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Gravitational collapse of the galactic dark matter

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Abstract
A spherically symmetric matter collapses under its own gravity. For the Milky Way, the time scale of collapse for a spherical halo with NFW density is ~ 26 My which is very small compared to the age of the Milky Way. The effect of a pressure, obeying a polytrope equation of state is investigated. It is shown that for $p \propto \rho^{9/8}$, the time scale would be of the order of 1 Gy. It is also argued that such a polytrope, if considered to be in equilibrium, could explain the rotation curve of the milky way.

Keywords: Milky Way, galaxy, dark matter, gravitational collapse, polytrope

1. Introduction

According to the current standard cosmology, the so-called ΛCDM, the energy- matter content of the universe is almost 70% dark energy, almost 25% dark matter, and almost 5% baryonic matter [13, p. 55]. The cosmological observations showing the existence of non-baryonic dark matter include the standard cosmological nucleosynthesis [13, p. 185]. Also, detailed studies of the dynamics of clusters of galaxies, that is, the large scale structure of the universe, indicate that this dark matter must be cold [13, p. 186]. The dark matter shows its effects in the rotation curves of the spiral galaxies [9], the velocity dispersion of the elliptical galaxies [1], the velocity dispersion of the clusters of galaxies [5], and other effects such as the gravitational lensing [6]. It should be noted that there is another paradigm for explaining all these phenomena by modifying the Newtonian dynamics [2], but that's beyond the scope of this article.

Although the majority of researches agree on the existence of dark matter, there is no consensus on what the dark matter is. In the standard framework of quantum fields, dark matter is composed of particles of one or several quantum fields. Several models for these fields are proposed. Different fields result in different dynamics of the dark matter. bulk property of the dark matter which is needed to describe the dynamics. For the spiral galaxies such as the Milky Way, the standard description of the dark matter is a *halo* with spherical density profile $\rho(r)$, where r is the distance from the center of the galaxy.

A spherically symmetric distribution of matter, produces a spherically symmetric gravitational field. If there is no pressure, the gravity will cause the matter to collapse. The time scale of the collapse could be obtained from the the density profile, and, as we shall see in a moment, it is very short compared to the ages of the galaxies. Therefore, a static distribution of the dark matter is possible, only if there is sufficient pressure. In this article we would like to address this problem.

2. The Collapse Time Scale of the NFW Dark Matter

Several profiles for the density of the halo are proposed [10]. One of the profiles which is mostly used, and is very successful in explaining the plateau of the rotation curves of spiral galaxies, is the Navarro-Frenk-White (NFW) profile introduced in [8]. (remove dot)

$$\rho_{NFW}(r) = \frac{\rho_0}{\frac{r}{a}(1+\frac{r}{a})^2} \tag{1}$$

For the Milky Way galaxy the parameters are [10]:

$$\rho_0 \simeq 1.8\times10^{-21}\,\frac{\text{kg}}{\text{m}^3} = 0.36\,\frac{\text{Gev}}{\text{c}^2\text{cm}^3} = 0.26\,\frac{10^8\text{M}_\odot}{(\text{kpc})^3}, \tag{2}$$

$$a \simeq 3.1\times10^{20}\,\text{m} = 10\text{kpc}. \tag{3}$$

It is easy to see that for such a distribution of mass, the mass inside a sphere of radius r is

$$M(r) = 4\pi\int_0^r \rho_{NFW}(s)s^2\,\text{d}s$$
$$= M_0\left[\ln(1+\frac{r}{a}) - \frac{r}{a+r}\right], \tag{4}$$

Where

$$M_0 = 6.7\times10^{41}\,\text{kg} = 3.4\times10^{11}\,\text{M}_\odot. \tag{5}$$

$M(r)$, drawn in figure1, is logarithmically divergent as $r \to \infty$.

