have $x \in f(a_{1k}, \dots, a_{(i-1)k}, b_k, a_{(m+i)k}, \dots, a_{(2m-1)k}) = f(a_{1k}, \dots, a_{(i-1)k}, y, a_{(m+i)k}, \dots, a_{(2m-1)k}) \subseteq f(a_{1k}, \dots, a_{(i-1)k}, f(a_{ik}, \dots, a_{(m+i-1)k}), a_{(m+i)k}, \dots, a_{(2m-1)k}) = f(a_{1k}, \dots, a_{(j-1)k}, f(a_{jk}, \dots, a_{(m+j-1)k}), a_{(m+j)k}, \dots, a_{(2m-1)k})$. So there exist $y' \in f(a_{jk}, \dots, a_{(m+j-1)k})$ such that $x \in f(a_{1k}^{(j-1)k}, y', a_{(m+j)k}^{(2m-1)k})$. Then, $[y'] \in F([a]_j^{m+j-1})$ and since $\phi_k^k y' = y'$, we have $[x] \in F([a]_l^{j-1}, [y'], [a]_{m+j}^{2m-1})$. Hence $[x] \in F([a]_l^{j-1}, F([a]_j^{m+j-1}), [a]_{m+j}^{2m-1})$ and then for any $m \geq j > i \geq 1$ $F([a]_l^{i-1}, F([a]_i^{m+i-1}), [a]_{m+i}^{2m-1}) \subseteq F([a]_l^{j-1}, F([a]_j^{m+j-1}), [a]_{m+j}^{2m-1})$. Similarly, we can prove the converse of above inclusion. So (\bar{X}, F) is associative.

Now, we prove the reproduction axiom. It is clear that

$$F([a]_l^{i-1}, \bar{X}, [a]_{i+1}^m) \subseteq \bar{X}, \text{ for all } [a]_l^m \in \bar{X}.$$

Assume that $[a]_l^m \in \bar{X}$. Then

$\phi_k^j a_j = a_{jk} \in R_k$ for any $k \geq 1, \dots, m$ where $1 \leq j \leq m$. Since R_k is an m -ary hypergroup, so for all $1 \leq i \leq m$, $a_{ik} \in R_k = f(a_{1k}, \dots, a_{(i-1)k}, R_k, a_{(m+i)k}, \dots, a_{mk})$.

Then, there exists $a'_k \in R_k$ such that $a_{ik} \in f(a_{1k}^{(i-1)k}, a'_k, a_{(m+i)k}^{mk})$ and $\phi_k^k a'_k = a'_k$. Thus $[a_i] \in F([a]_1^{i-1}, [a'_k], [a]_{i+1}^m) \subseteq F([a]_1^{i-1}, \bar{X}, [a]_{i+1}^m)$. Thus, $\bar{X} = F([a]_1^{i-1}, \bar{X}, [a]_{i+1}^m)$ for all $[a]_1^{i-1}, [a]_{i+1}^m \in \bar{X}$. So (\bar{X}, F) is an m -ary hypergroup.

Since (R_k, f) is a canonical n -ary hypergroup, there exists a unique element $0_k \in R_k$ such that $\{a_k\} = f(a_k, 0_k^{(m-1)})$ for all $a_k \in R_k$. Now let $[a_i] \in \bar{X}$. Then there exists $k \geq i$ such that $\phi_k^i a_i = a_k \in R_k$, since (R_k, f) is canonical. Hence $\{a_k\} = f(a_k, 0_k^{(m-1)})$ such that $\phi_k^k 0_k = 0_k$. Since 0_k is unique, so $\{[a_i]\} = F([a_i], [0_k]^{(m-1)})$. Also, since (R_k, f) has canonical property, there exists uniquely $a' \in R_k$ such that $0_k \in f(a_k, a'_k, 0_k^{(m-2)})$.

