
CRB

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(// - // //)

CRB

- (DOA)

CRB

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CRB

DOA

ML $\hat{\theta}$

[]

[] Ziv-Zakai [] Weiss-Weinstein :

$\lambda/2$

[] Barankin

$$\bar{x}(t) = \alpha(t)e^{j\phi(t)}e^{j\omega_c t} \quad (1)$$

$$x(t) = \alpha(t)e^{j\phi(t)}e^{-j\omega_c t} \quad (2)$$

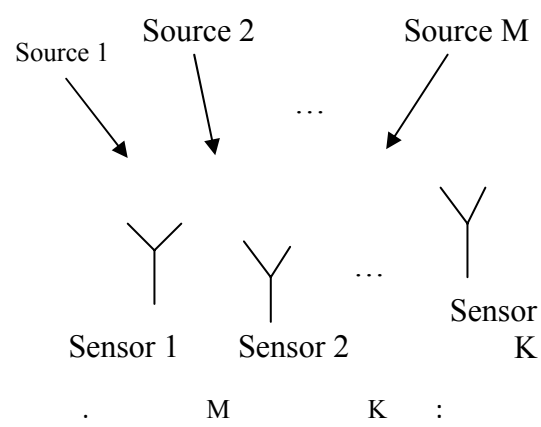
CRB

$$y_k(t) = h_k(t) * x(t - \tau_k) \quad (3)$$

CRB

$$B(\theta_1, \theta_2) = \underline{a}_1^H(\theta_1)\underline{a}_1(\theta_2) \quad (4)$$

$$\underline{a}_1(\theta) = \begin{bmatrix} 1 \\ e^{-j2\pi\frac{d}{\lambda}\sin\theta} \\ \dots \\ e^{-j2\pi(K-1)\frac{d}{\lambda}\sin\theta} \end{bmatrix}^T \quad (5)$$



) bore sight

$\phi(t)$ ω_c $\alpha(t)$

($\bar{x}_r(t)$) $\bar{x}(t)$ $\bar{x}_r(t)$

$$G_{44} = \frac{2}{\sigma^2} \text{Re}[X^H X D^H D] \quad ()$$

$$A = [a_1(\theta_1) \ \dots \ a_1(\theta_M)] \quad ()$$

$$D = \left[\begin{array}{ccc} \frac{\partial a_1(\theta)}{\partial \theta} \Big|_{\theta=\theta_1} & \dots & \frac{\partial a_1(\theta)}{\partial \theta} \Big|_{\theta=\theta_M} \end{array} \right] \quad ()$$

$$X = [x_1 \ \dots \ x_M] \quad ()$$

$$\underline{x}_k = [b_k e^{j\omega_k \cdot 1} \ \dots \ b_k e^{j\omega_k \cdot N}]^T \quad ()$$

; N=

$$X_1 = \left[\begin{array}{cc} \frac{\partial x_1}{\partial \omega_1} & \dots \ \frac{\partial x_M}{\partial \omega_M} \end{array} \right] \quad ()$$

$$X_2 = \left[\begin{array}{cc} \frac{\partial x_1}{\partial b_1} & \dots \ \frac{\partial x_M}{\partial b_M} \end{array} \right] \quad ()$$

$$X_3 = \left[\begin{array}{cc} \frac{\partial x_1}{\partial b_1} & \dots \ \frac{\partial x_M}{\partial b_M} \end{array} \right] \quad ()$$

CRB
k

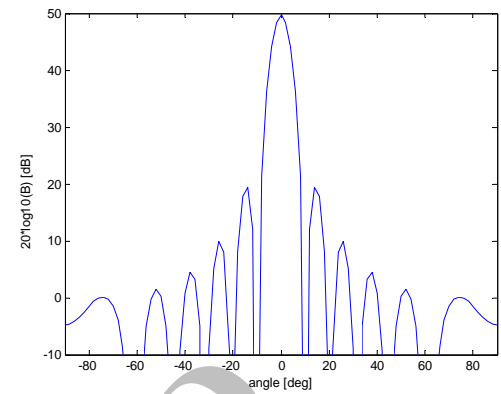
$$CRB(\theta_k) = \frac{6\sigma^2}{NK(K^2-1)|b_k|^2} \cdot \frac{1}{\pi^2 \cos^2 \theta_k} \quad ()$$

$$CRB(\omega_k) = \frac{6\sigma^2}{NK(N^2-1)|b_k|^2} \quad ()$$

K N
 θ_k σ^2

k b_k k
() ()

