Robust Decentralized Control System Design based on Nash Equilibrium Point using Linear Quadratic Regulators

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Abstract

Non-cooperative intelligent control agents (ICAs) with dedicated cost functions, can lead the system to poor performance and in some cases, closed-loop instability. A robust solution to this challenge is to place the ICAs at the feedback Nash equilibrium point (FNEP) of the differential game between them. This paper introduces the designation of a robust decentralized infinite horizon LQR control system based on the FNEP for a linear time-invariant system. For this purpose, two control strategies are defined. The first one is a centralized infinite horizon LQR (CIHLQR) problem (i.e. a supervisory problem), and the second one is a decentralized control problem (i.e. an infinite horizon linear-quadratic differential game). Then, while examining the optimal solution of each of the above strategies on the performance of the other, the necessary and sufficient conditions for the equivalence of the two problems are presented. In the absence of the conditions, by using the least-squares error criterion, an approximated CIHLQR controller is presented. It is shown that the theorems could be extended from a two-agent control system to a multi-agent system. Finally, the results are evaluated using the simulation results of a Two-Area non-reheat power system.

Keywords

Non-cooperative Differential Game, Nash-based Decentralized Control System, Infinite Horizon Linear Quadratic Regulator, Feedback Nash Equilibrium Point, Two-Area Power System.

1. Introduction

Linear-quadratic regulation (LQR) problem and its related optimal state feedback controller have been well explored in reference books and literary works in continuous and discrete-time linear systems [1, 2]. This controller recommends optimal, stabilizing, and robust control inputs in the form of a gained state feedback by optimizing a quadratic performance index and solving a Riccati equation.

The LQR achieves infinite gain margin and a guaranteed $\mp 60^{\circ}$ phase margin in each input channel independently and simultaneously. These properties are in agreement with the practical guidelines for control system design and make this controller a recommended robust controller implemented in linear systems with parametric uncertainties [3]. For example, The LQR controllers have been successfully implemented for a large number of complex systems such as the unmanned bicycle robot [4], load frequency control of interconnected power systems [5], Unmanned aerial vehicles (UAVs) [6, 7], and aircraft [8]. Nevertheless, in cases where the controller is used to control a nonlinear system, the characteristics of stability, optimality, and robustness should be reconsidered [6, 9, 10].

In an LQR controller designation stage, determination of the weighting matrices, matrices Q and R, is the key issue that directly affects the control action. Therefore, the development of new methods for autotuning the weighting matrices has been considered by researchers. Some of these methods include sensitivity-based methods [11], adaptive methods [6], evolutionary techniques [7], metaheuristic optimization algorithms [12], and correlation analysis [13].

In the implementation stage, the selected proper control structure has an important role in the real efficiency of the controller. This is especially true for large-scale or coupled linear systems that do not use a centralized control structure to implement the controller or in multi-agent systems [14].

Centralized, decentralized, and distributed control are three main structures in large-scale control systems. However, the technical difficulties of implementing a centralized control strategy have made the control system designers more inclined to use decentralized and distributed control structures [15]. In these two structures, the control system designer accomplishes the overall objectives of the system by assigning control of system inputs to local control systems (LCS) and setting specific local objectives for each.

The question that arises is that when the LCSs have internal cost functions, what constraints will the system designer have on fulfilling the overall objectives of the system? One of the main challenges in answering this question will be mentioned when the LCSs have conflicts of interest. These challenges have become more objective with the advent of smart controllers in industrial areas as well as the introduction of free trade policy in managerial and macroeconomic debates. In these cases, the system inputs are controlled by intelligent control agents (ICAs) with different control objectives [16-19]. Therefore, behavioral analysis of the control agents seems necessary.

In interconnected dynamics systems controlled by ICAs, the dedicated cost of each agent not only depends on its input choice but also on the choice of inputs by the other ICAs too. Game theory deals with solving such problems as a game.

Nash's papers have presented robust solutions for the games with non-cooperative ICAs [20, 21]. In these games, the agents do not cooperate or share their control information and strategy.

Introducing differential games by Isaacs established a deep link between the topics of optimal control and game theory [22]. In the meantime, linear quadratic differential games (LQDGs) have been favored by more engineers and researchers in the areas of optimization and control due to the linear dynamics of the game environment and quadratic form of the agents' cost functions as well as the simpler solution.

The non-cooperative solutions of an LQDG (i.e. the Nash equilibrium points) depend on solving the coupled algebraic Riccati equations (AREs) of the game [23]. Therefore, in some studies, necessary and sufficient conditions for the existence of the solutions and determination of them have been investigated. In recent years, some online and offline methods have been proposed to calculate Nash equilibrium points of zero and non-zero-sum games. These methods include numerical methods [24], adaptive dynamic programming [25], reinforcement learning [26], and extremum seeking control [27]. If the number of ICAs increases or some isolated games connect and form a networked game, the computational complexity of determining the equilibrium points increases exponentially.

This paper deals with the design of a robust decentralized infinite horizon LQR (IHLQR) controller based on the FNEP of an infinite horizon LQDG (IHLQDG). This design is a flexible design against the change of the LQR controller implementation method from single-agent centralized mode (i.e. a centralized IHLQR (CIHLQR)) to intelligent multi-agent decentralized mode with individual cost functions and non-cooperative behavior.

The next sections of this paper are organized as follows. Section 2 addresses the problem motivation and formulation by introducing two different strategies for controlling a linear time-invariant (LTI) system. The first strategy is using a CIHLQR derived from the optimal solution of a supervisory problem. The second one is a decentralized non-cooperative agent-based controller in the form of a feedback Nash IHLQDG. The effects of using each of these controllers on the cost of the supervisor and the agents are discussed. Section 3 provides a solution for designing an IHLQR controller based on the LFNE of a two-agent IHLQDG and then the results extend for a multi-agent problem. Section 4 deals with a numerical example of a Two-Area power system to illustrate the effectiveness of the proposed method. Conclusions are provided in Section 5.

2. Problem Formulation

Consider an LTI system with two control input vectors:

$$\dot{\boldsymbol{x}}(\mathrm{t}) = \mathbf{A}\boldsymbol{x}(\mathrm{t}) + \mathbf{B}\boldsymbol{u}(\mathrm{t}),\ \ \boldsymbol{x}(\mathrm{t}_0) = \boldsymbol{x}_{\mathbf{0}} \tag{1}$$

where $\mathrm{x}(\mathrm{t}) \in \mathbb{R}^{\mathrm{n}}$, $\mathrm{u}(\mathrm{t}) \in \mathbb{R}^{\mathrm{m}}$, and x_0 are state vectors, input vector, and initial conditions of the system, respectively. Input vector u(t) consists of two vectors $\mathrm{u}_1(\mathrm{t})$ controlled by agent 1 and $\mathrm{u}_2(\mathrm{t})$ controlled by agent 2 such that $\mathrm{u}(\mathrm{t})=[\mathrm{u}_1^{\mathrm{T}}(\mathrm{t})\ \ \mathrm{u}_2^{\mathrm{T}}(\mathrm{t})]^{\mathrm{T}}$. Also system matrix A and input matrix $\mathrm{B} = [\mathrm{B}_1\ \ \mathrm{B}_2]$ are constant matrices of appropriate dimensions where $\mathrm{Bu}(\mathrm{t}) = \mathrm{B}_1\mathrm{u}_1(\mathrm{t}) + \mathrm{B}_2\mathrm{u}_2(\mathrm{t})$.

