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CLEANNESS AND SHELLABLITY

Ali Soleyman Jahan
Department of Mathematics University of Kurdistan
416, Sanandaj, Iran
A.solaimanjahan@uok.ac.ir

Abstract. We study some basic property of cleanness. We show that if R is a Notherian ring and M is an almost clean R-module with the property that R/P is Cohen–Macaulay for any $P \in \mathrm{Ass}(M)$, then $\mathrm{depth}(M) = \min\{\dim(R/P) : p \in \mathrm{Ass}(M)\}$. Using this fact show that if M is a clean R-module and all minimal prime ideals of M are Cohen–Macaulay and have equal height, then M is Cohen–Macaulay. We also discuss the relation between cleanness and shalleblity and give a simple proof for a theorem of Dress. Finally we give an easy proof for the well known fact that a pure shellable simplicial complex is Cohen–Macaulay.

1. Introduction

Let R be a Noetherian ring and M an R-module. A chain $\mathcal{F}$: $(0) = M_0 \subset M_1 \subset \ldots \subset M_r = M$ of submodules of M is called a prime filtration of M, if for all $i = 1, \ldots, r$, there exists a prime ideal $P_i \in \mathrm{Spec}(R)$ such that $M_i/M_{i-1} \cong R/P_i$. If M is finitely generated such a prime filtration of M always exists. The set of prime ideals $P_1, \ldots, P_r$ which define the cyclic quotients of $\mathcal{F}$ will be denoted by $\mathrm{Supp}(\mathcal{F})$. It is easy to see that if $\mathcal{F}$ is a prime filtration of M, then $\mathrm{Ass}(M) \subset \mathrm{Supp}(\mathcal{F}) \subset \mathrm{Supp}(M)$.

Dress [3] called the prime filtration $\mathcal{F}$ *clean* if $\mathrm{Supp}(\mathcal{F}) = \mathrm{Min}(M)$. The R-module M is called clean if it has a clean filtration.

Herzog and Popescu [4] generalized this concept and they called a prime filtration $\mathcal{F}$ *pretty clean*, if for all $i < j$ which $P_i \subseteq P_j$ it follows that $P_i = P_j$. The R-module M is called pretty clean if it admits a pretty clean filtration. It follows from [4, Corollary 3.4] that if $\mathcal{F}$ is a pretty clean filtration of M, then $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}(M)$. The converse of the above fact is not true, see [4, Example 4.4]. We call an R-module M *almost clean* if it admits a prime filtration $\mathcal{F}$ with

$$\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}(M).$$

It is easy to see that

$$\text{clean} \Rightarrow \text{pretty clean} \Rightarrow \text{almost clean},$$

and if an R-module M has no embedded associated prime ideals, then

$$M \text{ is clean} \Leftrightarrow M \text{ is pretty clean} \Leftrightarrow M \text{ is almost clean}.$$

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables. Let I be a monomial ideal in S. We say that I is (pretty) clean if S/I is pretty clean. Cleanness is the algebraic counterpart of shellability for simplicial complexes.

A *simplicial complex* Δ over a set of vertices $[n] = \{1, \ldots, n\}$ is a collection of subsets of $[n]$ with the property that $i \in \Delta$ for all $i \in [n]$, and if $F \in \Delta$, then all the subsets of F are also in

Δ (including the empty set). An element of Δ is called a *face* of Δ, and the maximal faces of Δ under inclusion are called *facets*. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. The *dimension* of a face F is defined as $\dim F = |F| - 1$, where $|F|$ is the number of vertices of F. The dimension of the simplicial complex Δ is the maximal dimension of its facets. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension. We denote the simplicial complex Δ with facets $F_1, \ldots, F_t$ by $\Delta = \langle F_1, \ldots, F_t \rangle$.

If Δ is a simplicial complex on vertex set $[n]$, then the Stanley–Reisner ideal, I_Δ, is a squarefree monomial ideal generated by all monomials $x_{i_1} x_{i_2} \cdots x_{i_t}$ such that $\{i_1, i_2, \ldots, i_t\} \notin \Delta$. If $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$, then $I_\Delta = \bigcap_{i=1}^{m} P_{F_i}$, where $P_{F_i} = (x_j : j \notin F_i)$, see [1, Theorem 5.4.1]. We say the simplicial complex Δ is Cohen–Macaulay if S/I_Δ is Cohen–Macaulay ring.