() ()



ULA :

Cramer-Rao

CRB

[] []

CRB

$$\underline{\eta} = [\omega_1, \dots, \omega_M, \tilde{b}_1, \dots, \tilde{b}_M, \bar{b}_1, \dots, \bar{b}_M, \theta_1, \dots, \theta_M]^T \quad ()$$

θ_i b_i ω_i

$$\bar{b}_i = \text{Im}[b_i] \quad \tilde{b}_i = \text{Re}[b_i] \quad i$$

:

$$E[(\hat{\underline{\eta}} - \underline{\eta}_0)(\hat{\underline{\eta}} - \underline{\eta}_0)^T] \geq G^{-1} \quad ()$$

$$\underline{\eta} \quad \underline{\eta}_0 \quad \hat{\underline{\eta}}$$

() G

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{12}^T & G_{22} & G_{23} & G_{24} \\ G_{13}^T & G_{23}^T & G_{33} & G_{34} \\ G_{14}^T & G_{24}^T & G_{34}^T & G_{44} \end{bmatrix} \quad ()$$

$$G_{mn} = \frac{2}{\sigma^2} \text{Re}[X_m^H X_n A^H A]; \quad m, n = 1, 2, 3 \quad ()$$

$$G_{m4} = \frac{2}{\sigma^2} \text{Re}[X_m^H X A^H D]; \quad m = 1, 2, 3 \quad ()$$

$$1/D_n = \lambda/D \quad ()$$

D

DOA

CRB

CRB

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$$d \leq \lambda/2$$

λ

DOA

(XOR)

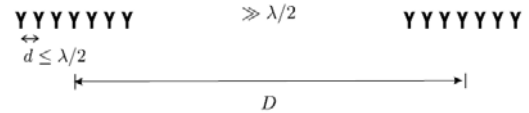
CRB

B_c



B_I

CRB



CRB

B_c

[]

$$B_c = \frac{1}{2NS^2 \underline{d}^H P_a^{-1} \underline{d} a^H R^{-1} a}$$

()

N

S

$$\underline{d} = \frac{\partial a(\theta)}{\partial u}$$

()

$$P_a^{-1} = I - \underline{a} \underline{a}^H / \underline{a}^H \underline{a} = I - \frac{1}{2K} \underline{a} \underline{a}^H$$

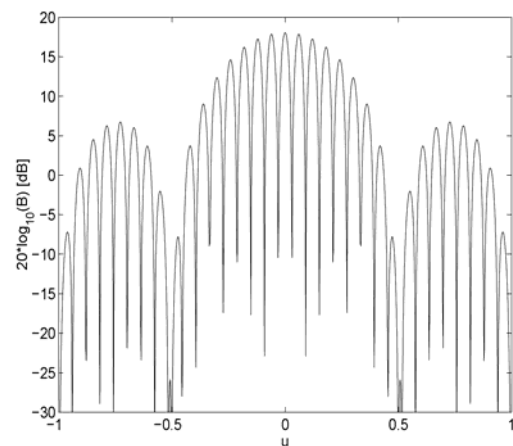
()

$$\underline{a} = \begin{bmatrix} a_1(\theta) \\ e^{-j2\pi D_n u} a_1(\theta) \end{bmatrix}$$