Suppose that three cost functions are defined in the system as follows:

$$J = \int_{t_0}^{\infty} \left(\boldsymbol{x}^T(\boldsymbol{t})\mathbf{Q}\boldsymbol{x}(t) + \boldsymbol{u}^T(\boldsymbol{t})\mathbf{R}\boldsymbol{u}(t)\right)dt \tag{2}$$

$$J_1 = \int_{t_0}^{\infty} \left(\boldsymbol{x}^T(\boldsymbol{t})\mathbf{Q}_1\boldsymbol{x}(t) + \boldsymbol{u}_1^T(\boldsymbol{t})\mathbf{R}_{11}\boldsymbol{u}_1(t) + \boldsymbol{u}_2^T(\boldsymbol{t})\mathbf{R}_{12}\boldsymbol{u}_2(t)\right)dt \tag{3a}$$

$$J_2 = \int_{t_0}^{\infty} \left(\boldsymbol{x}^T(\boldsymbol{t})\mathbf{Q}_2\boldsymbol{x}(t) + \boldsymbol{u}_2^T(\boldsymbol{t})\mathbf{R}_{22}\boldsymbol{u}_2(t) + \boldsymbol{u}_1^T(\boldsymbol{t})\mathbf{R}_{21}\boldsymbol{u}_1(t)\right)dt \tag{3b}$$

where the weighting matrices $\mathrm{Q} \geq 0$, $\mathrm{R}>0$, $\mathrm{Q}_{\mathrm{i}} \geq 0$, $\mathrm{R}_{\mathrm{ii}}>0$ and $\mathrm{R}_{\mathrm{ij}} \geq 0$, i,j=1,2, i≠j are symmetric and cost functions (2), (3a), and (3b) are the cost functions of a supervisor, agent 1, and agent 2, respectively.

Assumption 1: $\left(\mathrm{A}, \mathrm{B}, \sqrt{\mathrm{Q}}\right)$, $\left(\mathrm{A}, \mathrm{B}_1, \sqrt{\mathrm{Q}_1}\right)$ and $\left(\mathrm{A}, \mathrm{B}_2, \sqrt{\mathrm{Q}_2}\right)$ are stabilizable and detectable.

Definition 1: An intelligent agent is an agent that aims to minimize its cost function.

Assumption 2: Agents 1 and 2 are intelligent with non-cooperative behavior.

Definition 2: The supervisory Problem is defined as CIHLQR problem (1) and (2) and the agents' problem is defined as IHLQDG (1), (3a), and (3b).

Definition 3: Optimality in the sense of supervisor is the optimal solution to the CIHLQR problem (1) and (2).

According to Assumption 1, Since $\left(\mathrm{A}, \mathrm{B}, \sqrt{\mathrm{Q}}\right)$ are stabilizable and detectable, the optimal solution for the supervisory problem can be calculated as follows:

$$\boldsymbol{u}_{\boldsymbol{LQ}}(\mathrm{t}) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\boldsymbol{x}(\mathrm{t}) \tag{4}$$

where P is the symmetric stabilizing solution of the following algebraic Riccati equation(ARE)

$$-\mathbf{PA} - \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{PSP} - \mathbf{Q} = \mathbf{0} \tag{5}$$

while $S = \mathrm{BR}^{-1}\mathrm{B}^{\mathrm{T}}$.

In this case, the optimal cost of the supervisor can be determined by

$$J_{LQ} = \boldsymbol{x}_{\mathbf{0}}^{\boldsymbol{T}}\mathbf{P}\boldsymbol{x}_{\mathbf{0}} \tag{6}$$

By applying the optimal solution (4), the closed-loop system (1) is defined as:

$$\dot{\boldsymbol{x}}(\mathrm{t}) = \mathbf{A}_{\mathbf{LQ}}\boldsymbol{x}(\mathrm{t}),\ \ \boldsymbol{x}(\mathrm{t}_0) = \boldsymbol{x}_{\mathbf{0}} \tag{7}$$

where $\mathrm{A}_{\mathrm{LQ}} := \mathrm{A} - \mathrm{SP}$ is stable [1].

Suppose, given the limitations of implementing centralized control for large-scale systems and the need to use decentralized or distributed control structures, controlling the inputs (i.e. $u_1(t)$ and $u_2(t)$) are assigned to agents 1 and 2, respectively.

According to Assumption 2 and the theory of differential games, the optimality in the sense of agents' game is defined by the following definition:

Definition 4: *Optimality in the sense of agents' game* is the stationary linear feedback Nash equilibria (LFNE) of IHLQDG (1), (3a), and (3b).

According to Assumption 1, Since $\left(A, B_1, \sqrt{Q_1}\right)$ and $\left(A, B_2, \sqrt{Q_2}\right)$ are stabilizable and detectable, LFNE of the agents' game can be calculated as follows:

$$\begin{bmatrix} u_{1N}(t) \\ u_{2N}(t) \end{bmatrix} = \begin{bmatrix} -R_{11}^{-1}B_1^T P_1 x(t) \\ -R_{22}^{-1}B_2^T P_2 x(t) \end{bmatrix} \quad (8)$$

where (P_1,P_2) is the symmetric stabilizing solution of the coupled AREs:

$$0=-\left(A-S_2P_2\right)^T P_1-P_1\left(A-S_2P_2\right)+P_1S_1P_1-Q_1-P_2S_{21}P_2 \quad (9\text{ a})$$

$$0=-\left(A-S_1P_1\right)^T P_2-P_2\left(A-S_1P_1\right)+P_2S_2P_2-Q_2-P_1S_{12}P_1 \quad (9\text{ b})$$

while $S_i = B_iR_{ii}^{-1}B_i^T$, $S_{ij} = B_iR_{ii}^{-1}R_{ji}R_{ii}^{-1}B_i^T$, $i,j=1,2$ and $i\neq j$. the cost of the agent i in the LFNE (8) is determined by

$$J_{iN} = x_0^T P_i x_0, i = 1,2 \quad (10)$$

and the closed-loop system (1) after applying the optimal Nash solution (8) is as follows:

$$\dot{x}(t) = A_{LQ}x(t),\ x(t_0) = x_0 \quad (11)$$

where $A_N := A - S_1P_1 - S_2P_2$ is stable [28].

Remark 1: As can be seen, the determination of LFNE (8) is tied to the solution of the coupled AREs (9).

Li and Gajic in [29] proved that if $\left(A, B_1, \sqrt{Q_1}\right)$ and $\left(A, B_2, \sqrt{Q_2}\right)$ are stabilizable and detectable, then there exist unique positive semidefinite solutions for the coupled AREs (9).

Definition 5: The supervisory problem (1) and (2) and the agents' problem (1), (3a), and (3b) are equivalent if and only if the optimal solution of two problems be equal (i.e. $u_{LQ}=[u_{1N}^T \quad u_{2N}^T]^T$).

Block diagrams of the system (1) in the supervisory-based and agent-based control structures are shown in Fig. 1. In Fig. 1(a), the system inputs are controlled by the agents. So the cost of the supervisor depends on the agents' actions. However, in Fig. 1(b), the supervisor governs the inputs and the agents should implement what the supervisor determined.

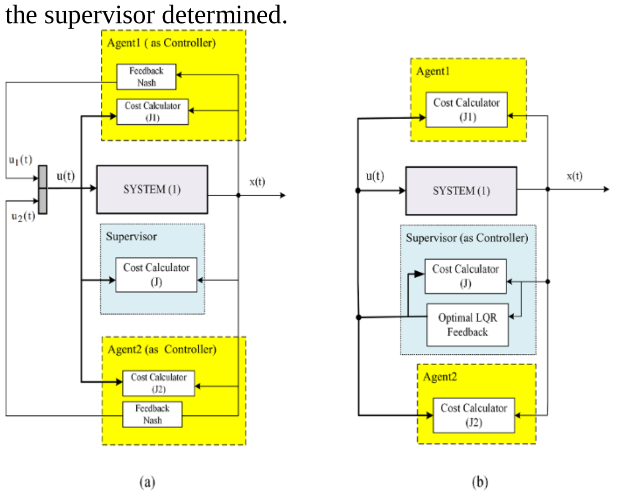

Fig. 1: Block diagram of the control system structures. (a) agent-based control system (b) supervisor-based control system.

In the following, two theorems are presented to examine the effect of LFNE (8) on the supervisor cost (2) and, the effect of LQR optimal action (4) on the agents' cost (3a) and (3b).