According to Björner and Wachs [2] an order $F_1, \ldots, F_t$ of the facets of Δ is called a (non-pure) shelling of Δ if the simplicial complex $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ is pure and $(\dim F_i - 1)$-dimensional for all $i = 2, \ldots, t$. Given a shelling $F_1, \ldots, F_t$ of Δ. We denote by Δ_i the simplicial complex with facets $F_1, \ldots, F_i$. We follow the notation in [2] and define the restriction of facet F_k by

$$R(F_k) = \{i \in F_k \colon F_k \setminus \{i\} \in \Delta_{k-1}\}.$$

Then $R(F_k) \subset F_k$ is the unique minimal face which is not in Δ_{k-1}, see [2, lemma 2.4]. In an other words

$$\langle F_k \rangle \setminus \Delta_{k-1} = [R(F_k), F_k] = \{B \colon R(F_k) \subset B \subset F_k\}.$$

Therefore the simplicial complex Δ splits up into disjoint union Boolean intervals $\Delta = \bigcup_{i=1}^{t} [R(F_i), F_i]$.

It is easy to see from the definition of Stanley–Reisner ideal the dicution above that

$$I_{\Delta_{k-1}} = (I_{\Delta_k}, u)_{\operatorname{supp}(u) \in (F_k \setminus \Delta_{k-1})} = (I_{\Delta_k}, X_{R(F_k)}), \quad \text{where} \quad X_{R(F_k)} = \prod_{j \in R(F_k)} x_j$$

In Section 1 we give an upper bound for the depth of an R-module M in terms of the minimum of $\dim(R/)$, where $P \in \operatorname{Supp}(\mathcal{F})$ and $\mathcal{F}$ is a prime filtration of M. Then we determine the depth of an almost clean R-module. We also show that if M is a clean R-module and for all $P \in \operatorname{Min}(M)$, R/P is Cohen–Macaulay, then M is Cohen–Macaulay. In Section 2 we give a easy proof for a Theorem of Dress. Then as a corollary we get the well known fact that if Δ is a pure shellable simplicial complex, then Δ is Cohen–Macaulay.

2. Depth of clean modules

Let R be a Notherian ring and M an R-module. First we show the following:

Theorem 2.1. *Let M be an R-module and $\mathcal{F}$ is prime filtration of M. Suppose that R/P is a Cohen–Macaulay ring for all $P \in \operatorname{Supp}(\mathcal{F})$. Then*

$$\operatorname{depth}(M) \geq \min\{\dim(R/P) : \ P \in \operatorname{Supp} \mathcal{F}\}.$$

It is well known that for an R-module M one has $\operatorname{depth}(M) \leq \min\{\dim(R/P) : \ P \in \operatorname{Ass}(M)\}$. If we combine this fact with Theorem 2.1 we get

Corollary 2.2. *Let M be an almost clean R-module with the property that R/P is Cohen–Macaulay for all $P \in \mathrm{Ass}(M)$. Then*

$$\mathrm{depth}(M) = \min\{\dim(R/P) : \ P \in \mathrm{Ass}(M)\}.$$

Corollary 2.3. *Let M be a clean R-module with the property that all prime ideals in $\mathrm{Min}(M)$ have the same height and R/P is Cohen–Macaulay for all $P \in \mathrm{Min}(M)$. Then M is Cohen–Macaulay.*

3. The relation between cleanness and shellablity

The following fact was proved by Dress [3]. Here we present an easy proof for it.

Theorem 3.1. *Let Δ be a simplicial complex with facets $F_1, \dots, F_t$. The following are equivalent:*

(a) *An order $F_1, \dots, F_t$ of facets of Δ is a shelling of Δ ;*

(b) *$\mathcal{F}: I = I_\Delta \subset I_{\Delta_{t-1}} \subset \cdots \subset I_{\Delta_1} \subset I_{\Delta_0} = S$ is a clean filtration of S/I_Δ with $I_{\Delta_{i-1}}/I_{\Delta i} \cong S/P_{F_i}$ for $i = 1, \dots, t$.*

If we combine Theorem 3.1 with Corollary 2.3 we get

Corollary 3.2. *Let Δ be a pure shellable simplicial complex. Then Δ is Cohen–Macaulay.*

References

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