()

$$R = S \underline{a}(u) \underline{a}^H(u) + I$$

()



$$u = \sin \theta$$

()

$$B_I = \frac{1}{4NS^2 \underline{d}_1^H P_{a_1}^{-1} \underline{d}_1 a_1^H R^{-1} a_1} \quad ()$$

$$\underline{d}_1 = \frac{\partial a_1(\theta)}{\partial u} = -j2\pi b_1 \Theta a_1 \quad ()$$

$$\underline{b}_1 = [0 \quad d_n \quad \dots \quad (K-1)d_n] \quad ()$$

$$\underline{a}_1(\theta) = \begin{bmatrix} 1 & e^{-j2\pi \frac{d}{\lambda} \sin \theta} & \dots & e^{-j2\pi(K-1) \frac{d}{\lambda} \sin \theta} \end{bmatrix}^T \quad ()$$

$$P_{a_1}^{-1} = I - \underline{a}_1 \underline{a}_1^H / \underline{a}_1^H \underline{a}_1 = I - \frac{1}{2K} \underline{a}_1 \underline{a}_1^H \quad ()$$

: CRB

$$\begin{aligned} \underline{d}_1^H P_{a_1}^{-1} \underline{d}_1 &= 4\pi^2 \sum_{k=1}^K b_{1k}^2 - \frac{2\pi^2}{K} \left(\sum_{k=1}^K b_{1k} \right)^2 \\ &= \frac{\pi^2 d_n^2 K(K^2 - 1)}{3} \end{aligned} \quad ()$$

$$\underline{a}_1^H R^{-1} \underline{a}_1 = \frac{K}{1 + KS} \quad ()$$

$$B_I = \frac{3(SK + 1)}{4\pi^2 d_n^2 NS^2 K^2 (K^2 - 1)} \quad ()$$

$$() \quad K^2 \gg 1 \quad 2KS \gg 1$$

$$B_I \approx \frac{3}{4\pi^2 d_n^2 NSK^3} \quad ()$$

CRB

$$d_n \quad S \quad K \quad N \quad () \quad ()$$

CRB

$$1 + 3D_n^2 / d_n^2 K^2 \quad ()$$

CRB

$$3D_n^2 / d_n^2 K^2 \gg 1 \quad ()$$

$$3D_n^2 / d_n^2 K^2 \quad () \quad S$$

$$\begin{aligned} & \cdot \quad 2K \times 2K \quad I \quad u = \sin \theta \\ & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \underline{b} \\ \underline{b} & \stackrel{\Delta}{=} [0 \quad d_n \quad \dots \quad (K-1)d_n \quad D_n \quad D_n + d_n \quad \dots \quad D_n + (K-1)d_n] \end{aligned} \quad ()$$

$$\underline{d} \quad d_n = d / \lambda \quad ()$$

$$\underline{d} = -j2\pi \underline{b} \Theta \underline{a} \quad ()$$

$$\underline{a} = [a_1 \quad a_2 \dots a_n] ; \underline{b} = [b_1 \quad b_2 \dots b_n] ; \underline{c} = [c_1 \quad c_2 \dots c_n] \quad ()$$

$$\underline{c} = \underline{b} \Theta \underline{a} ; c_i = b_i a_i \quad ()$$

$$\underline{d}^H P_a^{-1} \underline{d} = 4\pi^2 (\underline{b}^T \Theta \underline{a}^H) (\underline{b} \Theta \underline{a}) \quad ()$$

$$= 2 \frac{\pi^2}{K} (\underline{b}^T \Theta \underline{a}^H) \underline{a} \underline{a}^H (\underline{b} \Theta \underline{a})$$

$$= 4\pi^2 \sum_{k=1}^{2K} b_k^2 - \frac{2\pi^2}{K} \left(\sum_{k=1}^{2K} b_k \right)^2$$