The acceleration of gravity due to this mass is $\vec{g}(r) = -g(r)\hat{r}$:

$$g(\mathrm{r}) = g_0 \left[\frac{a^2}{r^2} \ln(1+\frac{r}{a}) - \frac{a^2}{r(a+r)} \right], \quad (6)$$

$$g_0 = 4\pi G \rho_0 a = 4.6\times10^{-10}\,\mathrm{m\,s^{-2}}, \quad (7)$$

$$g(a) = 8.9\times10^{-11}\,\mathrm{m\,s^{-2}}, \quad (8)$$

If an object of mass M is at the origin, and a test particle starts radial motion from distance r, the test particle reaches the origin in a time of the order $\sqrt{r^3/G\,M}$.

Now consider a spherical shell of dark matter with radius r and thickness dr. Since the halo is spherically symmetric, and ignoring the gravitational field of disk of the galaxy, this spherical shell experiences the gravitational field of a point mass of mass M(r). For the NFW density, it is straightforward to see that the time scale of collapse of the shell $[r,\ r+\mathrm{d}r]$ is

$$t_c = f\left(\frac{r}{a}\right)\tau, \quad (9)$$

Where

$$\tau = \frac{1}{\sqrt{4\pi G \rho_0}} \simeq 8.1\times10^{14}\,\mathrm{s} \ = 26\ \mathrm{My}, \quad (10)$$

And

$$f(x) = \frac{\pi}{\sqrt{8}} x^{3/2} \left[\ln(1+x) - \frac{x}{1+x} \right]^{-\frac{1}{2}} \quad (11)$$

$f(3) = 7.24$, therefore, the time scale of the collapse of the spherical shell of radius $r = 3a = 30$ kpc is 190 My.

In summary, a pressure-less spherically symmetric mass with the NFW density for the Milky Way, would collapse in a time scale of order 0.2 Gy. This is small compared to the age of the Milky Way, and even the time scale of the appreciable changes in the structure of the Milky Way.

One way out of this dilemma is to postulate a pressure. Newton's equation for a spherically symmetric *static* fluid under its own gravity and pressure is (see [12], p. 308)

$$-r^2 \frac{dp}{dr} = GM(r)\rho(r). \quad (12)$$

For the NFW density $\rho(r)$ and $M(r)$, we get

$$p(r) = p_1 h(x), \quad (13)$$

Where $x = r/a$,

$$p_1 = 4\pi G \rho_0 a^2, \quad (14)$$

And

$$h'(x) = \frac{1}{x^2(1+x)^3} - \frac{\ln(1+x)}{x^3(1+x)^2}. \quad (15)$$

This could be integrated, and the result is

$$p(x) = p_1 \int_x^\infty \left[\frac{\ln(1+s)}{s^3(1+s)^2} - \frac{1}{s^2(1+s)^3} \right] \mathrm{ds}. \quad (16)$$

The functions $\rho(x)$ and $p(x)$ are drawn in figure 2. In figure 3 we draw the Log-Log diagram of $P(\rho)$.

Assuming the existence of such a pressure is an ad hoc assumption. It would be better if we could start from an equation of state and derive the pressure.

3. Polytrope equation of state

One simple equations of state is the polytrope equation

$$\frac{p(r)}{p_0} = \left[\frac{\rho(r)}{\rho_0} \right]^{1+\frac{1}{n}}, \quad (17)$$

where n is called the polytrope index. The condition for the hydrostatic equilibrium is (see [12, p. 308]):

$$\rho(r) = \rho_0 [\theta_n(x)]^n, \quad (18)$$

$$p(r) = p_0 [\theta_n(x)]^{n+1}, \quad (19)$$

Where

$$p_0 = 2\pi G \rho_0 a^2, \quad (20)$$

$$x = \frac{r}{a}, \quad (21)$$

$$a = \sqrt{\frac{(n+1)p_0}{4\pi G \rho_0^2}}, \quad (22)$$

and $\theta_n(x)$ satisfies the Lane-Emden equation

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2 \frac{d\theta_n}{dx} \right) + [\theta_n(x)]^n = 0, \quad (23)$$

with initial conditions

$$\theta_n(0) = 1, \quad (24)$$

$$\theta_n'(0) = 0, \qquad \theta_n'(\xi) = \frac{d\theta_n}{dx}. \quad (25)$$

It is known that for

$$1 + \frac{1}{n} > \frac{6}{5},$$

that is for $n < 5$, the solution satisfies $\theta_n(x_1) = 0$, for some positive x_1 [12, p. 310]. Therefore, only the polytropes with $n < 5$ are considered for modeling the stars.