Theorem 1: Consider IHLQDG (1) with cost functions (3a) and (3b). At LFNE (8), the supervisor cost (2) is determined as follows:

$$J_N := J(x_0, F) = x_0^T P_N x_0 \quad (12)$$

where $F = \begin{bmatrix} R_{11}^{-1}B_1^T P_1 \\ R_{22}^{-1}B_2^T P_2 \end{bmatrix}$ and P_N is obtained from the Lyapunov equation

$$-(Q + F^T RF) = A_N{}^T P_N + P_N A_N \cdot \quad (13)$$

Proof: If the agents place in LFNE (8), the control input u(t) could be written as

$$u(t) = \begin{bmatrix} u_{1N}(t) \\ u_{2N}(t) \end{bmatrix} = \begin{bmatrix} -R_{11}^{-1}B_1^T P_1 x(t) \\ -R_{22}^{-1}B_2^T P_2 x(t) \end{bmatrix} = -\begin{bmatrix} R_{11}^{-1}B_1^T P_1 \\ R_{22}^{-1}B_2^T P_2 \end{bmatrix} x(t) = -Fx(t) \quad (14)$$

where (P_1,P_2) is a symmetric stabilizing solution of coupled AREs (9). Therefore, the supervisor's cost function (2) can be written as

$$J(x_0, F) = \int_{t_0}^{\infty} x^T(t)(Q + F^T RF)x(t)dt \quad (15)$$

where $F = \begin{bmatrix} R_{11}^{-1}B_1^T P_1 \\ R_{22}^{-1}B_2^T P_2 \end{bmatrix}$. Since the closed-loop system (11) is stable, the state vector x(t) can be written as

$$x(t) = x_0 e^{A_N(t-t_0)} \quad (16)$$

Hence, Eq. (15) could be rewritten as follows:

$$J(x_0, F) = x_0^T \left(\int_{t_0}^{\infty} \left(e^{A_N(t-t_0)}\right)^T (Q + F^T RF) e^{A_N(t-t_0)}\, dt \right) x_0 \quad (17)$$

By defining $P_N = \int_{t_0}^{\infty} \left(e^{A_N(t-t_0)}\right)^T (Q + F^T RF) e^{A_N(t-t_0)}\, dt$, Eq. (17) can be rewritten as:

$$J(x_0, F) = x_0^T P_N x_0 \cdot$$

Since A_N is stable, we have

$$0 - (Q + F^T RF) = \int_{t_0}^{\infty} \frac{d}{dt}\left\{\left(e^{A_N(t-t_0)}\right)^T (Q + F^T RF) e^{A_N(t-t_0)}\right\} dt$$

$$= \int_{t_0}^{\infty} \left\{A_N{}^T \left(e^{A_N(t-t_0)}\right)^T (Q + F^T RF) e^{A_N(t-t_0)} + \left(e^{A_N(t-t_0)}\right)^T (Q + F^T RF) e^{A_N(t-t_0)} A_N\right\} dt$$

$$= A_N{}^T P_N + P_N A_N. \quad \blacksquare$$

Theorem 2: Consider CIHLQR problems (1) and (2). If the supervisor's optimal solution (4) is applied to the system (1), then agents' costs (3a) and (3b) are calculated as follows:

$$J_{iLQ} = J_i(x_0, K) = x_0^T P_{iLQ} x_0, i = 1,2 \quad (18)$$

where J_{iLQ} is the agent i's cost, $K = R^{-1}B^T P$ and P_{iLQ} is the solution of the Lyapunov equation

$$A_{LQ}{}^T P_{iLQ} + P_{iLQ} A_{LQ} = -(Q_i + P^T S_{iLQ} P) \quad (19)$$

while $A_{LQ} := A - BR^{-1}B^T P$ and $S_{iLQ}=BR^{-1}\begin{bmatrix} R_{ii} & 0 \\ 0 & R_{ij} \end{bmatrix} R^{-1}B^T$, $i,j=1,2$ and $i\neq j$.

Proof: Agent i's cost function could be rewritten as follows:

$$J_{iLQ} = J_i(\boldsymbol{x_0}, \mathbf{K}) = \int_{t_0}^{\infty}\left(\boldsymbol{x}^T(t)\mathbf{Q}_i\boldsymbol{x}(t) + \begin{bmatrix}\boldsymbol{u_1}(t)\\ \boldsymbol{u_2}(t)\end{bmatrix}^T \begin{bmatrix}\mathbf{R_{ii}} & \mathbf{0}\\ \mathbf{0} & \mathbf{R_{ij}}\end{bmatrix}\begin{bmatrix}\boldsymbol{u_1}(t)\\ \boldsymbol{u_2}(t)\end{bmatrix}\right)dt =$$

$$= \int_{t_0}^{\infty}\left(\boldsymbol{x}^T(t)\mathbf{Q}_i\boldsymbol{x}(t) + (-\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\boldsymbol{x}(t))^T \begin{bmatrix}\mathbf{R_{ii}} & \mathbf{0}\\ \mathbf{0} & \mathbf{R_{ij}}\end{bmatrix}(-\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\boldsymbol{x}(t))\right)dt$$

$$= \int_{t_0}^{\infty}\boldsymbol{x}^T(t)(\mathbf{Q}_i + \mathbf{P}\underbrace{\mathbf{B}\mathbf{R}^{-1}\begin{bmatrix}\mathbf{R_{ii}} & \mathbf{0}\\ \mathbf{0} & \mathbf{R_{ij}}\end{bmatrix}\mathbf{R}^{-1}\mathbf{B}^T}_{\mathbf{S_{iLQ}}}\mathbf{P})\boldsymbol{x}(t)\,dt$$

$$, i,j = 1,2 \ \ and\ i \neq j \qquad (20)$$

Since the closed-loop system (7) is stable, the state vector x(t) can be written as

$$\boldsymbol{x}(t) = \boldsymbol{x_0}\boldsymbol{e}^{\boldsymbol{A_{LQ}}(t-t_0)} \qquad (21)$$

Hence, Eq. (20) could be rewritten as follows:

$$J_{iLQ}(\boldsymbol{x_0}, \boldsymbol{P}) = x_0^T\int_{t_0}^{\infty}\left(\boldsymbol{e}^{\boldsymbol{A_{LQ}}(t-t_0)}\right)^T(\boldsymbol{Q_1} + \boldsymbol{PS_{iLQ}P})\boldsymbol{e}^{\boldsymbol{A_{LQ}}(t-t_0)}\,dt\, x_0 \quad , i = 1,2 \qquad (22)$$

Assume $\mathbf{P_{iLQ}} = \int_{t_0}^{\infty}(\mathbf{e}^{\mathbf{A_{LQ}}(t-t_0)})^T(\mathbf{Q}_i + \mathbf{PS_{iLQ}P})\mathbf{e}^{\mathbf{A_{LQ}}(t-t_0)}\,\mathbf{dt}$, i=1,2 , so $J_i(\boldsymbol{x}_0, \mathbf{K}) = \boldsymbol{x_0^T}\mathbf{P_{iLQ}}\boldsymbol{x_0}$, $i = 1,2$ and since A_{LQ} is stable, similar to the proof of theorem 1, it could be written

$$-(\mathbf{Q} + \mathbf{PS_iP}) = \mathbf{A_{LQ}}^T\mathbf{P_{iLQ}} + \mathbf{P_{iLQ}}\mathbf{A_{LQ}} \ , i = 1,2. \quad \blacksquare$$

Remark 2: Since the supervisor's optimal cost (6) is the minimum cost of supervisor for all of linear state feedback strategies, then it could be claimed that the supervisor's cost never decreases at the LFNE of the game (1), (3a) and (3b). So

$$J_{LQ} \leq J_N \cdot \qquad (23)$$

According to Remark 2, by changing the control strategy of the system (1) from a CIHLQR control to a non-cooperative intelligent agent-based control, the system policy designer should have expected more cost for the supervisor. On the other hand, it cannot be said with certainty that if the optimal centralized LQR controller derives the inputs (Fig. 1(a)), the cost of the agents will necessarily increase. So a question arises whether the agents could be replaced by an optimal CIHLQR controller that fully tracks their behaviors. One of the results of answering the question is finding a simpler and more well-known model (i.e. a CIHLQR model), as an alternate for the feedback Nash IHLQDG problem. Another result is the ability to design a Nash-based LQR controller in the systems where the control inputs are governed by intelligent non-cooperative agent-based systems. The next section discusses this question.