$$= \frac{2\pi^2}{3} K (d_n^2 (K-1) + 3D_n^2)$$

$$R^{-1} = I - \frac{S}{1 + 2SK} \underline{a} \underline{a}^H \quad ()$$

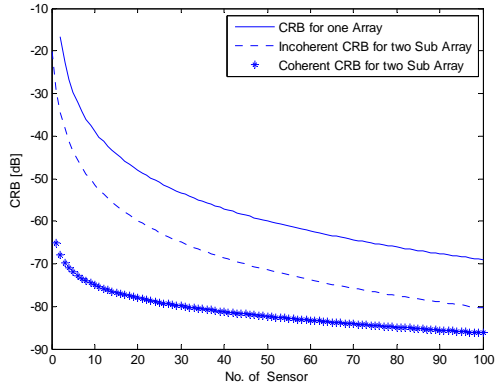
$$\underline{a}^H R^{-1} \underline{a} = 2K - \frac{S}{1 + 2SK} \underline{a}^H \underline{a} \underline{a}^H \underline{a} = \frac{2K}{1 + 2SK} \quad ()$$

$$B_C = \frac{3(2SK + 1)}{8\pi^2 NS^2 K^2 (d_n^2 (K^2 - 1) + 3D_n^2)} \quad ()$$

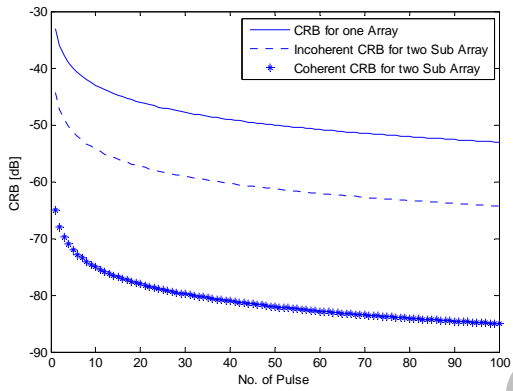
$$() \quad K^2 \gg 1 \quad 2KS \gg 1$$

$$B_C \approx \frac{3}{4\pi^2 NKS (d_n^2 K^2 + 3D_n^2)} \quad ()$$

$$\text{CRB} \quad D_n \quad d_n \quad S \quad K \quad N$$



$S = 2, d_n = 0.5, D_n = 50, \sigma = 1, N = 16$



$S = 2, d_n = 0.5, D_n = 50, \sigma = 1, K = 16$

() ()

CRB
()

SNR

CRB

() ()

CRB () λ

$$3D_n^2 / d_n^2 K^2$$

CRB

CRB

CRB

$D_n = 50, d_n = 0.5, S = 2$

$\sigma = 1$

$\pi/9$

θ_k
()

CRB

CRB

CRB

()

SNR

CRB

CRB

CRB

()

CRB

CRB

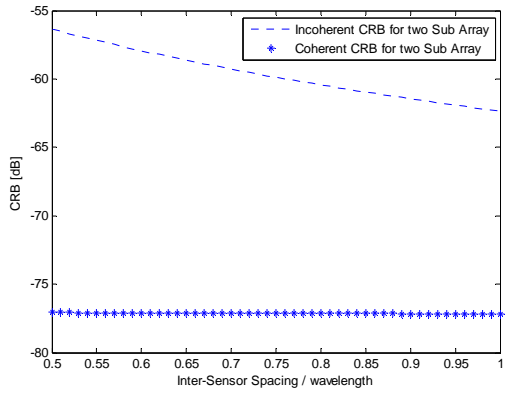
()

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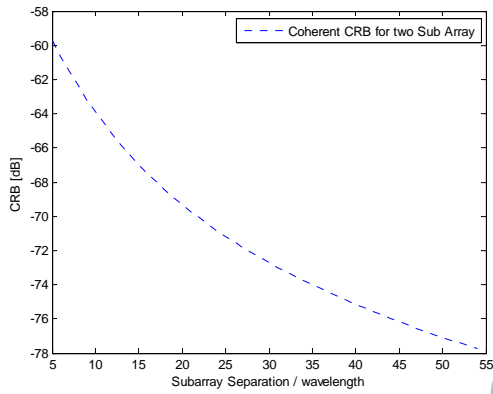
CRB

CRB ()



:

