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# COFINITENESS OF LOCAL COHOMOLOGY MODULES OF DIMENSION ONE IDEALS IN NON-LOCAL CASE

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Abstract. Let $R$ be a noetherian ring, $\mathfrak{a}$ an ideal of $R$ such that $\dim R/\mathfrak{a} = 1$ and $M$ a finite $R$–module. We will study cofiniteness and some other properties of the local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M)$.

## 1. Introduction

Throughout $R$ is a commutative noetherian ring. By a finite module we mean a finitely generated module. For basic facts about commutative algebra see [2] and [8] and for local cohomology we refer to [1].

Grothendieck [6], made the following.

**Conjecture:** For every ideal $\mathfrak{a}$ and every finite $R$–module $M$, the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M))$ is finite for all $n$.

Hartshorne [7] showed that this is false in general. However, he defined $\mathfrak{a}$–*cofinite* modules and he asked the following question.

**Question**: Suppose that $\mathfrak{a}$ is an ideal of $R$ and $M$ is a finite $R$–module. When is $\mathrm{Ext}^i_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ finite for every $i$ and $j$ ?

Hartshorne [7] showed that if $(R, \mathfrak{m})$ is a complete regular local ring and $M$ a finite $R$–module, then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$–cofinite in the following two cases.

(a) $\mathfrak{a}$ is a nonzero principal ideal.

(b) $\mathfrak{a}$ is a prime ideal with $\dim R/\mathfrak{a} = 1$.

Yoshida [9] and Delfino and Marley [3] extended (b) to all dimension one ideals $\mathfrak{a}$ of an arbitrary local ring $R$.

An $R$–module $M$ has *finite Goldie dimension* if $M$ contains no infinite direct sum of submodules. For a commutative noetherian ring, this can be expressed in two other ways, namely that the injective hull $\mathrm{E}(M)$ of $M$ decomposes as a finite direct sum of indecomposable injective modules or that $M$ is an essential extension of a finite submodule.

A module $M$ is *weakly Laskerian*, whenever for each submodule $N$ of $M$, the quotient $M/N$ has just finitely many associated primes, see [5]. A module $M$ is $\mathfrak{a}$–*weakly cofinite* if $\mathrm{Supp}_R(M) \subset \mathrm{V}(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian

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for all $i$. Clearly each $\mathfrak{a}$–cofinite module is $\mathfrak{a}$–weakly cofinite, but the converse is not true in general see [4, Example 3.5 (i) and (ii)].

A module M is *soclefree* if it has no simple submodules, or in other words $\operatorname{Ass} M \cap \operatorname{Max} R = \varnothing$. For example if $M$ is a module over the local ring $(R, \mathfrak{m})$ then the module $M/\Gamma_{\mathfrak{m}}(M)$, where $\Gamma_{\mathfrak{m}}(M)$ is the submodule of $M$ consisting of all elements of $M$ annihilated by some high power $\mathfrak{m}^n$ of the maximal ideal $\mathfrak{m}$, is always soclefree.

In 2.3, we give a characterization of $\mathfrak{a}$–cofiniteness of local cohomology modules with case where $\mathfrak{a}$ is a one-dimensional ideal in a non-local ring. In this situation we also prove in 2.7, that the local cohohomology modules always belong to a class introduced by Zöschinger in [10].

## 2. Main results

**Proposition 2.1.** *Let $M$ be a module over the noetherian ring $R$. The following statements are equivalent.*

(i) *$M$ is a finite $R$–module.*
(ii) *$M_{\mathfrak{m}}$ is a finite $R_{\mathfrak{m}}$–module for all $\mathfrak{m} \in Max\, R$ and $Min_R(M/N)$ is a finite set for all finite submodules $N \subset M$.*

**Corollary 2.2.** *Let $M$ be an $R$–module such that $Supp\, M \subset V(\mathfrak{a})$ and $M_{\mathfrak{m}}$ is $\mathfrak{a}R_{\mathfrak{m}}$–cofinite for each maximal ideal $\mathfrak{m}$. The following statements are equivalent.*

