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ON THE COMMUTATIVITY DEGREE IN ALGEBRAIC STRUCTURES

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ABSTRACT. The commutativity degree of groups and rings has been studied by certain authors since 1973, and the main result obtained is $Pr(A) \leq \frac{5}{8}$, where Pr(A) is the commutativity degree of a non-abelian group (or ring) A. Verifying this inequality for an arbitrary semigroup A is a natural question and in this paper by presenting an infinite class of finite noncommutative semigroups we prove that the commutativity degree may be arbitrarily close to 1. We name this class of semigroups the almost commutative or approximately abelian semigroups.

1. INTRODUCTION

For a given finite algebraic structure A, the *commutativity degree* of A, denoted by Pr(A), is defined as the probability of choosing a pair (x, y) of the elements of A such that x commutes with y. So,

$$Pr(A) = \frac{|\{(x,y) \in A^2 \mid xy = yx\}|}{|A^2|} = \frac{\sum_{x \in A} |C_A(x)|}{|A^2|},$$

where $C_A(x)$ is the centralizer of x in A. For a finite group A it is proved that $Pr(A) = \frac{k(A)}{|A|}$, where k(A) is the number of conjugacy classes of A (see [2, 4, 3, 1, 5], for example). The computational results on Pr(A) are mainly due to Gustafson [2] who shows that $Pr(A) \leq \frac{5}{8}$ for a finite non-abelian group A, and MacHale [4] who proves this inequality for a finite non-abelian ring. The groups studied by Lescot [3] mainly satisfy $\frac{1}{2} \leq Pr(A) \leq \frac{5}{8}$ and the recently obtained results of Doostie [1] concern the groups with the property $d(A) < \frac{1}{2}$, where in that paper d(A) is used instead of Pr(A).

For a finite non-abelian semigroup A, is $Pr(A) \leq \frac{5}{8}$? This is a natural question and by considering the presentations

$$\pi_1 = \langle a, b \mid a^m = b^n, aba^l b^k = 1 \rangle,$$

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$$\pi_2 = \langle a, b \mid a^m = b^n, a^2 b a^l b^k = a \rangle,$$

and

$$\pi_3 = \langle a, b \mid a^m = b^n, a^2 b a^l b^{k+1} = a b \rangle,$$

of the groups, monoids and or, semigroups, where m, n, l and k are any positive integers, we show that $\frac{5}{8}$ is not an upper bound for Pr(A), by providing a class of finite and non-abelian semigroups. Indeed, we show that Pr(A) is arbitrarily close to 1 and we name these kinds of semigroups the *almost commutative* or *approximately abelian* semigroups.

We recall the notion of a presentation $\langle A \mid R \rangle$ of a semigroup. For an alphabet A let A^+ be the free semigroup over A. For a subset R of $A^+ \times A^+$, let ρ be a congruence relation generated by R, then the semigroup $S = A^+/\rho$ will be denoted by $\langle A \mid R \rangle$ which is called a semigroup presentation for S. To lessen the likelihood of confusion, for $\omega_1, \omega_2 \in A^+$ we write $\omega_1 \equiv \omega_2$ if ω_1 and ω_2 are identical words, and $\omega_1 = \omega_2$ if they represent the same element of S (i.e. if $(\omega_1, \omega_2) \in \rho$). Thus, for example, if $A = \{a, b\}$ and R is $\{ab = ba\}$, then $aba = a^2b$ but $aba \not\equiv a^2b$.

To avoid confusion we denote a semigroup presentation by $Sg(\pi)$ and a group presentation by $Gp(\pi)$.

2. Main results

Our main results are:

Theorem 2.1. Let $S = Sg(\pi_2)$ and $G = Gp(\pi_2)$. If G is abelian then so is S and Pr(S) = 1. If G is non-abelian and finite then S is also non-abelian and finite. Moreover,

$$Pr(S) = \frac{|G|^2 \cdot Pr(G) + 2(n-1) \cdot |\langle b \rangle| + (n-1)^2}{|S|^2}.$$

Theorem 2.2. Let $S = Sg(\pi_3)$. If S is finite and the minimal two-sided ideal of S is abelian then,

$$Pr(S) = \frac{m^2 + n^2 + 4mn - mn(m+n) - 3(m+n) + 2 + |S|^2}{|S|^2}.$$

Moreover, the semigroup S is never abelian and for all positive integers m, n, land k if $m \mid l$ and $n \mid k$, then S is finite and $Pr(S) > \frac{5}{8}$.

As a corollary of Theorem 2.2 we conclude that, for all fixed values of m and n,

$$\lim_{k \to \infty} \Pr(S) = 1.$$

For this reason we call the family of semigroups $S = Sg(\pi_3)$ (where the minimal two-sided ideal is abelian) almost commutative or approximately abelian.

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