3. Controller Design: An LQR Approach

In this section, the designation of a robust decentralized infinite horizon LQR (IHLQR) controller based on the FNEP of an infinite horizon LQDG (IHLQDG) is discussed. To that end, we present an LQR approach to model a feedback Nash IHLQDG. For this purpose, at first, necessary and sufficient conditions for LQR modeling of a feedback Nash IHLQDG are examined and if there is not any fully compatible model, an LSE method for finding the closest LQR model is provided.

3.1 Feedback Nash IHLQDG Modeling: Alternate LQR Problem Design

Consider IHLQDG (1), (3a), and (3b) and CIHLQR problem (1) and (2). the problem is how to replace the two non-cooperative intelligent agent-based controllers with one centralized Linear quadratic regulator, where the state response of the system (1) be equal or closely equal to the Nash equilibrium behavior of the game.

Suppose that Q (i.e. states' weighting matrix of the cost function (2)) is recommended by the supervisor. The following theorem defines the conditions of existence of the alternate CIHLQR problem.

Theorem 3: The feedback Nash IHLQDG game (1), (3a), and (3b) is equivalent to LQR problem (1) and (2) assuming the recommended Q matrix, if and only if the following conditions hold:

(a) vec(-Q) is at the null space of $(A^T \oplus A_N^T)^{-1} - (A_N^T \oplus A^T)^{-1}$ where vec(.) and $\oplus$ are vectorization of a matrix and Kronecker sum operator, respectively.

(b) $A_N + A$ is stable.

(c) Weighting matrix R is determined as a symmetric positive definite solution of the

$$\boldsymbol{R^{-1}} = \boldsymbol{FP^{-1}}(\boldsymbol{B^T})^+ \qquad (24)$$

where $F = \begin{bmatrix}R_{11}^{-1}B_1^TP_1\\ R_{22}^{-1}B_2^TP_2\end{bmatrix}$ and $(\cdot)^+$ is the Moore-Penrose inverse operator and P is the solution of the Lyapunov equation

$$(\boldsymbol{A_N} + \boldsymbol{A})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A_N} + \boldsymbol{A}) = -2\boldsymbol{Q} \cdot \qquad (25)$$

Proof of necessity: If the optimal LQR solution (4) is equal to the LFNE (8), then the closed-forms of the system (1) after applying LFNE (8) and the LQR solution (4) are equal, so $A_{LQ} = A_N$. Therefore, ARE (5) could be rewritten as a Sylvester Equation

$$\boldsymbol{A_N^T P} + \boldsymbol{PA} = -\boldsymbol{Q} \qquad (26)$$

Since P should be symmetric and Q is symmetric too, transposing the Eq. (26) yields the result

$$\boldsymbol{A^T P} + \boldsymbol{PA_N} = -\boldsymbol{Q} \cdot \qquad (27)$$

On the other hand, the Kronecker forms of Eqs. (26) and (27) are [30]:

$$\begin{cases}(\boldsymbol{A^T} \oplus \boldsymbol{A_N^T})\boldsymbol{vec}(\boldsymbol{P}) = \boldsymbol{vec}(-\boldsymbol{Q})\\ (\boldsymbol{A_N^T} \oplus \boldsymbol{A^T})\boldsymbol{vec}(\boldsymbol{P}) = \boldsymbol{vec}(-\boldsymbol{Q})\end{cases} \qquad (28)$$

So $\begin{cases}\mathrm{vec}(P) = (A^T \oplus A_N^T)^{-1}\mathrm{vec}(-Q)\\ \mathrm{vec}(P) = (A_N^T \oplus A^T)^{-1}\mathrm{vec}(-Q)\end{cases}$, thus if there is any P as the solution of Eqs. (26) and (27), vec(-Q) should be in the null space of $((A^T \oplus A_N^T)^{-1} - (A_N^T \oplus A^T)^{-1})$.

On the other hand, by summing up the sides of Eqs. (26) and (27), it follows that

$$(\mathbf{A_N} + \mathbf{A})^T\mathbf{P} + \mathbf{P}(\mathbf{A_N} + \mathbf{A}) = -2\mathbf{Q} \qquad (29)$$

If $A_N + A$ is stable, the Lyapunov Eq. (29) has a symmetric positive definite (PD) solution for P. If condition (a) holds, then the PD solution of Eq. (29) satisfies Eq. (26). Consequently, the R matrix of the LQR problem could be determined by using the fact that $u_{LQ}(t) = \begin{bmatrix}u_{1N}(t)\\ u_{2N}(t)\end{bmatrix}$, therefore $R^{-1}B^TP = \begin{bmatrix}R_{11}^{-1}B_1^TP_1\\ R_{22}^{-1}B_2^TP_2\end{bmatrix}$ or

$R^{-1}=\begin{bmatrix}R_{11}^{-1}B_1^TP_1\\ R_{22}^{-1}B_2^TP_2\end{bmatrix}(B^TP)^+=FP^{-1}(B^T)^+$, where $u_{LQ}(t)$, $u_{1N}(t)$. and $u_{2N}(t)$ are LQR control input and feedback Nash control input of agent 1 and agent 2, respectively.

Proof of sufficiency: If conditions (a) to (c) are met, a sufficient condition can be easily met by reversing the necessary condition proof procedure. ■

3.2 Feedback Nash IHLQDG Modeling: Approximate LQR Problem Design

If the recommended Q does not satisfy the condition (a) in Theorem 3, new Q (i.e. Q_{LS}), could be determined based on the least square error (LSE) criterion.

Algorithm1 provides the method of determining the alternative approximated CIHLQR for IHLQDG (1), (3a), and (3b). This algorithm has been extracted from the proof of Theorem 3.

Algorithm 1: Least Square Error approximation for the alternative CIHLQR Problem when the supervisor has a recommended Q.

Step 1:

Determine matrix S where the columns are the unitary orthogonal Null space vectors of

$$\left(\mathbf{A}^T\oplus\mathbf{A}_N^T\right)^{-1}-\left(\mathbf{A}_N^T\oplus\mathbf{A}^T\right)^{-1}$$

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Step 2:

If vec(-Q) is linearly dependent on the column vectors of S, then

$$\boldsymbol{vec}(\boldsymbol{P})=\left(\mathbf{A}^T\oplus\mathbf{A}_N^T\right)^{-1}\boldsymbol{vec}(-\mathbf{Q})$$

else if vec(-Q) is linearly independent to S column vectors, at first, find the Q_{LS} matrix by solving the least square problem:

$$S\,vec(-Q_{LS})=vec(-Q)$$

Then

$$\boldsymbol{vec}(\mathbf{P})=\left(\mathbf{A}^T\oplus\mathbf{A}_N^T\right)^{-1}\boldsymbol{vec}(-\boldsymbol{Q}_{LS})$$

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Step 3:

The R matrix is calculated as follow:

$$\mathbf{R}^{-1}=\begin{bmatrix}\mathbf{R}_{11}^{-1}\mathbf{B}_1^T\mathbf{P}_1\\ \mathbf{R}_{22}^{-1}\mathbf{B}_2^T\mathbf{P}_2\end{bmatrix}\mathbf{P}^{-1}(\mathbf{B}^T)^+.$$

3.3 Generalizing the controller design for n>2 ICAs

In this section, the design problem for n>2-agents is discussed. For this purpose, at first, LFNE of n-agents IHLQDG is defined and then the design method is developed.

Consider system (1) with the cost function of the supervisor (2). Suppose that $u(t)=[u_1^T(t)\;\; u_2^T(t)\;\; \dots\;\; u_n^T(t)]^T$ and $B=[B_1\;\; B_2\;\; \dots\;\; B_n]$. So the system dynamic equation (1) can be written as

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{Ax}(t)+\sum_{i=1}^{n}\mathbf{B}_i\boldsymbol{u}_i(t) \qquad (30)$$

Assume that input $u_i(t)$, i=1,…,n is controlled by the ith agent and each agent has the following cost function:

$$J_i=\int_0^\infty(\boldsymbol{x}^T\mathbf{Q}_i\boldsymbol{x}+\sum_{j=1}^{n}\boldsymbol{u}_j^T\mathbf{R}_{ij}\boldsymbol{u}_j)dt, \qquad i=1,\dots,n \quad (31)$$

Assumption 3: $(A,\ B,\sqrt{Q})$, $(A, B_i, \sqrt{Q_i})$, i=1,…,n are stabilizable and detectable.