Since $\phi_k^k a'_k = a'_k$, and hence $[0_k] \in F([a_i], [a'_k], [0_k]^{(m-2)})$. Therefore, we show $[a'_k]$ as the inverse of $[a_i]$ is in \bar{X} . Finally, we investigate the reversibility property. Let $[x] \in F([a]_1^m)$. Then there exist $k \geq 1, \dots, n$ such that $x \in f(a_{1k}^{mk})$ and $\phi_k^1 a_1 = a_{1k}, \dots, \phi_k^m a_m = a_{mk}$. Since $x \in R_k$, so $\phi_k^k x = x$ and since (R_k, f) is canonical, we have $a_{ik} \in f(a'_{1k}, \dots, a'_{(i-1)k}, x, a'_{(i+1)k}, a'_{mk})$ for all $1 \leq i \leq m$. Hence $[a_i] \in F([a'_1], \dots, [a'_{i-1}], [x], [a'_{i+1}], \dots, [a'_m])$.

Theorem 1.2. (\bar{X}, F, G) is a Krasner (m, n) -hyperring where $G([b_1], \dots, [b_n]) = [g(b_{1k}^{nk})]$ such that $b_{lk} = \phi_k^l b_l$ for $k \geq l$ and $1 \leq l \leq n$.

Proof. By Theorem 1.1. (\bar{X}, F) is an m -ary canonical hypergroup. Similarly, we can prove (\bar{X}, G) is an n -ary semigroup. We prove that G is distributive with respect to F . Assume that $[a]_1^{i-1}, [x]_1^m, [a]_{i+1}^n \in \bar{X}$ and for any $1 \leq i \leq n$, $[y] \in G([a]_1^{i-1}, F([x]_1^m), [a]_{i+1}^n)$. Then there exists $[b] \in F([x]_1^m)$ such that $[y] = G([a]_1^{i-1}, [b], [a]_{i+1}^n)$.

Hence, $b \in f(x_{1k}, \dots, x_{mk})$ for $\phi_k^1 x_1 = x_{1k}, \dots, \phi_k^m x_m = x_{mk}$ with

$k \geq 1, \dots, m$. Also $y = g(a_{1k}^{(i-1)k}, c_k, a_{(i+1)k}^{nk})$, such that $\phi_k^1 a_1 = a_{1k}, \dots, \phi_k^{i-1} a_{i-1} = a_{(i-1)k}, \phi_k^{i+1} a_{i+1} = a_{(i+1)k}, \dots, \phi_k^n a_n = a_{nk}$ for some $k \geq 1, \dots, i-1, i+1, \dots, n$. Since, $b \in R_k$,

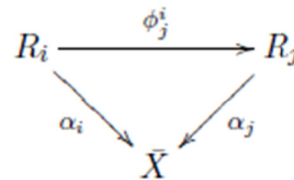
$c_k = \phi_k^k b = b$. Therefore, by distributivity on R_k , we obtain that $y = g(a_{1k}^{(i-1)k}, c_k, a_{(i+1)k}^{nk}) = g(a_{1k}^{(i-1)k}, b, a_{(i+1)k}^{nk}) \in g(a_{1k}^{(i-1)k}, f(x_{1k}^{mk}), a_{(i+1)k}^{nk}) = f(g(a_{1k}^{(i-1)k}, x_{1k}, a_{(i+1)k}^{nk}), \dots, g(a_{1k}^{(i-1)k}, x_{mk}, a_{(i+1)k}^{nk}))$, for all $1 \leq i \leq n$.

Since, $x_{1k}, \dots, x_{mk} \in R_k$, $\phi_k^k x_i = x_i$. It follows that $[y] \in F(G([a]_1^{i-1}, [x]_1^m, [a]_{i+1}^n), \dots, G([a]_1^{i-1}, [x]_m, [a]_{i+1}^n))$.

Similarly, the converse of above inclusion can be obtained.

Theorem 1.3. \bar{X} is $\varinjlim R_i$ in (R_i, ϕ_j^i) , the direct system of Krasner (m, n) -hyperrings indexed by I .

