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COMMUTATIVITY PRESERVING MAPS ON MATRICES OVER DIVISION RINGS

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Abstract. Let $\mathbf{D}$ be an arbitrary division ring. We denote by $\mathbf{M}_n(\mathbf{D})$ and $\mathbf{P}_n(\mathbf{D})$ the set of all $n \times n$ matrices and the set of all $n \times n$ idempotent matrices over $\mathbf{D}$ respectively. In this talk we discuss about characterizations of commutativity preserving linear maps on $\mathbf{M}_n(\mathbf{D})$ and commutativity preserving (not necessarily linear) maps on $\mathbf{P}_n(\mathbf{D})$.

1. Introduction

Let $\mathcal{A} \in \{\mathbf{M}_n(\mathbf{D}), \mathbf{P}_n(\mathbf{D})\}$. It is said that a map $\varphi : \mathcal{A} \to \mathcal{A}$ preserves commutativity (respectively preserves commutativity in both direction) if $\varphi(X)\varphi(Y) = \varphi(Y)\varphi(X)$ whenever $XY = YX$ (respectively $\varphi(X)\varphi(Y) = \varphi(Y)\varphi(X) \iff XY = YX$). The study of symmetry transformations in quantum mechanics is closely related to the problem of characterizing maps on idempotents or projections preserving orthogonality [3]. In [4] it was shown that in the infinite-dimensional case the problem of characterizing automorphisms of the poset of idempotents is equivalent to the problem of characterizing orthogonality preserving bijective maps on idempotent operators. The solutions of both problems follow from a more general result on bijective maps on idempotents preserving commutativity in both directions.

Example 1.1. Let $\mathcal{A}$ be an algebra with identity $\mathbf{1}$ over a ring $\mathbf{R}$. A map of the below form

$$\theta(x) = \lambda\psi(x) + \tau(x)\mathbf{1} \ , \ \forall \ x \in \mathcal{A}$$

is a commutativity preserving map on $\mathcal{A}$ where $\lambda \in \mathbf{R}$, ψ either is an automorphism or an antiautomorphism of $\mathcal{A}$ and τ is a linear functional on $\mathcal{A}$. We will say that such map is a standard commutativity preserving map.

2. Commutativity preserving linear maps on $\mathbf{M}_n(\mathbf{D})$

In this section we state the relationships between the set of central simple Algebras and the set of matrices over a division ring. Then we use some phenomenon about linear preservers of commutativity on central simple Algebra to find a characterization of commutativity preserving maps on $\mathbf{M}_n(\mathbf{D})$. We recall the following concepts.

Definition 2.1. Let $\mathcal{A}$ be a nonzero algebra with identity over a field $\mathbb{F}$.
(i) $\mathcal{A}$ is simple if it has no nontrivial two-sided ideals.
(ii) the center of $\mathcal{A}$ is $Z(\mathcal{A}) = \{X \in \mathcal{A} : XY = YX, \forall\, Y \in \mathcal{A}\}$.
(iii) $\mathcal{A}$ is central if $Z(\mathcal{A}) = \mathbb{F}$.

Of special interest will be algebras with identity that are **central simple**, i.e., are both central and simple.

Example 2.2. If $\mathbf{D}$ is a division algebra over $\mathbb{F}$, then the ring $\mathbf{M}_n(\mathbf{D})$ is simple for all $n \geq 1$. It is easy to see that in a matrix ring the centre can only contain scalar multiples of the identity matrix, so the ring $\mathbf{M}_n(\mathbf{D})$ is a central simple algebra over $Z(\mathbf{D})$.

A main theorem on simple algebras over a field provides a converse to the above example. In fact a finite-dimensional central simple algebra is isomorphic to a matrix algebra over a division ring that is finite-dimensional over its center. In particular, this result covers the case of matrices over any field as well as the case of quaternionic matrices.

The following Proposition gives an equivalent condition for preserving commutativity maps on $\mathbf{M}_n(\mathbb{F})$, see [1,2].

Proposition 2.3. *Let $\mathbb{F}$ be an algebraically closed field with char$(\mathbb{F}) = 0$, and let φ be a linear map on $\mathcal{A} = \mathbf{M}_n(\mathbb{F})$. Then the following condition are equivalent.*

(i) $\varphi(X)\varphi(Y) = \varphi(Y)\varphi(X)$ *whenever* $XY = YX$.

(ii) $\varphi(X^2)\varphi(X) = \varphi(X)\varphi(X^2)$ *for all* $X \in \mathbf{M}_n$. (1)

Now we state the following Theorem which characterizes commutativity preserving maps on central simple algebras.

Theorem 2.4. *Let $\mathcal{A}$ be a finite-dimensional central simple algebra over a field $\mathbb{F}$ such that $\dim_{\mathbb{F}}\mathcal{A} \neq 4$. If a linear map φ satisfies condition* (1) (*in particular, if φ preserves commutativity*) *, then φ is either a standard commutativity preserving map or its range is commutative.*

Now, for a division ring $\mathbf{D}$ and integer $n \geq 3$, we can characterize linear maps on $\mathbf{M}_n(\mathbf{D})$ which satisfy (1).