$S = 2, d_n = 0.5, D_n = 50, \sigma = 1, N = 16, K = 16$



$S = 2, d_n = 0.5, D_n = 50, \sigma = 1, N = 16, K = 16$

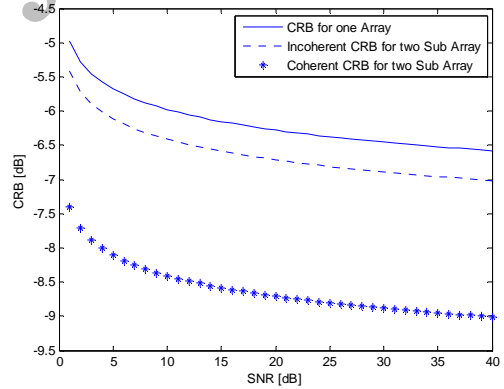
() ()
CRB CRB

() ()

()

CRB

() () ()



:

SNR

$d_n = 0.5, D_n = 50, \sigma = 1, N = 16, K = 16$

() ()

λ

$D \gg \lambda/2$

() ()

DOA

$$\bar{\Delta} \quad \bar{\Phi}$$

$$\bar{\Phi} = \lim_{K \rightarrow \infty} \frac{1}{K} \Phi = \begin{bmatrix} \bar{G}_{11} & \bar{G}_{12} & \bar{G}_{13} \\ \bar{G}_{12}^T & \bar{G}_{22} & \bar{G}_{23} \\ \bar{G}_{13}^T & \bar{G}_{23}^T & \bar{G}_{33} \end{bmatrix} \quad (-)$$

SNR

$$\bar{\Delta} = \lim_{K \rightarrow \infty} \frac{1}{K^2} \Delta = \begin{bmatrix} \bar{G}_{14} \\ \bar{G}_{24} \\ \bar{G}_{34} \end{bmatrix} \quad (-)$$

()

$$\bar{G}_{mn} = \lim_{K \rightarrow \infty} \frac{1}{K} G_{mn}; \quad m, n = 1, 2, 3 \quad (-)$$

G

()

$$\bar{G}_{m4} = \lim_{K \rightarrow \infty} \frac{1}{K^2} G_{m4}; \quad m = 1, 2, 3 \quad (-)$$

G_{mn}

N

$K \rightarrow \infty$

(-)

K

G

$$\overline{CRB}(\theta) = \left(\bar{G}_{44} - \bar{\Delta}^T \bar{\Phi}^{-1} \bar{\Delta} \right)^{-1} \quad (-)$$

$K \rightarrow \infty$

:

\overline{CRB}

$$G = \begin{bmatrix} \Phi & \Delta \\ \Delta^T & G_{44} \end{bmatrix} \quad (-)$$

(-)

$$\begin{aligned} [\bar{\Delta}^T \bar{\Phi}^{-1} \bar{\Delta}]_{kk} &= g_{14}(g_{14}\psi_{11} + g_{24}\psi_{12} + g_{34}\psi_{13}) \\ &+ g_{24}(g_{14}\psi_{12} + g_{24}\psi_{22} + g_{34}\psi_{23}) \\ &+ g_{34}(g_{14}\psi_{13} + g_{24}\psi_{23} + g_{34}\psi_{33}) \end{aligned} \quad (-)$$

$$\Delta = \begin{bmatrix} G_{14} \\ G_{24} \\ G_{34} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{12}^T & G_{22} & G_{23} \\ G_{13}^T & G_{23}^T & G_{33} \end{bmatrix} \quad (-)$$

$$\psi_{11} = \frac{1}{g_{11} - \frac{g_{12}^2}{g_{22}} - \frac{g_{13}^2}{g_{33}}} \quad (-)$$

DOA

CRB

$$\psi_{12} = -\frac{1}{g_{22} - g_{12} - \frac{g_{22}g_{13}^2}{g_{12}g_{33}}} \quad (-)$$

$$CRB(\theta) = (G_{44} - \Delta^T \Phi^{-1} \Delta)^{-1} \quad (-)$$

$K \rightarrow \infty$

G

$$\psi_{13} = -\frac{1}{\frac{g_{33}g_{11}}{g_{13}} - \frac{g_{33}g_{12}}{g_{13}g_{22}} - g_{13}} \quad (-)$$