Proof. Consider $\alpha_i : R_i \rightarrow \bar{X}$ defined by $\alpha_i(a_i) = [a_i]$ and the following diagram:



For all $a_i \in R_i$ we have $\alpha_j \circ \phi_j^i(a_i) = [\phi_j^i a_i] = [a_i] = \alpha_i(a_i)$ and so the diagram is commutative. Now, let R be a Krasner (m, n) -hyperring and $\{\mu_i \mid \mu_i : R_i \rightarrow R\}$ a family of strong homomorphism with $\mu_i = \mu_j \circ \phi_j^i$. Define $\beta : \bar{X} \rightarrow R$ by

$\beta([a_i]) = \mu_i(a_i)$. We show that β is a strong homomorphism and so the universal mapping property holds. First we show that β is well defined. Let $[a_i] = [b_j]$ then there exists $k \geq i, j$ such that

$\phi_k^i a_i = \phi_k^j b_j$. Thus $\mu_k(\phi_k^i a_i) = \mu_k(\phi_k^j b_j)$, and so $\mu_i(a_i) = \mu_j(b_j)$. Hence, β is well defined. Now let $[a_1], \dots, [a_m] \in \bar{X}$ then $\beta(F([a_1]^m)) = \{\beta([x]) \mid x \in f(a_{1k}^{mk})\};$
 $a_{lk} = \phi_k^l a_l$ for $k \geq l, 1 \leq l \leq m\}$
 $= \mu_k(f(a_{1k}^{mk})); a_{lk} = \phi_k^l a_l$ for $k \geq l, 1 \leq l \leq m$
 $= f(\mu_k(a_{1k}), \dots, \mu_k(a_{mk}))$ (since μ_k is strong homomorphism) $= f(\mu_k(\phi_k^1 a_1), \dots, \mu_k(\phi_k^m a_m))$
 $= f(\mu_1(a_1), \dots, \mu_m(a_m)) = f(\beta([a_1]), \dots, \beta([a_m]))$.
 similarly, we can show that $\beta(G([b_1]^n)) = g(\beta([a_1]), \dots, \beta([a_n]))$ So, β is a strong homomorphism such that $\beta \circ \alpha_i = \mu_i$.

Γ^* -Relation and Direct systems

Let (R, f, g) be an Krasner (m, n) -hyperring. Mirvakili and Davvaz in [16] introduced the strongly compatible relation Γ^* on (m, n) -hyperrings as follow:

For every $k \in \mathbf{Z}^+$ and $l_1^s \in \mathbf{Z}^+$, where $s = k(m-1)+1$ we have $x \Gamma_{k;l_1^s} y$ if and only if $\{x, y\} \subseteq f_{(k)}(u_1, \dots, u_s)$ with $u_i = g_{(l_i)}(x_{i1}^{it_i})$ for some $x_{i1}^{it_i} \in R$, where $t_i = l_i(n-1)+1$ and $1 \leq i \leq s$. Now, set $\Gamma_k = \bigcup_{l_1^s \in \mathbf{N}} \Gamma_{k;l_1^s}$ and $\Gamma = \bigcup_{k \in \mathbf{N}} \Gamma_k$. It is shown that

the transitive closure of Γ , denoted by Γ^* , is the smallest strongly compatible relation on R , such that $(R/\Gamma^*, F, G)$ is an (m, n) -ring, where the m -ary and n -ary operations F and G in R/Γ^* are defined as follows: $F(\Gamma^*(x)_1^m) = \Gamma^*(c)$, for all $c \in f(\Gamma^*(x)_1^m)$; $G(\Gamma^*(x)_1^n) = \Gamma^*(d)$ for all $d \in g(\Gamma^*(x)_1^n)$. Consider $\phi: R \rightarrow R/\Gamma^*$ and suppose $\omega_R = \{x \in R \mid \phi(x) = 0_{R/\Gamma^*}\}$, where 0 is the unit element of m -ary group $(R/\Gamma^*, F)$. Then the unit element of $(R/\Gamma^*, F)$ is equal to ω_R , i.e.,