3. 1. $n = 1$ Polytrope dark matter

For $n = 1$, the Lane-Emden equation has the solution:

$$\theta_1(x) = \frac{\sin x}{x}. \quad (26)$$

This equation is the classical solution of a scalar field, satisfying the Gross- Pitaevskii-Poisson equations in the Thomas-Fermi approximation [3].

Using (26), let us define

$$\rho(r) = \begin{cases} \rho_0 \theta_1(x) & 0 \le r \le \pi a, \\ 0 & r \ge \pi a, \end{cases} \quad (27)$$

From this density, one can find $M(r)$:

$$M(r) = \begin{cases} 4\pi\rho_0 a^3 [\sin x - x\cos x] & 0 \le r \le \pi a, \\ 4\pi^2 \rho_0 a^3 & \pi \mathrm{a} \le r \le \infty, \end{cases} \quad (28)$$

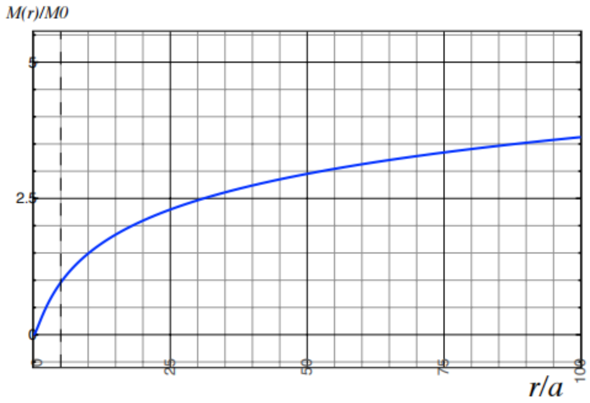

Figure 1. The function $M(r)$ for the NFW density profile. The vertical dashed line indicates the radius $r = 5\,a$ from the center. for $r < 5a$ the function $M(r)$ is almost linear.

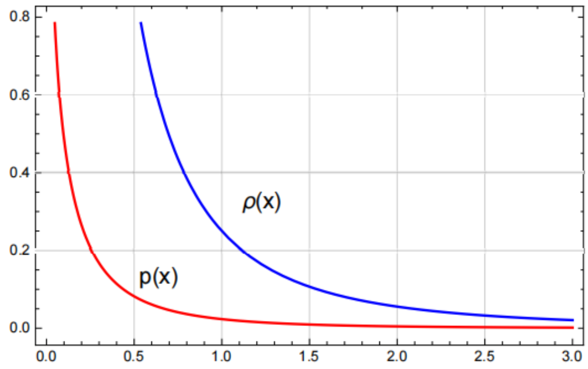

Figure 2. Pressure and density profile of the *static* NFW halo. $p(r) = p_1\, p(x)$, and $\rho(r) = \rho_0\, \rho(x)$, where $x = r/a$. The pressure is calculated from the Newtonian equations for the hydrostatic equilibrium of the NFW mass under its own gravity.

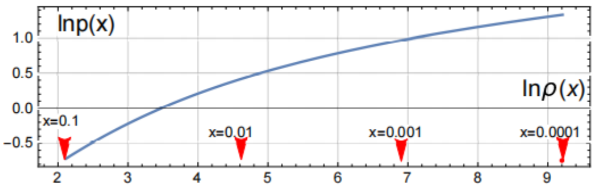

Figure 3: $\ln(p)$ vs. $\ln(\rho)$, $x \in [0.0001, 0.1]$, for a NFW dark matter which is static under its own gravity and pressure. The pressure is calculated from the Newtonian equation of hydrostatic equilibrium.