G_{mn}

()

K

G_{44}

Φ

$$\psi_{22} = \frac{1}{g_{22}} + \frac{1}{\frac{g_{22}g_{11}}{g_{12}^2} - g_{22} - \frac{g_{22}g_{13}^2}{g_{12}^2g_{33}}} \quad (-)$$

K^3

Δ

G_{m4}

K

$$\psi_{23} = \frac{1}{\frac{g_{11}g_{22}g_{33}}{g_{12}g_{13}} - \frac{g_{12}g_{33}}{g_{13}} - \frac{g_{22}g_{13}}{g_{12}}} \quad (-)$$

K^2

K^2

$$\psi_{33} = \frac{1}{g_{33}} + \frac{1}{\frac{g_{33}g_{11}}{g_{13}^2} - \frac{g_{12}^2g_{33}}{g_{13}^2g_{22}} - g_{33}} \quad (-)$$

$$\overline{CRB}(\theta) \stackrel{\Delta}{=} \lim_{K \rightarrow \infty} K^3 \cdot CRB(\theta) = \quad (-)$$

$$\lim_{K \rightarrow \infty} \left(\frac{1}{K^3} G_{44} - \frac{1}{K^2} \Delta^T \left(\frac{1}{K} \Phi \right)^{-1} \frac{1}{K^2} \Delta \right)^{-1}$$

$$\Gamma = \begin{bmatrix} G_{22} & G_{23} & G_{24} \\ G_{23}^T & G_{33} & G_{34} \\ G_{24}^T & G_{34}^T & G_{44} \end{bmatrix} \quad (-)$$

$$\Lambda = \begin{bmatrix} G_{12} & G_{13} & G_{14} \end{bmatrix}$$

DOA CRB

$$CRB(\theta) = (\Gamma - \Lambda^T G_{11}^{-1} \Lambda)^{-1} \quad (-)$$

$N \rightarrow \infty$ G

K G_{mn}

N^3 G_{11} N^2 Λ N Γ

$$\overline{CRB}(\theta) = \lim_{N \rightarrow \infty} \frac{1}{N} N.CRB(\theta) = \quad (-)$$

$$\lim_{N \rightarrow \infty} \left(\frac{1}{N} \Gamma - \frac{1}{N^2} \Lambda^T \left(\frac{1}{N^3} G_{11} \right)^{-1} \frac{1}{N^2} \Lambda \right)^{-1}$$

: $\overline{\Lambda}$ $\overline{\Gamma}$

$$\overline{\Gamma} = \lim_{N \rightarrow \infty} \frac{1}{N} \Gamma = \begin{bmatrix} \overline{G}_{22} & \overline{G}_{23} & \overline{G}_{24} \\ \overline{G}_{23}^T & \overline{G}_{33} & \overline{G}_{34} \\ \overline{G}_{24}^T & \overline{G}_{34}^T & \overline{G}_{44} \end{bmatrix} \quad (-)$$

$$\overline{\Lambda} = \lim_{N \rightarrow \infty} \frac{1}{N^2} \Lambda = \begin{bmatrix} \overline{G}_{12} & \overline{G}_{13} & \overline{G}_{14} \end{bmatrix} \quad (-)$$

$$\overline{G}_{mn} = \lim_{N \rightarrow \infty} \frac{1}{N} G_{mn}; \quad m, n = 1, 2, 3 \quad (-)$$

$$\overline{G}_{1n} = \lim_{N \rightarrow \infty} \frac{1}{N^2} G_{1n}; \quad m = 1, 2, 3 \quad (-)$$

$$\overline{G}_{11} = \lim_{N \rightarrow \infty} \frac{1}{N^3} G_{11}; \quad m = 1, 2, 3 \quad (-)$$