$$F(\Gamma^*(x), \omega_R^{(m-1)}) = \Gamma^*(x) \text{ for all } x \in R.$$

Proposition 2.1. If (R_i, ϕ_j^i) is a direct system of Krasner (m, n) -hyperrings indexed by a direct set I , then $(R_i/\Gamma_{R_i}^*, \phi_j^{*i})$ is a direct system of (m, n) -rings, where $\phi_j^{*i}: R_i/\Gamma_{R_i}^* \rightarrow R_j/\Gamma_{R_j}^*$ defined by $\phi_j^{*i}(\Gamma_{R_i}^*(a_i)) = \Gamma_{R_j}^*(\phi_j^i a_i)$.

Proof. It is clear that $(R_i/\Gamma_{R_i}^*, \phi_j^{*i})$ is a family of (m, n) -rings and strong homomorphisms. Clearly, ϕ_i^{*i} is the identity for all $i \in I$. Now, for $i \leq j \leq k$, we have

$$\begin{aligned} (\phi_k^j \circ \phi_j^i)^*(\Gamma_{R_i}^*(a_i)) &= \phi_k^{*i}(\Gamma_{R_i}^*(a_i)) \\ &= \Gamma_{R_k}^*(\phi_k^i a_i) = \Gamma_{R_k}^*(\phi_k^j \circ \phi_j^i(a_i)) \\ &= \Gamma_{R_k}^*(\phi_k^j(\phi_j^i a_i)) = \phi_k^{*j}(\Gamma_{R_j}^*(\phi_j^i a_i)) \\ &= \phi_k^{*j} \circ \phi_j^{*i}(\Gamma_{R_i}^*(a_i)). \end{aligned}$$

Therefore, one concludes $(\phi_k^j \circ \phi_j^i)^* = \phi_k^{*i} = \phi_k^{*j} \circ \phi_j^{*i}$.

Proposition 2.2. For $[a_i], [b_j] \in \bar{X}$ let $[a_i] \theta [b_j]$ if $\phi_k^i a_i \Gamma_{R_k} \phi_k^j b_j$ for $k \geq i, j$. Then $\theta = \Gamma_{\bar{X}}$.

Proof. Let $[a_j] \Gamma_{\bar{X}} [b_j]$, then there exist $[x]_{i1}^{it_i} \in \bar{X}$ and $h, l_1^s \in \mathbf{Z}^+$ where $t_i = l_i(m-1)$ and $1 \leq i \leq s$ such that $\{[a_j], [b_j]\} \subseteq F_{(h)}([u_1], \dots, [u_s])$ where $[u_i] = G_{(l_i)}([x]_{i1}^{it_i})$. Suppose $x_{i1} \in R_{i1}, \dots, x_{it_i} \in R_{it_i}$ where $1 \leq i \leq s$. Since $G_{(l_i)}([x]_{i1}^{it_i}) = [g_{(l_i)}(x_{i1k}^{it_i k})]$ such that $x_{i1k} = \phi_k^{i1} x_{i1}, \dots, x_{it_i k} = \phi_k^{it_i} x_{it_i}$ for some $k \geq i1, \dots, it_i, j', j$, then $\{[a_j], [b_j]\} = \{[\phi_k^{j'} a_j], [\phi_k^j b_j]\} \subseteq F_{(h)}(G_{(l_1)}([x]_{11}^{l_1}), \dots, G_{(l_s)}([x]_{s1}^{l_s}))$.