Assumption 4: The agents are non-cooperative intelligent players of IHLQDG (30) and (31).

Optimality in the sense of the supervisor is defined as Section 2 and Eqs. (4) -(7). On the other hand, regarding Assumptions 3 and 4, the *optimality in the sense of the agents* is defined as stationary linear feedback Nash equilibrium (LFNE) point of IHLQDG (30), (31) as follows:

$$\boldsymbol{u}_N^*(\boldsymbol{t})=\left[\boldsymbol{u}_1^{*T}(\boldsymbol{t}),\boldsymbol{u}_2^{*T}(\boldsymbol{t}),\cdots,\boldsymbol{u}_n^{*T}(\boldsymbol{t})\right]^T \qquad (32)$$

where $u_i^*(t)=-R_{ii}^{-1}B_i^TP_ix$, i=1,…,n and P_i,i=1,…,n are symmetric stabilizing solutions of the set of coupled AREs

$$\left(\mathbf{A}-\sum_{\substack{j=1\\ j\neq i}}^{n}\mathbf{S}_j\mathbf{P}_j\right)^T\mathbf{P}_i+\mathbf{P}_i\left(\mathbf{A}-\sum_{\substack{j=1\\ j\neq i}}^{n}\mathbf{S}_j\mathbf{P}_j\right)+\mathbf{Q}_i-\mathbf{P}_i\mathbf{S}_i\mathbf{P}_i+\sum_{\substack{j=1\\ j\neq i}}^{n}\mathbf{P}_j\mathbf{S}_{ij}\mathbf{P}_j=\mathbf{0}, \quad \mathbf{i},\mathbf{j}=\mathbf{1},\dots,\mathbf{n} \qquad (33)$$

where

$\mathbf{S_i}:=\mathbf{B_iR_{ii}^{-1}B_i^T}$ and $\mathbf{S_{ij}}:=\mathbf{B_iR_{ii}^{-1}R_{ji}R_{ii}^{-1}B_i^T}$, i,j=1,…,n , i≠j.

Agent i's LFNE cost is determined by

$$J_{iN}=\boldsymbol{x}_0^T\mathbf{P}_i\boldsymbol{x}_0,\ i=1,\dots,n \qquad (34)$$

and the closed-loop system (1) after applying the LFNE (32) is as follow:

$$\dot{\boldsymbol{x}}=\mathbf{A_N}\boldsymbol{x},\quad \boldsymbol{x}(\boldsymbol{t_0})=\boldsymbol{x_0} \qquad (35)$$

where $A_N:=A-\sum_{i=1}^{n}S_iP_i$ is stable [26].

In generalizing the design problem to multi-agent (n>2), as can be seen, the proof presented in Theorem 3 has the least dependence on the number of agents. Therefore, the following proposition can be presented as a generalization of Theorem 3.

Proposition 1: The feedback Nash IHLQDG game (30) and (31) is equivalent to LQR problem (1) and (2) assuming the recommended Q matrix by the supervisor, if and only if the following conditions hold:

(a) vec(-Q) is at the null space of
$$\left(\left(\mathbf{A}^T\oplus\mathbf{A}_N^T\right)^{-1}-\left(\mathbf{A}_N^T\oplus\mathbf{A}^T\right)^{-1}\right)$$
where vec(.) and ⊕ are vectorization of a matrix and Kronecker sum operator, respectively.

(b) $\mathbf{A_N}+\mathbf{A}$ is stable.

(c) R is determined as a symmetric positive definite solution of the following Equation:

$$R^{-1}=FP^{-1}(B^T)^+ \qquad (36)$$

where $\mathbf{F}=\begin{bmatrix}\mathbf{R}_{11}^{-1}\mathbf{B}_1^T\mathbf{P}_1\\ \mathbf{R}_{22}^{-1}\mathbf{B}_2^T\mathbf{P}_2\\ \vdots\\ \mathbf{R}_{nn}^{-1}\mathbf{B}_n^T\mathbf{P}_n\end{bmatrix}$ and $(\cdot)^+$ is the Moore-Penrose inverse operator and P is the solution of the Lyapunov equation

$$(\mathbf{A_N} + \mathbf{A})^{\mathbf{T}}\mathbf{P} + \mathbf{P}(\mathbf{A_N} + \mathbf{A}) = -2\mathbf{Q}\cdot \qquad (37)$$

Proof: Proofs of necessity and sufficiency are similar to Theorem 3. ■

Remark 3: If the recommended Q does not satisfy the condition (a) in Proposition 1, an approximated LSE LQR supervisory problem could be determined by an algorithm similar to Algorithm 1, except Step 3 where the matrix R should be determined by Eq. (36).

4. Numerical example: Smart Automatic Generation Control design

In this section, we study the design of optimal Automatic Generation Control (AGC) based on linear differential game theory for a Two-Area power system (TAPS) with non-Reheat Thermal units (Fig. 2). The assumption is that each of the areas is controlled by an independent and non-cooperative company. To increase the reliability and accessibility of electric energy to consumers, the power areas are connected by a power line known as Tie-Line. Therefore, unwanted changes in each area affect the behavior of the system-state variables and the companies' cost functions. A Block diagram of a typical non-Reheat TAPS has been shown in Fig. 3.

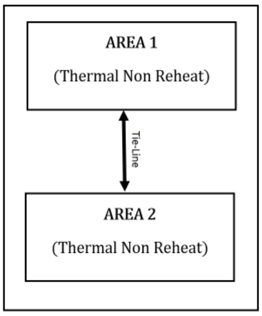

Fig. 2. Non-Reheat TAPS

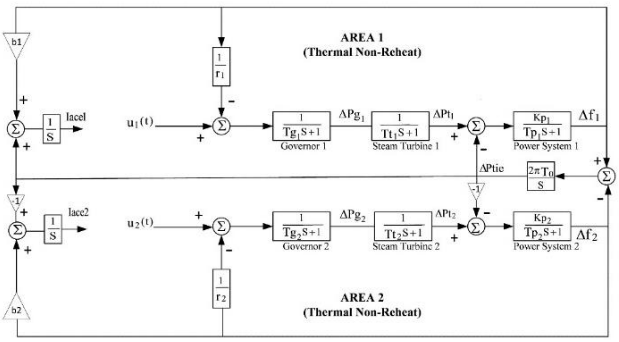

Fig. 3. Block diagram of a typical non-Reheat TAPS

According to Fig. 3, the state-space model of the typical non-Reheat TAPS could be determined as follows:

$$\dot{x}(t)=\begin{bmatrix} \frac{-1}{T_{P1}} & \frac{K_{P1}}{T_{P1}} & 0 & 0 & 0 & 0 & \frac{-K_{P1}}{T_{P1}} & 0 & 0 \\ 0 & \frac{-1}{T_{t1}} & \frac{1}{T_{t1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{r_1 T_{g1}} & 0 & \frac{-1}{T_{g1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{T_{P2}} & \frac{K_{P2}}{T_{P2}} & 0 & \frac{K_{P2}}{T_{P2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{T_{t2}} & \frac{1}{T_{t2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{r_2 T_{g2}} & 0 & \frac{-1}{T_{g2}} & 0 & 0 & 0 \\ 2\pi T_0 & 0 & 0 & -2\pi T_0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b_2 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} x(t)+\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{T_{g1}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{T_{g2}} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(t),\ x0=\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where

$x(t)=[\Delta f_1, \Delta P_{g1}, \Delta P_{t1}, \Delta f_2, \Delta Pg_2, \Delta P_{t2}, \Delta P_{tie}, I_{ace1}, I_{ace2}]^T$ and $u(t)=[u_1(t)\ \ u_2(t)]^T$.