Since this halo is in hydrostatic equilibrium, the model we can construct using this density for the halo does not suffer from the collapse problem we mentioned. For example, let us introduce a very simple model for the Milky Way:

a) A spherical mass, M_b representing the central black hole and a spherical bulge,

b) an exponential disk with surface density σ_0 and radius scale d, a halo with density (27).

The rotation curve for this simplified model, is drawn in figure 4.

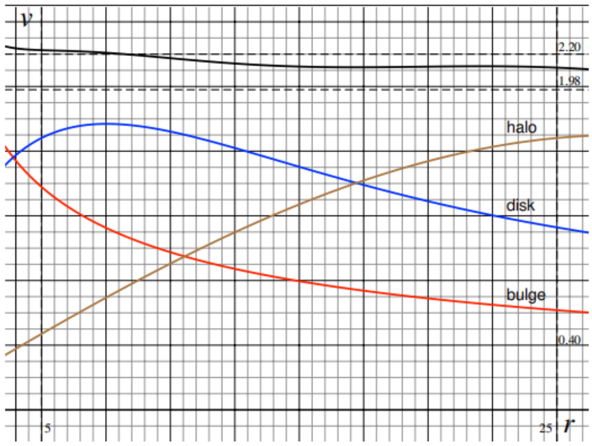

Figure 4. Rotation curve of the simple model consisting of a spherical central bulge of mass $M_b = 2.2\times10^{10} M_\odot$; an exponential disk with $\sigma = 0.85\, M_\odot \mathrm{pc}^{-2}$, $d = 8$ kpc, and a halo with density (27) with $\rho_0 = 0.18$ GeV c^{-2} cm^{-3}, and $a = 10$ kpc. Velocity, in 100 km s^{-1}, vs. distance from the center in kpc. The two horizontal dashed lines indicate the range of the observed values.

4. Polytrope NFW halo

Suppose at time $t = 0$, there is a spherically symmetric halo with an NFW density profile, and suppose that this dark matter obeys a polytrope equation of state with index n. Let us find the time scale for the gravitational collapse of this halo.

Consider an element of the fluid between the shells r and $r + \mathrm{d}r$, and in an infinitesimal solid angle $\mathrm{d}\Omega$. The mass of this element is

$$\mathrm{d}m = \rho(r) r^2 \mathrm{d}r\ \mathrm{d}\Omega. \tag{29}$$

$$\rho(r)\frac{d^2 r}{dt^2} = -\frac{G\ M(r)\rho(r)}{r^2} - \frac{dp}{dr}. \tag{30}$$

It is more convenient to write this equation as

$$\frac{d^2 r}{dt^2} = -\frac{G\ M(r)}{r^2} - \frac{1}{\rho(r)}\frac{dp}{dr}. \tag{31}$$

The gravitational potential Φ satisfies

$$\frac{d\Phi}{dr} = \frac{G\ M(r)}{r^2}. \tag{32}$$

Using (4), by integration it follows that Writing Newton's equation we get

$$\Phi(r) = -\frac{p_1}{\rho_0}\frac{\ln(1+x)}{x}. \tag{33}$$

If we assume a polytrope equation of state of the form[1]

$$\frac{p(r)}{p_1} = w\left(\frac{\rho(r)}{\rho_0}\right)^{1+\frac{1}{n}}. \tag{34}$$

It is then straightforward to see that

$$-\frac{1}{\rho}\frac{dp}{dr} = -\frac{1}{\rho}\frac{\partial}{\partial r}\left(w\ p_0 f^{1+\frac{1}{n}}(r)\right) \tag{35}$$

[1] We introduced w, so that $p_0 = w\, p_1$

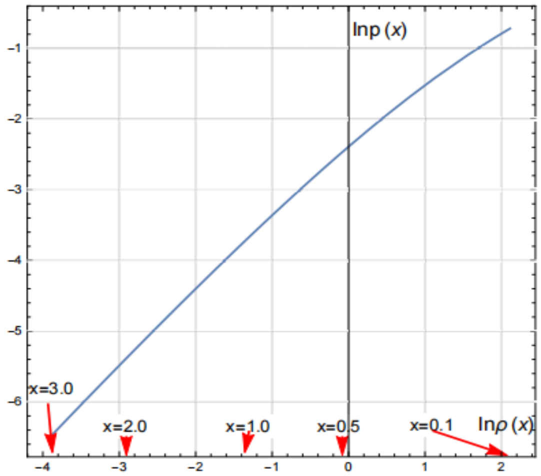

Figure 5. $\ln(p)$ vs $\ln(\rho)$ for $x \in [0.1, 3.0]$. The graph is close to a line with the slope 9/8.