Hence for some $n \geq k$ we obtain $\{\phi_n^{j'} a_j, \phi_n^j b_j\} \subseteq F_{(h)}(G_{(l_1)}([x]_{11}^{l_1}), \dots, G_{(l_s)}([x]_{s1}^{l_s}))$, which implies

that $\phi_n^j a_j, \Gamma_{R_n} \phi_n^j b_j$. Thus $[a_j] \theta [b_j]$. Conversely, if $[a_j] \theta [b_j]$ then there exists $k \geq j, j$ such that $\phi_k^j a_j, \Gamma_{R_k} \phi_k^j b_j$ and so there exist $x_{i_1}^{i_1} \in R_k \subseteq \bigcup R_i$ and $h, l_1^s \in \square$ where $t_i = l_i(m-1)$ and $1 \leq i \leq s$ such that $\{\phi_k^j a_j, \phi_k^j b_j\} \subseteq f_{(h)}(u_1, \dots, u_s)$, where $u_i = g_{(l_i)}(x_{i_1}^{i_1})$. Thus $\{[\phi_k^j a_j], [\phi_k^j b_j]\} \subseteq F_{(h)}([u_1], \dots, [u_s])$ where $[u_i] = G_{(l_i)}([x_{i_1}^{i_1}])$. Hence $[a_j] \Gamma_{\bar{X}} [b_j]$, and so $\Gamma_{\bar{X}} = \theta$.

Theorem 2.3. Let (R_i, ϕ_j^i) be a direct system of Krasner (m, n) -hyperrings indexed by a direct set I , and let Γ^* be the fundamental relation of $\varinjlim R_i$. Therefore, $\varinjlim (R_i / \Gamma_{R_i}^*) \cong (\varinjlim R_i) / \Gamma^*$.

Proof. Define $\mu: \varinjlim (R_i / \Gamma_{R_i}^*) \rightarrow (\varinjlim R_i) / \Gamma^*$ by $[\Gamma_{R_i}^*(a_i)] \mapsto \Gamma^*([a_i])$ for any $a_i \in R_i$. Since $\Gamma^*([a_i]) = \Gamma^*([b_j]) \Leftrightarrow \phi_k^i a_i \Gamma_{R_k}^* \phi_k^j b_j$ for some $k \geq i, j$

$$\begin{aligned} &\Leftrightarrow \Gamma_{R_k}^*(\phi_k^i a_i) = \Gamma_{R_k}^*(\phi_k^j b_j) \\ &\Leftrightarrow \phi_k^{*i} \Gamma_{R_i}^*(a_i) = \phi_k^{*j} \Gamma_{R_j}^*(b_j) \\ &\Leftrightarrow [\Gamma_{R_i}^*(a_i)] = [\Gamma_{R_j}^*(b_j)], \end{aligned}$$

μ is well defined and one to one. Now, we show that μ is a strong homomorphism:

$$\mu(F([\Gamma_{R_i}^*(a_i)]_1^m)) = \mu([F(\phi_k^{*i} \Gamma_{R_i}^*(a_i))_1^m])$$

for some $k \geq i_1, \dots, i_m = \mu([F(\Gamma_{R_k}^*(\phi_k^i a_i))_1^m]) = \mu([\Gamma_{R_k}^*(f((\phi_k^i a_i)_1^m))]) = \mu([\Gamma_{R_k}^*(x)])$, for $x \in f((\phi_k^i a_i)_1^m) = \Gamma^*([x])$ where $x \in f((\phi_k^i a_i)_1^m)$. On the other hands, one obtains $F(\mu([\Gamma_{R_i}^*(a_i)]_1^m)) = F(\Gamma^*([a_i])_1^m) = \Gamma^*([x])$, $[x] \in f([a_i]_1^m) = \Gamma^*([x])$, where $x \in f((\phi_k^i a_i)_1^m)$ for some $k \geq i_1, \dots, i_m$. So $\mu(F([\Gamma_{R_i}^*(a_i)]_1^m)) = F(\mu([\Gamma_{R_i}^*(a_i)]_1^m))$. Similarly, $\mu(G([\Gamma_{R_i}^*(a_i)]_1^n)) = G(\mu([\Gamma_{R_i}^*(a_i)]_1^n))$.

Proposition 2.4. If $a_i \in \omega_{R_i}$ then $[\Gamma_{R_i}^*(a_i)]$ is the zero element in $\varinjlim (R_i / \Gamma_{R_i}^*)$ and $[a_i] \in \omega_{\varinjlim R_i}$.

