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RELATIONS BETWEEN BLOCKING SETS AND $\mathfrak{C}_n$-GROUPS

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Abstract. A blocking set in $PG(n,q)$ is a set of points that has nonempty intersection with every hyperplane of $PG(n,q)$. A blocking set that contains a line is called trivial. A blocking set is called minimal if none of its proper subsets are blocking sets. A cover $\mathcal{C}$ for a group G is called a $\mathfrak{C}_n$-cover whenever $\mathcal{C}$ is an irredundant maximal core-free n-cover for G and in this case we say that G is a $\mathfrak{C}_n$-group.

In this paper we give relations between blocking sets and $\mathfrak{C}_n$-groups.

1. Introduction

Let G be a group. A set $\mathcal{C}$ of proper subgroups of G is called a cover for G if its set-theoretic union is equal to G. If the size of $\mathcal{C}$ is n, we call $\mathcal{C}$ an n-cover for the group G. A cover $\mathcal{C}$ for a group G is called irredundant if no proper subset of $\mathcal{C}$ is a cover for G. A cover $\mathcal{C}$ for a group G is called core-free if the intersection $D = \bigcap_{M\in\mathcal{C}} M$ of $\mathcal{C}$ is core-free in G, i.e. $D_G = \bigcap_{g\in G} g^{-1}Dg$ is the trivial subgroup of G. A cover $\mathcal{C}$ for a group G is called maximal if all the members of $\mathcal{C}$ are maximal subgroups of G. A cover $\mathcal{C}$ for a group G is called a $\mathfrak{C}_n$-cover whenever $\mathcal{C}$ is an irredundant maximal core-free n-cover for G and in this case we say that G is a $\mathfrak{C}_n$-group.

A blocking set in $PG(n,q)$ is a set of points that has nonempty intersection with every hyperplane of $PG(n,q)$. A blocking set that contains a line is called trivial. A blocking set is called minimal if none of its proper subsets are blocking sets. For a blocking set B in $PG(n,q)$ we denote by $d(B)$ the least positive integer d such that B is contained in a d-dimensional subspace of $PG(n,q)$. Thus $d(B)$ is equal to the (projective) dimension of the subspace $\langle B\rangle$ in $PG(n,q)$.

Nontrivial minimal blocking sets in $PG(2,p)$ of size $\frac{3(p+1)}{2}$ exist for all odd primes p. Indeed, an example is given by the projective triangle: the set consisting of the points $(0,1,-s^2)$, $(1,-s^2,0)$, $(-s^2,0,1)$ with $s\in\mathbb{F}_p$.

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For further studies in the topic of blocking sets see Chapter 13 of the second edition of Hirschfeld's book [4] and also see [3] and for further studies in the topic of covering groups by subgroups see [1] and [2].

In this paper we give relations between blocking sets and $\mathfrak{C}_n$-groups.

2. Main results

As we mentioned in the last section, by a blocking set in $PG(n,q)$, we mean a blocking set with respect to hyperplanes in $PG(n,q)$.

Now we give some notations and definitions as needed in the sequel. We denote the product of n copies of $\mathbb{F}_q$ by $(\mathbb{F}_q)^n$. We note that $(\mathbb{F}_q)^n$ is a vector space of dimension n over $\mathbb{F}_q$. If $b=(b_1,\dots,b_n)\in(\mathbb{F}_q)^n$, we denote by M_b the set of elements $x=(x_1,\dots,x_n)\in(\mathbb{F}_q)^n$ such that $b\cdot x=\sum_{i=1}^n b_ix_i$ is equal to zero. Note that if $0\neq b$, then M_b is an $(n-1)$-dimensional subspace of the vector space $(\mathbb{F}_q)^n$ and every $(n-1)$-dimensional subspace of $(\mathbb{F}_q)^n$ equals to M_b for some non-zero $b\in(\mathbb{F}_q)^n$. Since for every $0\neq\lambda\in\mathbb{F}_q$, $M_b=M_{\lambda b}$, $M_{\mathfrak{p}}$ is well-defined for every point $\mathfrak{p}$ of $PG(n-1,q)$, and $M_{\mathfrak{p}}$ may be considered as a hyperplane in $PG(n-1,q)$. We now give some results which clarify the relations between non-trivial minimal blocking sets of size n and $\mathfrak{C}_n$-covers for groups.

Proposition 2.1. *Let B be a set of points in $PG(n,q)$. Then B is a blocking set in $PG(n,q)$ if and only if the set $\mathcal{C}=\{M_b\mid b\in B\}$ is a $|B|$-cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

Proposition 2.2. *Let B be a set of points in $PG(n,q)$. Then B is a minimal blocking set in $PG(n,q)$ if and only if the set $\mathcal{C}=\{M_b\mid b\in B\}$ is an irredundant $|B|$-cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

Remark 2.3. *Note that if q is prime, then the cover $\mathcal{C}$ in the statements of Propositions 2.1 and 2.2 is a maximal cover for $(\mathbb{F}_q)^{n+1}$.*

Remark 2.4. *It is easy to see that a (minimal) blocking set B with $d(B)=d$ in $PG(n,q)$ can be obtained from a (minimal) blocking set in $PG(d,q)$. So if we adopt an induction process on n to find all minimal blocking sets B in $PG(n,q)$, we must find only all those minimal blocking sets with $d(B)=n$.*

Proposition 2.5. *Let B be a set of points in $PG(n,q)$. Then B is a blocking set with $d(B)=n$ if and only if the set $\mathcal{C}=\{M_b\mid b\in B\}$ is a core-free $|B|$-cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

Theorem 2.6. *Let p be a prime number and n be a positive integer. Then a finite p-group G admits a $\mathfrak{C}_{n+1}$-cover if and only if $G\cong(C_p)^{m+1}$ for some positive integer m such that $PG(m,p)$ has a minimal blocking set B with $d(B)=m$ and $|B|=n+1$.*

Proof. Let G be a finite p-group admitting a $\mathfrak{C}_{n+1}$-cover. Then G has a maximal irredundant core-free $(n+1)$-cover, $\mathcal{C}=\{M_i\mid i=1,\dots,n+1\}$ say. Since the Frattini subgroup $\Phi(G)$ of G is contained in M_i for every $i\in\{1,\dots,n+1\}$, $\Phi(G)\le D_G=1$, where D is the intersection of the cover $\mathcal{C}$. Hence $\Phi(G)=1$ and so G is isomorphic to $(C_p)^{m+1}$ for some positive integer m. Now Propositions 2.2 and 2.5 and Remark 2.3 complete the proof.

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