(i) *$M$ is $\mathfrak{a}$–cofinite.*
(ii) *For all $j$, $Min_R(Ext^j_R(R/\mathfrak{a}, M)/T)$ is a finite set for each finite submodule $T$ of $Ext^j_R(R/\mathfrak{a}, M)$.*

**Corollary 2.3.** *Let $\mathfrak{a}$ an ideal of $R$ such that $dim\, R/\mathfrak{a} = 1$, $M$ a finite $R$–module and $i \geq 0$. The following statements are equivalent:*

(i) *$H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$–cofinite.*
(ii) *For all $j$, $Min_R(Ext^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))/T)$ is a finite set for each finite submodule $T$ of $Ext^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$.*

**Corollary 2.4.** *If $H^i_{\mathfrak{a}}(M)$ (with $dim\, R/\mathfrak{a} = 1$) is an $\mathfrak{a}$–weakly cofinite module, then it is also $\mathfrak{a}$–cofinite.*

Next we will introduce a subcategory of the category of $R$–modules that has been studied by Zöschinger in [10, Satz 1.6].

**Theorem 2.5.** (***Zöschinger***) *For any $R$–module $M$ the following statements are equivalent.*

(i) *$M$ satisfies the minimal condition for submodules $N$ such that $M/N$ is soclefree.*
(ii) *For any descending chain $N_1 \supset N_2 \supset N_3 \supset \dots$ of submodules of $M$, there is $n$ such that the quotients $N_i/N_{i+1}$ have support in $Max\, R$ for all $i \geq n$.*
(iii) *With $L(M) = \bigoplus_{\mathfrak{m} \in Max\, R} \Gamma_{\mathfrak{m}}(M)$, the module $M/L(M)$ has finite Goldie dimension, and $dim\, R/\mathfrak{p} \leq 1$ for all $\mathfrak{p} \in Ass_R(M)$.*

*If they are fulfilled, then for each monomorphism* $f : M \longrightarrow M$,

$$\mathit{Supp}_R(\mathit{Coker}\, f) \subset \mathit{Max}\, R.$$

We will say that $M$ is in the class $\mathcal{Z}$ if $M$ satisfies the equivalent conditions in 2.5.

**Proposition 2.6.** *The class* $\mathcal{Z}$ *is a Serre subcategory of the category of* $R$–*modules, that is* $\mathcal{Z}$ *is closed under taking submodules, quotients and extensions.*

**Theorem 2.7.** *Let* $N$ *be a module over a noetherian ring* $R$ *and* $\mathfrak{a}$ *an ideal of* $R$ *such that* $\dim R/\mathfrak{a} = 1$. *If* $N_{\mathfrak{m}}$ *is* $\mathfrak{a}R_{\mathfrak{m}}$–*cofinite for all* $\mathfrak{m} \in \mathit{Max}\, R$, *then* $N$ *is in the class* $\mathcal{Z}$. *In particular, if* $M$ *is a finite* $R$–*module then* $H^i_{\mathfrak{a}}(M)$ *is in the class* $\mathcal{Z}$ *for all* $i$.

## References


[1] M.P. Brodmann, R.Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge University Press, 1998.

[2] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, revised ed., 1998.

[3] D. Delfino and T. Marley, *Cofinite modules and local cohomology*, J. Pure Appl. Alg. **121**(1997), 45–52.

[4] K. Divaani-Aazar, A. Mafi *Associated primes of local cohomology modules of weakly Laskerian modules* Comm. Algebra **34**(2006), 681–690.

[5] K. Divaani-Aazar, A. Mafi, *Associated primes of local cohomology modules*, Proc. Amer. Math. Soc. **133** (2005), 655–660.

[6] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, North-Holland, Amsterdam, 1968.

[7] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9** (1970), 145–164.

[8] H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.

[9] K. I. Yoshida, *Cofiniteness of local cohomology modules for ideals of dimension one*, Nagoya Math. J. **147**(1997), 179–191.

[10] H. Zöschinger, *Minimax Moduln*, J. Algebra. **102**(1986), 1–32.