Proof. Let $F([\Gamma_{R_i}^*(a_i)]^{(m-1)}, [\Gamma_{R_j}^*(b_j)]) = [X]$, where $X = F((\phi_k^{*i} \Gamma_{R_i}^*(a_i))^{(m-1)}, \phi_k^{*j} \Gamma_{R_j}^*(b_j))$ for $k \geq i, j$. So $((\phi_k^{*i} (\omega_{R_i}))^{(m-1)}, \phi_k^{*j} \Gamma_{R_j}^*(b_j)) = F((\omega_{R_k})^{(m-1)}, \phi_k^{*j} \Gamma_{R_j}^*(b_j)) = \phi_k^{*j} \Gamma_{R_j}^*(b_j)$. Thus $F([\Gamma_{R_i}^*(a_i)]^{(m-1)}, [\Gamma_{R_j}^*(b_j)]) = [\phi_k^{*j} \Gamma_{R_j}^*(b_j)] = [\Gamma_{R_j}^*(b_j)]$. Hence $[\Gamma_{R_i}^*(a_i)]$ is the zero element in $\varinjlim (R_i / \Gamma_{R_i}^*)$. By Theorem 3.3. it concludes $\Gamma^*([a_i])$ is the zero element in $(\varinjlim R_i) / \Gamma^*$ and so $[a_i] \in \omega_{\varinjlim R_i}$.

It is easy to verify that if (R, f, g) is an Krasner (m, n) -hyperring, then ω_R , is an (m, n) -subhyperring.

Corollary 2.5. Let (R_i, ϕ_j^i) be a direct system of Krasner (m, n) -hyperrings indexed by a direct set I . Then $(\omega_{R_i}, \phi_j^i |_{\omega_{R_i}})$ is a direct system and $(\varinjlim \omega_{R_i}) / \Gamma^*$ is a zero (m, n) -ring.

Proof. Using Theorem 2.3. $(\varinjlim \omega_{R_i}) / \Gamma^* \cong \varinjlim (\omega_{R_i} / \Gamma_{R_i}^*) = 0$

Lemma 2.6. $\omega_{\varinjlim R_i} = \{[a_i] | a_i \in \bigcup R_i, \phi_k^i a_i \in \omega_{R_k} \text{ for some } k \geq i\}$.

Proof. Set $D = \{[a_i] | a_i \in \bigcup R_i, \phi_k^i a_i \in \omega_{R_k} \text{ for some } k \geq i\}$ Clearly, $D \subseteq \omega_{\varinjlim R_i}$. Let $[a_i] \in \omega_{\varinjlim R_i}$, then $a_i \in \bigcup R_i$ and $\Gamma^*([a_i]) = 0_{(\varinjlim R_i) / \Gamma^*}$.

By Theorem 2.3, it concludes that $[\Gamma^*(a_i)] = [\Gamma^*(0_{R_i})]$. Thus $[\phi_k^i a_i] = [\Gamma^*(0_{R_k})]$ for some $k \geq i$ and so $\phi_k^i a_i \in \omega_{R_k}$ for some $k \geq i$. Therefore, $\omega_{\varinjlim R_i} \subseteq D$.

Theorem 2.7. Let $Dir(I)$ be the category of all direct systems of Krasner (m, n) -hyperrings and strong homomorphisms over direct set I and $(m, n) - KH_r$ be the category of Krasner (m, n) -hyperrings and strong homomorphisms. Suppose $\varinjlim: Dir(I) \rightarrow (m, n) - KH_r$ by

$\{R_i, \phi_j^i\} \mapsto \{[a] \mid a \in \bigcup R_i\}$ and whenever $t : \{R_i, \phi_j^i\} \rightarrow \{H_i, \phi_j^i\}$ is a morphism in $Dir(I)$ $\underline{\lim} t = \vec{t} : \underline{\lim} R_i \rightarrow \underline{\lim} H_i$ where $a = a_i \in R_i$ and $\vec{t}([a]) = [t_i a_i]$, where $t_i : R_i \rightarrow H_i$. Then $\underline{\lim}$ is an exact functor.