Assuming $i = 1,2$, $\Delta f_i(t)$, $\Delta P_{gi}(t)$, ΔP_{ti} and I_{acei} respectively represent frequency deviation, variations in governor position output, variations in turbine mechanical output deviation, Area Control Error signal (ACE) integral in the ith control area and $\Delta P_{tie}(t)$ is variations in power exchange between two power areas through Tie-line.

Suppose that the system parameters and the weighting matrices of cost functions for the commissioner (i.e. the supervisor) and the companies (i.e. the non-cooperative intelligent control agents) are defined as follows:

$T_{g1}=T_{g2}=0.08$ s, $T_{t1}=T_{t2}=4$ s, $T_{P1}=T_{P2}=20$s

$K_{P1}=K_{P2}=120\,\frac{Hz}{PuMW}$, $r_1=r_2=2.4\,\frac{Hz}{PuMW}$, $b_1=b_2=0.425\,\frac{PuMW}{Hz}$, $T_0=0.0707\,\frac{MW}{rad}$

$$Q = \mathrm{diag}([1000 \quad 1 \quad 1 \quad 1000 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1])$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q_1 = \mathrm{diag}([10 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 10 \quad 1 \quad 0]),\ R_1 = 2$$

$$Q_2 = \mathrm{diag}([0 \quad 0 \quad 0 \quad 100 \quad 10 \quad 10 \quad 10 \quad 0 \quad 1]),\ R_2 = 4$$

So the commissioner's optimal CIHLQR solution (4) could be determined analytically by $u_{LQ}=-Kx(t)$ where

$$K=-R^{-1}B^TP=\begin{bmatrix} 30.9112 & 19.2942 & 2.1173 & 0.1613 & 0.0043 & 0.0003 & -14.1125 & 1 & 0 \\ 0.1613 & 0.0043 & 0.0003 & 30.9112 & 19.2942 & 2.1173 & 14.1125 & 0 & 1 \end{bmatrix}$$

The LFNE (8) for the companies could be calculated by using the numerical method presented in [31] as

$$\begin{bmatrix} u_{1N}(t) \\ u_{2N}(t) \end{bmatrix} = \begin{bmatrix} -R_{11}^{-1}B_1^TP_1x(t) \\ -R_{22}^{-1}B_2^TP_2x(t) \end{bmatrix} = \begin{bmatrix} -K_{1N}x(t) \\ -K_{2N}x(t) \end{bmatrix}$$

where

$$K_{1N}= [1.9727 \quad 2.9312 \quad 0.6348 \quad 0.0036 \quad -0.0712 \quad -0.0078 \quad -1.7076 \quad 0.7971 \quad 0.1270]$$

$$K_{2N}= [0.3879 \quad 0.1344 \quad 0.0081 \quad 4.2254 \quad 5.5091 \quad 1.3883 \quad 6.4201 \quad 0.3996 \quad 0.5636]$$

.

Table I shows the costs of the commissioner and the companies in the LFNE and the optimal CIHLQR strategies. Theorems 1 and 2 are used to complete the cost Table I.

Table I. The Cost of Using LQR and Nash strategies

Cost	LFNE	CIHLQR
Commissioner as supervisor	942.4514	**330.1313**
Company 1 as agent 1	13.6643	**112.5181**
Company 2 as agent 2	26.5171	**110.9246**

The results show that using the LQR strategy will optimize the commissioner's cost, but the cost of companies increases several times. On the other hand, the feedback Nash strategy decreases the cost of companies, while increasing sharp the cost of the commissioner and reducing the system's performance in the main state variables. Figs. 4, 5, and 6 show the control inputs and the main state deviations for these design strategies.

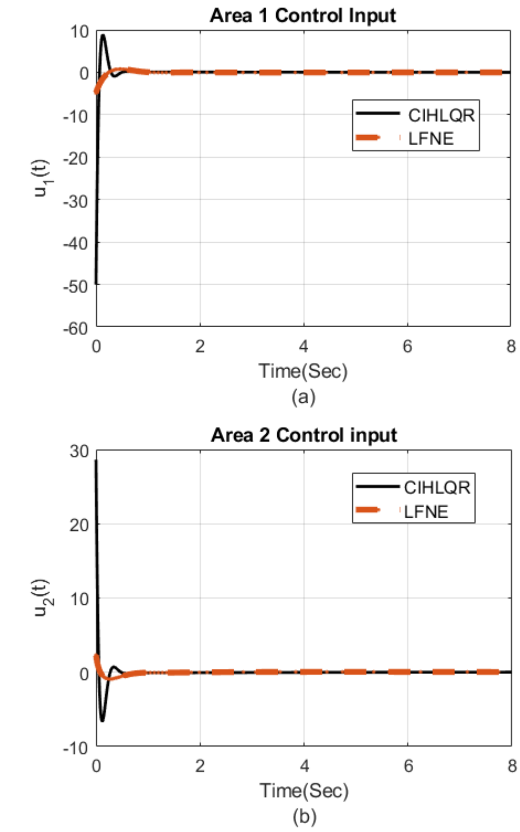

Fig. 4. Control inputs for CIHLQR and LFNE. (a) Area 1's control input, (b) Area 2's control input.

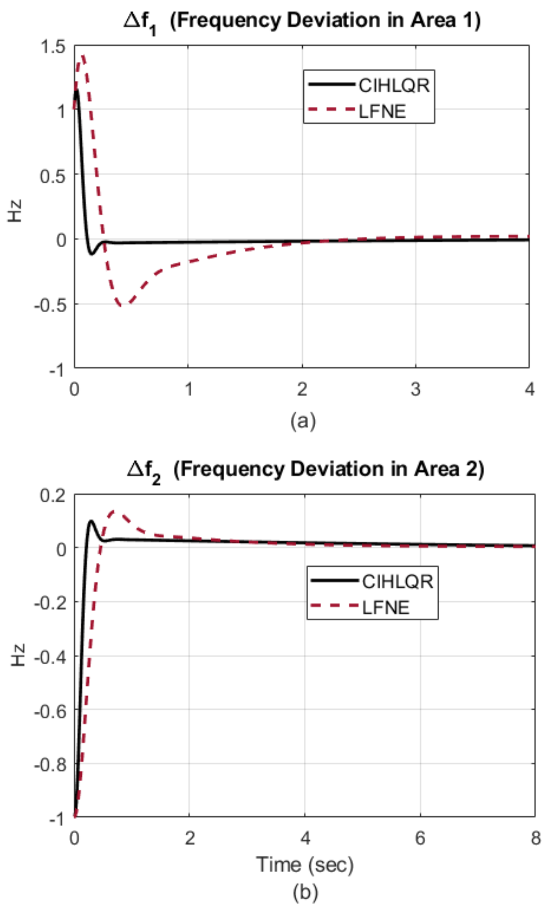

Fig. 5. Deviations of frequency by applying CIHLQR and LFNE Strategies. (a) Area 1, (b) Area 2.

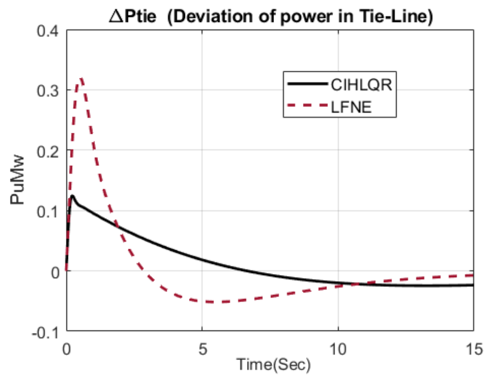

Fig. 6. (c) Deviation of power in Tie-Line.

If the control strategy is changed from a CIHLQR to a decentralized agent-based LQR control system, because of the self-interest of the companies and non-cooperative behavior between them, maybe the TAP becomes unstable. For this reason, designing a robust Nash-based CIHLQR not only has properties of the centralized LQR controllers but also has the flexibility property for changing a centralized controller to a non-cooperative decentralized intelligent agent-based controller without noticeable in the performance. Thus, the alternate CIHLQR of the game is investigated using Algorithm 1.