Theorem 2.5. *Let $\mathbf{D}$ be a division ring such that char$(\mathbf{D}) = 0$. For $n \geq 3$, if φ is a linear map preserving commutativity on $\mathbf{M}_n(\mathbf{D})$, then either the range of φ is commutative or there exist an invertible $S \in \mathbf{M}_n(\mathbf{D})$, a scalar $\alpha \in Z(\mathbf{D})$ and a linear functional f on $\mathbf{M}_n(\mathbf{D})$ such that:*

$$\varphi(A) = \alpha S^{-1}AS + f(A)I, \forall A \in \mathbf{M}_n(\mathbf{D}) \text{ or } \varphi(A) = \alpha S^{-1}A^tS + f(A)I, \forall A \in \mathbf{M}_n(\mathbf{D}).$$

Example 2.6. For $n \geq 3$, every linear map peserving comutativity on the algera of $n \times n$ matrices over the ring of real quaternions is of the form φ in Theorem 2.5.

3. Commutativity preserving maps on $\mathbf{P}_n(\mathbf{D})$

In this section we give a complete description of maps on $\mathbf{P}_n(\mathbf{D})$ that preserve commutativity. We recall that a matrix $P \in \mathbf{M}_n(\mathbf{D})$ is idempotent if $P^2 = P$.

Example 3.1. Assume that $\varphi : \mathbf{P}_n(\mathbf{D}) \to \mathbf{P}_n(\mathbf{D})$ is any map which sends every idempotent either into itself, or into its orthocomplement, that is, $\varphi(P) \in \{P, I - P\}$ for every $P \in \mathbf{P}_n(\mathbf{D})$. Clearly, every such map preserves commutativity in both directions.

Every such map in Example 3.1 is called an **orthopermutation**. Obviously, an orthopermutation is bijective if and only if for every $P \in \mathbf{P}_n(\mathbf{D})$ we have either $\varphi(P) = P$ and $\varphi(I - P) = I - P$, or $\varphi(P) = I - P$ and $\varphi(I - P) = P$.

The following proposition slightly indicates the converse of the above statement.

Proposition 3.2. *Let $\mathbf{D}$ be any division ring, $n \geq 3$, $\varphi : \mathbf{P}_n(\mathbf{D}) \to \mathbf{P}_n(\mathbf{D})$ a map preserving commutativity in both directions, and $P, Q \in \mathbf{P}_n(\mathbf{D})$ idempotents such that $\varphi(P) = \varphi(Q)$. Then either $P = Q$, or $P = I - Q$.*

Some examples show that besides a preserving property we have to impose some additional conditions on maps $\varphi : \mathbf{P}_n(\mathbf{D}) \to \mathbf{P}_n(\mathbf{D})$ if we want to get nice structural results. The natural choices for these additional conditions are the assumption of injectivity, the assumption of surjectivity, or a stronger preserving (in both directions) property.

If $A \in \mathbf{M}_n(\mathbf{D})$ is any matrix and $\tau : \mathbf{D} \to \mathbf{D}$ an automorphism or an antiautomorphism (bijective additive map satisfying $\tau(xy) = \tau(x)\tau(y), \forall x, y \in \mathbf{D}$), then we denote by A^τ the matrix obtained from A by applying τ entry-wise, i.e. $A^\tau = [\tau(a_{ij})]$.

The injective maps preserving commutativity on $\mathbf{P}_n(\mathbf{D})$ are characterized in the following Theorem, see [5].

Theorem 3.3. *Let $\mathbf{D}$ be any division ring. Assume that $n \geq 3$ and let $\phi : \mathbf{P}_n(\mathbf{D}) \to \mathbf{P}_n(\mathbf{D})$ be an injective map preserving commutativity. Then there exist a nonsingular matrix $T \in \mathbf{M}_n$, a bijective orthopermutation $\eta : \mathbf{P}_n(\mathbf{D}) \to \mathbf{P}_n(\mathbf{D})$, and either a nonzero endomorphism $\tau : \mathbf{D} \to \mathbf{D}$ such that*

$$\phi(A) = T\eta(A)^\tau T^{-1}, \forall A \in \mathbf{P}_n(\mathbf{D})$$

or a nonzero anti-endomorphism $\sigma : \mathbf{D} \to \mathbf{D}$ such that

$$\phi(A) = T[\eta(A)^\sigma]^t T^{-1}, \forall A \in \mathbf{P}_n(\mathbf{D})$$

Let φ be a map preserving commutativity in both directions on an algebra of square matrices, then:

$$\{A : \varphi(A) = 0\} \subseteq Span\{I\}.$$

This together with Theorem 3.3 and Proposition 3.2 yields the characterization of maps on idempotents preserving commutativity in both directions.

Corollary 3.4. *Let $\mathbf{D}$ be any division ring. Assume that $n \geq 3$ and let $\phi : \mathbf{P}_n(\mathbf{D}) \to \mathbf{P}_n(\mathbf{D})$ be a map preserving commutativity in both directions. Then* *there exist a nonsingular matrix $T \in \mathbf{M}_n$, an orthopermutation $\eta : P_n(\mathbf{D}) \to P_n(\mathbf{D})$ and either a nonzero endomorphism $\tau : \mathbf{D} \to \mathbf{D}$ such that*

$$\phi(A) = T\eta(A)^{\tau}T^{-1}, \forall A \in \mathbf{P}_n(\mathbf{D}),$$

or a nonzero anti-endomorphism $\sigma : \mathbf{D} \to \mathbf{D}$ such that

$$\phi(A) = T[\eta(A)^{\sigma}]^{t}T^{-1}, \forall A \in \mathbf{P}_n(\mathbf{D}).$$

References

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