: (-)

$$\overline{CRB}(\theta) = \left(\overline{\Gamma} - \overline{\Lambda}^T \overline{G}_{11}^{-1} \overline{\Lambda} \right)^{-1} \quad (-)$$

$\overline{\Gamma}^{-1}$ \overline{CRB}

: []

(-)

$$\left[\overline{\Lambda}^{-1} \overline{\Lambda}^T \right]_{kk} = \frac{g_{14}^2 g_{22} g_{33} - 2g_{12} g_{14} g_{24} g_{33} + 2g_{13} g_{34} (g_{12} g_{24} - g_{14} g_{22})}{g_{22} (g_{33} g_{44} - g_{34}^2) - g_{24}^2 g_{33}} + \frac{g_{13}^2 (g_{22} g_{44} - g_{24}^2) + g_{12}^2 (g_{33} g_{44} - g_{34}^2)}{g_{22} (g_{33} g_{44} - g_{34}^2) - g_{24}^2 g_{33}}$$

$$\left[\overline{CRB}(\theta) \right]_{kk} = \frac{\sigma^2}{2|b_k|^2 \|P_{a_k}^\perp d_k\|^2} \quad (-)$$

$$g_{11}^K = \frac{N(2N+1)(N+1)}{3\sigma^2} |b_K|^2 \alpha_K \quad (-)$$

$$g_{12}^K = -\frac{N(N+1)}{\sigma^2} \bar{b}_K \alpha_K \quad (-)$$

$$g_{13}^K = \frac{N(N+1)}{\sigma^2} \tilde{b}_K \alpha_K \quad (-)$$

$$g_{22}^K = \frac{2N}{\sigma^2} \alpha_K \quad (-)$$

$$g_{23}^K = 0 \quad (-)$$

$$g_{33}^K = \frac{2N}{\sigma^2} \alpha_K \quad (-)$$

$$g_{14}^K = \frac{N(N+1)}{\sigma^2} |b_k|^2 \bar{\beta}_k \quad (-)$$

$$g_{24}^k = \frac{2N}{\sigma^2} (\tilde{b}_k \tilde{\beta}_k - \bar{b}_k \bar{\beta}_k) \quad (-)$$

$$g_{34}^k = \frac{2N}{\sigma^2} (\bar{b}_k \tilde{\beta}_k + \tilde{b}_k \bar{\beta}_k) \quad (-)$$

$$g_{44}^k = \frac{2}{\sigma^2} N |b_k|^2 \gamma_k \quad (-)$$

$$\alpha_K = \lim_{K \rightarrow \infty} \frac{1}{K} a^H(\theta_K) a(\theta_K) \quad (-)$$

$$\beta_k = \lim_{K \rightarrow \infty} \frac{1}{K^2} a^H(\theta_K) d(\theta_K) \quad (-)$$

$$\gamma_k = \lim_{K \rightarrow \infty} \frac{1}{K^3} d^H(\theta_K) d(\theta_K) \quad (-)$$

$$\left[\overline{CRB}(\theta) \right]_{kk} = \lim_{K \rightarrow \infty} K^3 \frac{\sigma^2}{2N |b_k|^2 \|P_{a_k}^\perp d_k\|^2} \quad (-)$$

: $K \rightarrow \infty$

$$CRB(\theta_k) = \frac{\sigma^2}{2N |b_k|^2 \|P_{a_k}^\perp d_k\|^2} \quad (-)$$

(-) () ()

. ()

$N \rightarrow \infty$ G

. K

K N

G

:

$$G = \begin{bmatrix} G_{11} & \Lambda \\ \Lambda^T & \Gamma \end{bmatrix} \quad (-)$$

:

$$CRB(\theta_k) = \frac{\sigma^2}{2N|b_k|^2 \|P_{a_k}^\perp d_k\|^2} \quad (-)$$

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- 1 - Direction of Arrival
- 2 - Cramer-Rao Band
- 3 - Unbiased
- 4 - Maximum Likelihood
- 5 - Side Lobes
- 6 - Uniform Linear Array