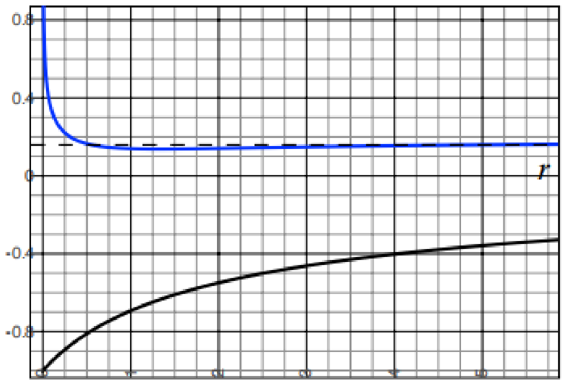

Figure 6. The potential Φ (black), due to the mass distribution of a NFW density, and the effective potential $\Phi+\Psi$ (blue), where Ψ is the potential defined by $-\vec{\nabla}\Psi = -\rho^{-1}\vec{\nabla} p$, if the dark matter satisfies the polytrope equation of state, $p = w\ p_0 \left(\frac{\rho}{\rho_0}\right)^{1+\frac{1}{n}}$ with $n = 8$ and $w = 0.11$. Note that the effective potential is almost flat for $x > 0.4$. The unit of the energy is $4\pi G \rho_0 a^2$, and the unit of radius is a.

$$= -\frac{w\ p_0}{\rho_0}\frac{1}{f}\left(1+\frac{1}{n}\right) f^{\frac{1}{n}}(r) f'(r) \tag{36}$$

$$= -\frac{\partial}{\partial r}\left(\frac{(n+1) w\ p_0}{\rho_0} f^{\frac{1}{n}}(r)\right) \tag{37}$$

or

$$-\frac{1}{\rho}\vec{\nabla} p = -\vec{\nabla}\Psi. \tag{38}$$

where

$$\Psi = \frac{w\ p_0}{\rho_0}\begin{cases} \frac{n+1}{\left[x(1+x)^2\right]^{1/n}} & n < \infty \\ -\ln\left[x(1+x)^2\right] & n = \infty. \end{cases} \tag{39}$$

Now the equation of motion reads

$$\frac{d^2 r}{dt^2} = -\frac{d\Phi}{dr} - \frac{d\Psi}{dr}. \tag{40}$$

From the diagram of $\ln(p)$ vs. $\ln(\rho)$, drawn in figure 5, we conclude that $n = 8$ could be a plausible candidate. For $n = 8$ and $w = 0.11$, the effective potential $\Phi+\Psi$ is drawn in figure 6. There is now a barrier near the origin which prevents the system to collapse to a point. Also, the effective potential is almost flat for $r > 0.5$ kpc. We can now calculate the period of oscillations of a spherical shell of thickness dr, initially at r. For the initial radius of $x = 5$, the final (smallest) radius would be 0.554, and collapse time is thus:

$$T = \frac{1}{\sqrt{2}}\int_{0.554}^{5.000} \frac{dx}{\sqrt{\phi(x_0)+\psi(x_0)-\phi(x)-\psi(x)}} \simeq 44\ \tau \simeq 1\ \text{Gy}, \tag{41}$$

where τ is given by (eq. 10)[2]. The period of oscillations is twice this value, 2 Gy. The conclusion is that an NFW dark matter satisfying an $n=8$ polytrope equation of state is not static, but the time scale of the gravitational collapse, or period of oscillations, is of the order of 2 Gy, which is more plausible than 0.2 Gy corresponding to a pressure-less NFW dark matter.