Since the Q matrix does not satisfy condition (a) of Theorem 3, weighting matrices Q_{LS} and R_{LS} are calculated using Algorithm 1 as follows:

$$Q_{LS}=\begin{bmatrix} 724.59 & 101.7 & 24.75 & 19.2 & -14.84 & 14.54 & 104 & -31.56 & -67.94 \\ 101.7 & 63.38 & 62.38 & 15.16 & -17.67 & -8.94 & 29.47 & 15.32 & 22.33 \\ 24.75 & 62.38 & 70.14 & -12.86 & -26.4 & 2.21 & 14.09 & -13.75 & 31.54 \\ 19.2 & 15.16 & -12.86 & 1024.54 & -103.78 & -36.16 & 104 & 83.23 & 16.28 \\ -14.84 & -17.67 & -26.4 & -103.78 & -22.73 & -23.73 & -5.31 & -31.41 & 8.6 \\ 14.54 & -8.94 & 2.21 & -36.16 & -23.73 & 103.48 & -21.11 & -26.76 & 24.12 \\ 104 & 29.47 & 14.09 & 104 & -5.31 & -21.11 & 7.79 & 13.41 & 6.92 \\ -31.56 & 15.32 & -13.75 & 83.23 & -31.14 & -26.76 & 13.41 & 97 & 24.48 \\ -67.96 & 22.33 & 31.54 & 16.28 & 8.6 & 24.12 & 6.92 & 24.48 & 15.69 \end{bmatrix}$$

$$R_{LS}=\begin{bmatrix} 140.35 & 1.9 \\ 1.9 & 41.39 \end{bmatrix}$$

The cost and some operation results of the alternative CIHLQR controller have been shown in Table II and Figs. 7 and 8. As can be seen in Table II, the alternative CIHLQR controller approximates the behavior of agents very well by enforcing a very high cost to itself. As a corollary, it could be deduced from the results that replacing an N-agents' controller system with a centralized controller to control the system as the same as the agents is not rational. However, as mentioned in the introduction of this article, this method of controller design in large systems, where control of system inputs is available to non-cooperative intelligent agents, is a necessity.

Since the LQR controllers have the inherent property of being robust to parametric uncertainties in the under control linear system, the proposed controller will also have this property. The robustness of IHLQR controllers in the presence of asymptotic and non-asymptotic disturbances has been discussed in detail in the authors' paper [32].

Table II. The cost of alternate CIHLQR

Cost	LFNE	Alternate CIHLQR
Commissioner	942.4514	**984.9**
Company 1	13.6643	**14.03**
Company 2	26.5171	**31.26**

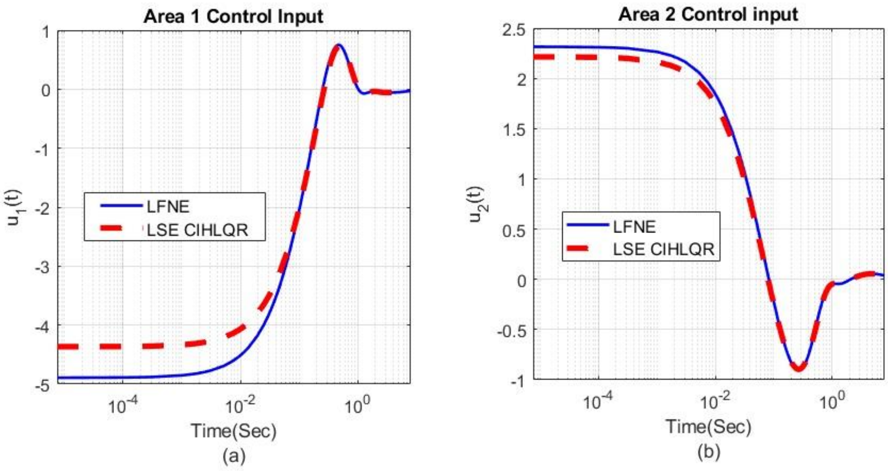

Fig. 7. Plotting the control inputs in the LFNE method and the alternate LSE approximated CIHLQR controller for (a) Area 1, (b) Area 2.

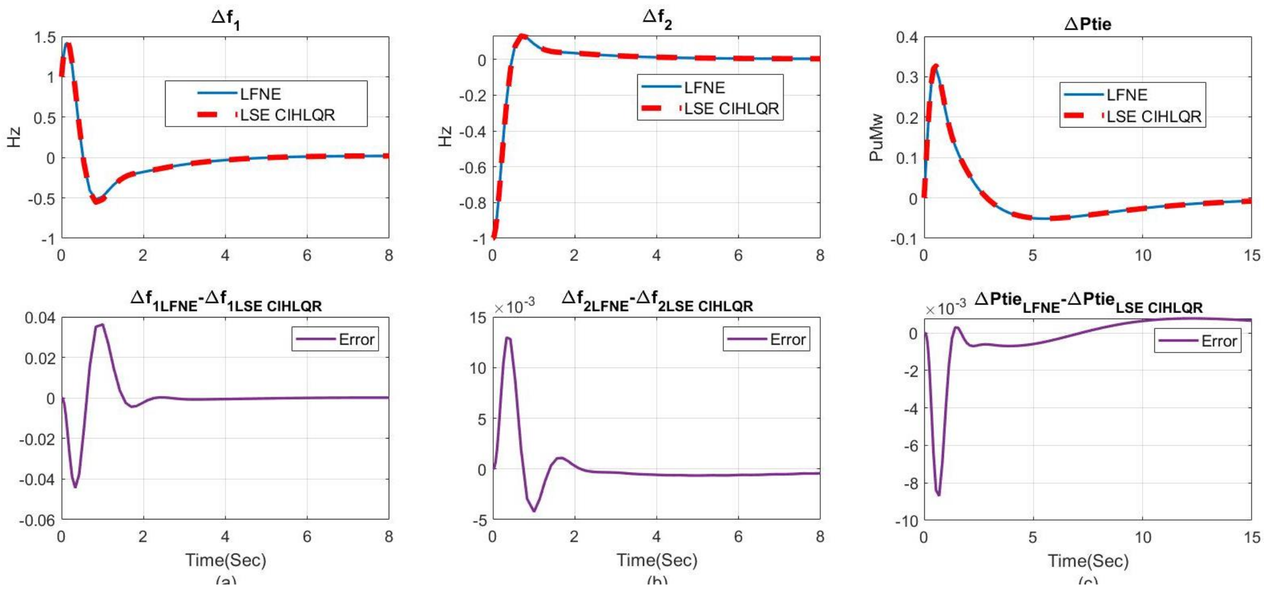

Fig. 8. Deviations of some important states by feedback Nash Strategy and its LQR alternative controller (a) Deviation of frequency in area 1, (b) Deviation of frequency in Area 2, (c) Deviation of power in Tie-Line.

5. Conclusion

This paper deals with the design of a two-agent decentralized robust control system based on the Nash equilibrium point. In this regard, to provide an analytical solution, the agent-based controller design problem of the game was changed to the problem of designing a centralized infinite horizon LQR centralized infinite horizon LQR (CIHLQR). For this purpose, the necessary and sufficient conditions for the existence of the alternate CIHLQR problem were investigated as a theorem. In the cases where fully tracking is not possible, the least-squares error method was used to extract the weighted matrices of the alternative centralized problem cost functions. The methods presented in this study were used in the design of automatic generation control of a two-area power system with two non-cooperative power generation companies. The results confirm the validity of the introduced theories.

6. References

[1] B. D. Anderson and J. B. Moore, *Optimal control: linear quadratic methods*. Courier Corporation, 2007.

[2] F. L. Lewis, D. Vrabie, and V. L. Syrmos, *Optimal control*. John Wiley & Sons, 2012.

[3] A. Sinha, *Linear systems: optimal and robust control*. CRC press, 2007.