Proof. It is easy to see that $\underline{\lim}$ is a functor. We prove that if $\{A_i, \phi_j^i\} \rightarrow \{B_i, \phi_j^i\} \rightarrow \{C_i, \phi_j^i\}$ is a sequence of morphisms of direct systems over I , such that $\omega_{A_i} \xrightarrow{n_i} A_i \xrightarrow{t_i} B_i \xrightarrow{s_i} C_i \xrightarrow{u_i} \omega_{C_i}$ (*) is exact for any $i \in I$, then $\omega_{\underline{\lim} A_i} \xrightarrow{\vec{n}} \underline{\lim} A_i \xrightarrow{\vec{t}} \underline{\lim} B_i \xrightarrow{\vec{s}} \underline{\lim} C_i \xrightarrow{\vec{u}} \omega_{\underline{\lim} C_i}$ is an exact sequence of Krasner (m, n) -hyperrings. Hence,

(i) $Ker \vec{t} = Im \vec{n}$; suppose $[a] \in Ker \vec{t}$, then $\vec{t}[a] \in \omega_{\underline{\lim} B_i}$. Let $a = a_i \in A_i$, then $\vec{t}[a] = [t_i a_i] \in \omega_{\underline{\lim} B_i}$ which implies $\phi_k^i(t_i a_i) \in \omega_{\underline{\lim} B_k}$ for some $k \geq i$, and so $\phi_k^i(t_i a_i) = t_k \phi_k^i(a_i) \in \omega_{B_k}$. Since the sequence (*) is exact for every $i \in I$, so $Ker t_k = Im n_k = \omega_{A_k}$ and then $t_k \phi_k^i(a_i) \in Im n_k = \omega_{A_k}$.

Hence $[a_i] = [\phi_k^i a_i] \in \underline{\lim} \omega_{A_k} = \underline{\lim} Im n_k = Im \vec{n}$. Therefore, $Ker \vec{t} \subseteq Im \vec{n}$. Conversely, if $[a] \in Im \vec{n} = \underline{\lim} \omega_{A_i}$, then $a_i \in \omega_{A_i}$ and so $t_i a_i \in \omega_{B_i}$ which implies $\vec{t}[a_i] = [t_i a_i] \in \omega_{\underline{\lim} B_i}$. Thus $[a_i] \in Ker \vec{t}$.

(ii) $Ker \vec{s} = Im \vec{t}$; let

$[x] = [b_i] \in Ker \vec{s}$, then

$$\vec{s}[b_i] = [s_i b_i] \in \omega_{\underline{\lim} C_i}.$$

Thus $\phi_k^i(s_i b_i) \in \omega_{\underline{\lim} C_k}$ for some $k \geq i$. We have $\phi_k^i(s_i b_i) = s_k \phi_k^i(b_i) \in \omega_{C_k}$. Since (*) is exact for every $i \in I$, we obtain that $\phi_k^i(b_i) \in Ker s_k = Im t_k$ and so there exists $a_k \in A_k$ such that $t_k a_k = \phi_k^i(b_i)$. Thus

$\vec{t}[a_k] = [t_k a_k] = [\phi_k^i b_i] = [b_i]$ and so $[b_i] \in Im \vec{t}$. Conversely, $\vec{s} \circ \vec{t}[a_i] = \vec{s}([t_i a_i]) = [s_i \circ t_i a_i]$. Since (*) is exact for any $i \in I$, then $s_i \circ t_i a_i \in \omega_{C_i}$, and hence $[s_i \circ t_i a_i] \in \omega_{\underline{\lim} C_i}$. Thus $\vec{s} \circ \vec{t}[a_i] \in \omega_{\underline{\lim} C_i}$ and $Im \vec{t} \subseteq Ker \vec{s}$.