5. *n* = 8 Polytrope

In the previous section we saw that if the halo's density is the NFW, and if it satisfies a polytrope equation of state with $n = 8$, then the mass could not collapse completely—it will oscillate, with a characteristic time of the order of 2 Gy. Now let us investigate the density profile of a polytrope mass, which is *in equilibrium* under its own gravity and pressure.

It is known that for $n > 5$, the solution of the Lane-Emden equation is such that, there is no positive root for $\theta_n(x)$.

Therefore, polytrope models for $n > 5$ are not used to model stars, and are rarely investigated.

To see the behavior of $\rho(r)$ for the $n = 8$ polytrope, we must first obtain $\theta_8(x)$, and then use (18). We used the algorithm given in (11) to calculate the series

$$\theta_8(x) = \sum_{k=0}^{\infty}(-1)^k c_k\ x^{2k}, \tag{42}$$

up to $k = 31$. Writing a code in Wolfram Mathematica 11.1, we calculated the coefficients up to $k = 31$.

[2] This integral is calculated using Wolfram Mathematica 11.1

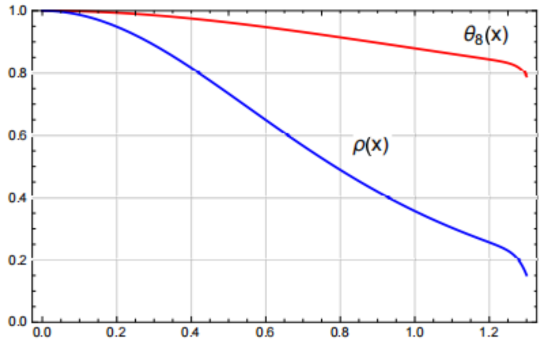

Figure 7. The function $\theta_8(x)$ (red) and the density $\rho=\theta_8^8(x)$ (blue), calculated by the series (42) up to $k=31$. The conversion of the series is very slow. The graph shown here represents $\theta_8(x)$ only for $0\le x\le 1.25$.

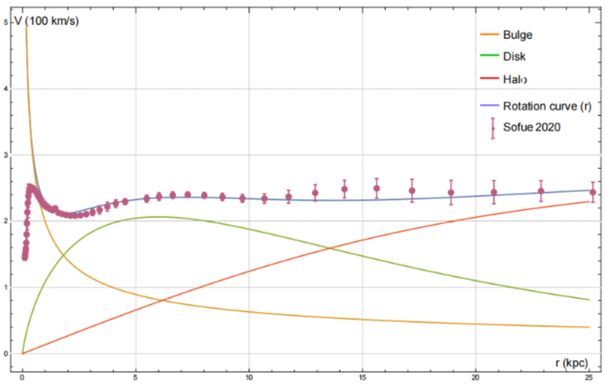

Figure 8. Rotation curve for the simple model composing of a spherical central bulge, an exponential disk, and polytrope halo with index $n=8$. The observed values as given by [10] are also shown. Note that between 5 kpc and 25 kpc the rotation of this simple model fits in the observed interval.

6. A simple model

To investigate the rotation curve due to a *static* polytrope dark matter with index $n=8$, we must use the function (42), the convergence of which is very slow. Here we present a simple *approximate* model for the $n=8$ polytrope, valid for $0\le x\le 1.2$ Using the function (42), we calculate $f(x)=\theta_8^8(x)$, and fit the polynomial

$$f(x)=1+c_2x^2+c_3x^3+c_4x^4 \tag{43}$$

to that. This, we did using Wolfram Mathematica 11.1, and we found the following coefficients:

$$c_2=-1.61064, \tag{44}$$

$$c_3=+1.22639, \tag{45}$$

$$c_4=-0.261527, \tag{46}$$

Now the rotation curve could be calculated easily from

$$v^2(r)=\frac{G\,M(r)}{r}=\frac{4\pi G}{r}\int_0^r \rho(s)\,s^2\,ds. \tag{47}$$

Using this, we can construct a simplified model for the milky way thus:

1) A spherical central bulge of mass $M_b=92\times10^8 M_\odot$.

