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COFINITENESS OF LOCAL COHOMOLOGY MODULES OF DIMENSION ONE IDEALS IN NON-LOCAL CASE

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ABSTRACT. Let R be a noetherian ring, \mathfrak{a} an ideal of R such that dim $R/\mathfrak{a} = 1$ and M a finite R-module. We will study cofiniteness and some other properties of the local cohomology modules $\mathrm{H}^{i}_{\mathfrak{a}}(M)$.

1. INTRODUCTION

Throughout R is a commutative noetherian ring. By a finite module we mean a finitely generated module. For basic facts about commutative algebra see [2] and [8] and for local cohomology we refer to [1].

Grothendieck [6], made the following.

Conjecture: For every ideal \mathfrak{a} and every finite *R*-module *M*, the module $\operatorname{Hom}_{R}(R/\mathfrak{a}, \operatorname{H}^{n}_{\mathfrak{a}}(M))$ is finite for all *n*.

Hartshorne [7] showed that this is false in general. However, he defined \mathfrak{a} -cofinite modules and he asked the following question.

Question: Suppose that \mathfrak{a} is an ideal of R and M is a finite R-module. When is $\operatorname{Ext}_{R}^{i}(R/\mathfrak{a}, \operatorname{H}_{\mathfrak{a}}^{j}(M))$ finite for every i and j?

Hartshorne [7] showed that if (R, \mathfrak{m}) is a complete regular local ring and M a finite R-module, then $\mathrm{H}^{i}_{\mathfrak{a}}(M)$ is \mathfrak{a} -cofinite in the following two cases.

(a) \mathfrak{a} is a nonzero principal ideal.

(b) \mathfrak{a} is a prime ideal with dim $R/\mathfrak{a} = 1$.

Yoshida [9] and Delfino and Marley [3] extended (b) to all dimension one ideals \mathfrak{a} of an arbitrary local ring R.

An R-module M has finite Goldie dimension if M contains no infinite direct sum of submodules. For a commutative noetherian ring, this can be expressed in two other ways, namely that the injective hull E(M) of M decomposes as a finite direct sum of indecomposable injective modules or that M is an essential extension of a finite submodule.

A module M is weakly Laskerian, whenever for each submodule N of M, the quotient M/N has just finitely many associated primes, see [5]. A module M is \mathfrak{a} -weakly cofinite if $\operatorname{Supp}_R(M) \subset V(\mathfrak{a})$ and $\operatorname{Ext}_R^i(R/\mathfrak{a}, M)$ is weakly Laskerian

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for all *i*. Clearly each \mathfrak{a} -cofinite module is \mathfrak{a} -weakly cofinite, but the converse is not true in general see [4, Example 3.5 (i) and (ii)].

A module M is *soclefree* if it has no simple submodules, or in other words Ass $M \cap \text{Max} R = \emptyset$. For example if M is a module over the local ring (R, \mathfrak{m}) then the module $M/\Gamma_{\mathfrak{m}}(M)$, where $\Gamma_{\mathfrak{m}}(M)$ is the submodule of M consisting of all elements of M annihilated by some high power \mathfrak{m}^n of the maximal ideal \mathfrak{m} , is always soclefree.

In 2.3, we give a characterization of \mathfrak{a} -cofiniteness of local cohomology modules with case where \mathfrak{a} is a one-dimensional ideal in a non-local ring. In this situation we also prove in 2.7, that the local cohomology modules always belong to a class introduced by Zöschinger in [10].

2. Main results

Proposition 2.1. Let M be a module over the noetherian ring R. The following statements are equivalent.

- (i) M is a finite R-module.
- (ii) $M_{\mathfrak{m}}$ is a finite $R_{\mathfrak{m}}$ -module for all $\mathfrak{m} \in Max R$ and $Min_R(M/N)$ is a finite set for all finite submodules $N \subset M$.

Corollary 2.2. Let M be an R-module such that $Supp M \subset V(\mathfrak{a})$ and $M_{\mathfrak{m}}$ is $\mathfrak{a}R_{\mathfrak{m}}$ -cofinite for each maximal ideal \mathfrak{m} . The following statements are equivalent.

- (i) M is \mathfrak{a} -cofinite.
- (ii) For all j, Min_R(Ext^j_R(R/a, M)/T) is a finite set for each finite submodule T of Ext^j_R(R/a, M).

Corollary 2.3. Let \mathfrak{a} an ideal of R such that $\dim R/\mathfrak{a} = 1$, M a finite R-module and $i \geq 0$. The following statements are equivalent:

- (i) $H^i_{\mathfrak{a}}(M)$ is \mathfrak{a} -cofinite.
- (ii) For all j, Min_R(Ext^j_R(R/a, Hⁱ_a(M))/T) is a finite set for each finite submodule T of Ext^j_R(R/a, Hⁱ_a(M)).

Corollary 2.4. If $H^i_{\mathfrak{a}}(M)$ (with $\dim R/\mathfrak{a} = 1$) is an \mathfrak{a} -weakly cofinite module, then it is also \mathfrak{a} -cofinite.

Next we will introduce a subcategory of the category of R-modules that has been studied by Zöschinger in [10, Satz 1.6].

Theorem 2.5. (*Zöschinger*) For any *R*-module *M* the following statements are equivalent.

- M satisfies the minimal condition for submodules N such that M/N is soclefree.
- (ii) For any descending chain $N_1 \supset N_2 \supset N_3 \supset \ldots$ of submodules of M, there is n such that the quotients N_i/N_{i+1} have support in Max R for all $i \ge n$.
- (iii) With $L(M) = \bigoplus_{\mathfrak{m} \in MaxR} \Gamma_{\mathfrak{m}}(M)$, the module M/L(M) has finite Goldie

dimension, and $\dim R/\mathfrak{p} \leq 1$ for all $\mathfrak{p} \in Ass_R(M)$.

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If they are fulfilled, then for each monomorphism $f: M \longrightarrow M$, $Supp_B(Coker f) \subset Max R.$

We will say that M is in the class \mathcal{Z} if M satisfies the equivalent conditions in 2.5.

Proposition 2.6. The class Z is a Serre subcategory of the category of R-modules, that is Z is closed under taking submodules, quotients and extensions.

Theorem 2.7. Let N be a module over a noetherian ring R and a an ideal of R such that dim $R/\mathfrak{a} = 1$. If $N_\mathfrak{m}$ is $\mathfrak{a}R_\mathfrak{m}$ -cofinite for all $\mathfrak{m} \in Max R$, then N is in the class \mathcal{Z} . In particular, if M is a finite R-module then $H^i_\mathfrak{a}(M)$ is in the class \mathcal{Z} for all i.

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