(iii) $Ker \vec{u} = Im \vec{s}$; the proof is similar to (ii).

References

1. Marty F. Sur une generalization de la notion de groupe, 8^{iem} congress des Mathematiciens Scandinaves, Stockholm . 45-49 (1934).
2. Corsini P., Leoreanu-Fotea V. Applications of hypersrtructure theory. Advances in Mathematics. Vol. 5: Kluwer Academic Publishers, (2003).
3. Connes A., Consani C. The hyperring of adèle classes. *J. Number Theory.* **131** (2): 159-194 (2011).
4. Cristea I., Jancic-Rasovic S. Compositions hyperrings. An. Stiint. Univ. 261 "Ovidius" Constanta Ser. *Mat.* **21**(2): 81-94 (2013).
5. Dudek W.A., and Mirvakili S. Neutral elements, fundamental relations and n -ary hypersemigroups. *Int. J. Algebra Comput.* **19**: 567-583 (2009).
6. Krasner M. A class of hyperrings and hyperfields. *Int. J. Math. Math. Sci.* **6** (2): 307-311 (1983).
7. Shojaei H., Ameri R. Various kinds of quotient of a canonical hypergroup. *Eng. & Nat Sci.* **9** (1): 133-141 (2018).
8. Soltani Z., Ameri R. An introduction to zero-divisor graphs of a commutative multiplicative hyperring. *Sigma J. Eng. & Nat Sci.* **9** (1): 101-106 (2018).
9. Vougiouklis T. Hyperstructures and their representations. *Riv. Mat. Pura e Appl.* **2**: 1-180 (1994).
10. Ameri R., Norouzi M. Prime and primary hyperideals in Krasner (m, n) -hyperrings. *Eur. J. Combin.* **34**: 379-390 (2013).
11. Ameri R., Aivazi M., Hoskova-Mayerov S. Superring of Polynomials over a Hyperring. *J. Math.* **7** (902): 1-15 (2019).
12. Davvaz B., Vougiouklis T. n -ary hypergroups. *Iran. J. Sci. Technol. Trans. A Sci.* **30** (A2): 165-174 (2006).
13. Dehkordi S. O., Davvaz B. A strong regular on Γ -semihyperrings. *J. Sci. I. R. Iran.* **22**(3): 257-266 (2011).
14. Mirvakili S., Davvaz B. Relations on Krasner (m, n) -hyperrings. *Eur. J. Combi.* **31**: 790-802 (2010).
15. Pelea C. Hyperrings and \mathcal{A}^* -relations: A general approach. *J. Algebra.* **383** 104-128 (2013).
16. Mirvakili S., Davvaz B. Constructions of (m, n) -hyperrings. *Mat. Vesnik.* **67** (1): 1-16 (2015).
17. Jafarzadeh N., Ameri R. On the relation between categories of (m, n) -ary hypermodules and (m, n) -ary modules. *Sigma J. Eng. Nat. Sci.* **9** (1): 85-99 (2018).
18. Pelea C. On the direct limit of the direct system of multialgebras. *Discrete Math.* **306**: 2916-2930 (2006).
19. Hoskova S. Topological hypergroupoids. *Compute. Math.*

- Appl.* **64**: 2845-2849 (2012).
20. Pelea C. A note on the direct limit of a direct system of multialgebras in a subcategory of multialgebras. *Carpathian J. Math.* **22**(1-2): 121-128 (2006).
 21. Pre-semihyperadditive Categories. *An. St. Univ. Ovidius Constanta.* **27**(1): 269-288 (2019).
 22. Leoreanu-Fotea V. The direct and the inverse limit of hyperstructures associated with fuzzy sets of type 2. *Iran. J. Fuzzy Syst.* **5**(3): 89-94 (2008).
 23. Leoreanu V. Direct limit and inverse limit of join spaces associated with fuzzy sets. *Pure Math. Appl.* **113**: 509-516 (2000).
 24. Awodey S. Category theory. Oxford University Press Inc. New York, Second Edition. 336 P. (2010).