2) An exponential disk with density $\sigma_0 e^{-r/d}$ where

$$\sigma_0=12\frac{10^8 M_\odot}{(\mathrm{kpc})^2}=2.5\frac{\mathrm{kg}}{\mathrm{m}^2}, \tag{48}$$

$$d=16.0\ \mathrm{kpc}. \tag{49}$$

3) A polytrope halo with index $n=8$ and parameters

$$\rho_0=0.1\frac{10^8 M_\odot}{(\mathrm{kpc})^3}=6.7\times10^{-22}\frac{\mathrm{kg}}{\mathrm{m}^3}, \tag{50}$$

$$b=22\ \mathrm{kpc}. \tag{51}$$

In figure 8 the rotation curves for this model is drawn. The total mass of the Milky Way, inside a sphere of radius 25 kpc, in this model is

$$\begin{aligned} M(25\ \mathrm{kpc}) &= M_{\mathrm{bulge}}+M_{\mathrm{disk}}+M_{\mathrm{halo}} \\ &= (92+386+3098)\times10^8 M_\odot \\ &\simeq 3.5\times10^{11} M_\odot. \end{aligned} \tag{52}$$

7. About the origin of the pressure

We are thankful to one of the referees of the manuscript that pointed to us that: "In the standard scheme of structure formation, the Boltzmann equation is used to obtain the dynamics of the phase space of dark matter particles. The non-linear effects which are caused by growing density contrast introduce a stress tensor in the Euler equation. This stress tensor is a key point to the non-linear structure formation. And that is why the collision-less non-relativistic dark matter particles start to act like a fluid with pressure."

In this regard, we would like to state some comments.

In a gas, the origin of pressure is the collisions of particles, which results from the interactions between the particles. The standard theoretical basis for the derivation of pressure is to use Boltzmann equation for the single-particle distribution function. If the dark matter consists of photons or neutrinos, one must use relativistic Boltzmann equation, and one has to consider Thomson scattering between electrons and photons [4].

For a scalar field φ coupled minimally to gravity, we have

$$p=-\frac{1}{2}g^{\mu\upsilon}\partial_\mu\partial_\upsilon\varphi-V(\varphi),$$

where $V(\varphi)$ is the potential function describing the field, and $g^{\mu\upsilon}$ is the spacetime metric [13]. Such fields have soliton solutions leading to anisotropic pressure[7].

8. Summary and conclusion

A spherically symmetric distribution of matter would collapse under its own gravity, unless there is sufficient pressure to prevent this. We wrote the Newtonian equation for the hydrostatic equilibrium of an NFW distribution to get the pressure, figure 2. The plot of ln p vs. ln ρ, figure 5, indicates that in the interval

$0 \le r \le a \sim 10$ kpc a simple relation of the form $p \propto \rho^{9/8}$ holds. Using the algorithm given in [11], we obtained the function $\theta_8(x)$ which is the solution of the Lane-Emden equation of index $n = 8$. Using a polynomial approximation for $f(x) = \theta^8(x)$, which is valid for $0 < x < 1.2$, we obtained an *approximate* expression for the rotation curve (figure 8). The total mass of the Milky Way, within 25 kpc from the center, is approximately 1.5 times that of the mass found using a NFW model as given by [10].

The time scale of the collapse of the NFW DM, initially at rest, could be calculated (eqs (9)-(11). Being 0.2 Gy for a shell of radius 30 kpc, it is very short compared to the age of the Milky Way. Assuming a pressure obeying the polytrope $n = 8$, that is $p \sim \rho^{9/8}$, but relaxing the assumption of hydrostatic equilibrium, we found the time scale of the oscillations to be 2 Gy (see eq (41).

It should be noted that a static $n = 1$ polytrope could also describe the rotation curve of the Milky Way, see §3.1, figure 4, and [1].

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Competing Interests

The authors have no relevant financial or non-financial interests to disclose.

Author Contribution

This work was done by E.F. as part of her PhD problem under the supervision of A.S. Both authors contribute to this work substantially

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