[4] A. Owczarkowski, D. Horla, and J. Zietkiewicz, "Introduction of Feedback Linearization to Robust LQR and LQI Control – Analysis of Results from an Unmanned Bicycle Robot with Reaction Wheel," *Asian Journal of Control*, vol. 21, no. 2, pp. 1028-1040, 2019, doi: https://doi.org/10.1002/asjc.1773.

[5] A. G. Pillai and E. Rita Samuel, "PSO based LQR-PID output feedback for load frequency control of reduced power system model using balanced truncation," *International Transactions on Electrical Energy Systems*, vol. 31, no. 9, p. e13012, 2021, doi: https://doi.org/10.1002/2050-7038.13012.

[6] A. Sir Elkhatem and S. Naci Engin, "Robust LQR and LQR-PI control strategies based on adaptive weighting matrix selection for a UAV position and attitude tracking control," *Alexandria Engineering Journal*, vol. 61, no. 8, pp. 6275-6292, 2022/08/01/ 2022, doi: https://doi.org/10.1016/j.aej.2021.11.057.

[7] E. Joelianto, D. Christian, and A. Samsi, "Swarm control of an unmanned quadrotor model with LQR weighting matrix optimization using genetic algorithm," *Journal of Mechatronics, Electrical Power, and Vehicular Technology*, vol. 11, no. 1, pp. 1-10, 2020.

[8] N. Cramer, S. S.-M. Swei, K. C. Cheung, and M. Teodorescu, "Extended discrete-time transfer matrix approach to modeling and decentralized control of lattice-based structures," *Structural Control and Health Monitoring*, vol. 23, no. 10, pp. 1256-1272, 2016, doi: https://doi.org/10.1002/stc.1837.

[9] J. Narayan and S. K. Dwivedy, "Robust LQR-Based Neural-Fuzzy Tracking Control for a Lower Limb Exoskeleton System with Parametric Uncertainties and External Disturbances," *Applied Bionics and Biomechanics*, vol. 2021, p. 5573041, 2021/06/12 2021, doi: 10.1155/2021/5573041.

[10] H. Calgan and M. Demirtas, "A robust LQR-FOPIλDμ controller design for output voltage regulation of stand-alone self-excited induction generator," *Electric Power Systems Research*, vol. 196, p. 107175, 2021/07/01/ 2021, doi: https://doi.org/10.1016/j.epsr.2021.107175.

[11] D. Masti, M. Zanon, and A. Bemporad, "Tuning LQR controllers: A sensitivity-based approach," *IEEE Control Systems Letters*, vol. 6, pp. 932-937, 2021.

[12] H. H. Bilgic, M. A. Sen, A. Yapici, H. Yavuz, and M. Kalyoncu, "Meta-heuristic tuning of the LQR weighting matrices using various objective functions on an experimental flexible arm under the effects of disturbance," *Arabian Journal for Science and Engineering*, vol. 46, no. 8, pp. 7323-7336, 2021.

[13] X. Huang and P. Li, "An Auto-tuning LQR based on Correlation Analysis," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 7148-7153, 2020/01/01/ 2020, doi: https://doi.org/10.1016/j.ifacol.2020.12.525.

[14] E. E. Vlahakis, L. D. Dritsas, and G. D. Halikias, "Distributed LQR-based Suboptimal Control for Coupled Linear Systems∗∗This work was supported by a City, University of London scholarship held by Eleftherios Vlahakis," *IFAC-PapersOnLine*, vol. 52, no. 20, pp. 109-114, 2019/01/01/ 2019, doi: https://doi.org/10.1016/j.ifacol.2019.12.139.

[15] W. Wang, F. Zhang, and C. Han, "Distributed linear–quadratic regulator control for discrete-time multi-agent systems," *IET Control Theory & Applications*, vol. 11, no. 14, pp. 2279-2287, 2017, doi: https://doi.org/10.1049/iet-cta.2016.1641.

[16] M. Wang, H. Xu, S. Yang, L. Yang, R. Duan, and X. Zhou, "Non-cooperative differential game based energy consumption control for dynamic demand response in smart grid," *China Communications*, vol. 16, no. 8, pp. 107-114, 2019, doi: 10.23919/JCC.2019.08.010.

[17] H. Xu and R. Lin, "Resource allocation for network security risk assessment: A non-cooperative differential game based approach," *China Communications*, vol. 13, no. 4, pp. 131-135, 2016, doi: 10.1109/CC.2016.7464130.

[۱۸] م. جوادی, س. میرحسینی مقدم و م. مرزبند, "مدیریت بهینه انرژی ریزشبکه‌ها در بازار خرده‌فروشی بر پایه روش تئوری بازی غیر همکارانه با در نظر گرفتن عدم قطعیت," مجله مهندسی برق دانشگاه تبریز, vol. 46, no. 1, pp. 63-74, 2016. [Online]. Available: https://tjee.tabrizu.ac.ir/article_4673_8577edbed16eabf47d026ba28b6c5e5b.pdf.

[۱۹] م. جوادی, س. س. قاضی‌میرسعید و م. مرزبند, "استراتژی قیمت‌گذاری استاتیکی برپایه مفهوم انرژی انتقالی در سیستم‌های چند-ریزشبکه‌ای خانگی با استفاده از روش تئوری بازی غیرهمکارانه," مجله مهندسی برق دانشگاه تبریز, vol. 48, no. 2, pp. 557-571, 2018. [Online]. Available: https://tjee.tabrizu.ac.ir/article_7784_02918d4438b4a3a5b8eb0fc2fd1c3322.pdf.

[20] J. Nash, "Non-cooperative games," *Annals of mathematics*, pp. 286-295, 1951.

[21] J. F. Nash Jr, "The bargaining problem," *Econometrica: Journal of the econometric society*, pp. 155-162, 1950.

[22] R. Isaacs, *Differential games: a mathematical theory with applications to warfare and pursuit, control and optimization*. Courier Corporation, 1999.

[23] J. Engwerda, "Min-Max Robust Control in LQ-Differential Games," *Dynamic Games and Applications*, 2022/01/18 2022, doi: 10.1007/s13235-021-00421-z.

[24] J. Engwerda, "A Numerical Algorithm to Find All Feedback Nash Equilibria in Scalar Affine Quadratic Differential Games," *IEEE Transactions on Automatic Control*, vol. 60, no. 11, pp. 3101-3106, 2015, doi: 10.1109/TAC.2015.2411914.

[25] D. Vrabie and F. Lewis, "Adaptive dynamic programming for online solution of a zero-sum differential game," *Journal of Control Theory and Applications*, vol. 9, no. 3, pp. 353-360, 2011.

[26] Y. Lv and X. Ren, "Approximate Nash solutions for multiplayer mixed-zero-sum game with reinforcement learning," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 49, no. 12, pp. 2739-2750, 2018.

[27] S. Krilašević and S. Grammatico, "Learning generalized Nash equilibria in multi-agent dynamical systems via extremum seeking control," *Automatica*, vol. 133, p. 109846, 2021/11/01/ 2021, doi: https://doi.org/10.1016/j.automatica.2021.109846.

[28] J. Engwerda, *LQ dynamic optimization and differential games*. John Wiley & Sons, 2005.

[29] T. Li and Z. Gajic, "Lyapunov Iterations for Solving Coupled Algebraic Riccati Equations of Nash Differential Games and Algebraic Riccati Equations of Zero-Sum Games," in *New Trends in Dynamic Games and Applications*: Springer, 1995, pp. 333-351.

[30] R. A. Horn and C. R. Johnson, "Topics in matrix analysis," *Cambridge University Presss, Cambridge*, pp. 242-295, 1991.

[31] G. Freiling, G. Jank, and H. Abou-Kandil, "On global existence of solutions to coupled matrix Riccati equations in closed-loop Nash games," *Automatic Control, IEEE Transactions on*, vol. 41, no. 2, pp. 264-269, 1996, doi: 10.1109/9.481532.

[32] S. Najafi Birgani, B. Moaveni, and A. Khaki - Sedigh, "Infinite horizon linear quadratic tracking problem: A discounted cost function approach," *Optimal Control Applications and Methods*, vol. 39, no. 4, pp. 1549-1572, 2018.