Tarbiat Moallem University, 20th Seminar on Algebra, 2-3 Ordibehesht, 1388 (Apr. 22-23, 2009) pp 37-39

THE PROPERTIES OF *n*- SINGER GENERATOR AND NON-COMMUTING SUBSETS OF FINITE THREE-DIMENSIONAL GENERAL LINEAR GROUPS

Azizollah Azad Department of Mathematics, Faculty of Sciences, Arak University 38156, Shariati Ave. Arak, Iran a-azad@Araku.ac.ir

ABSTRACT. Let G be a group. A subset N of G is said to be non-commuting if $xy \neq yx$ for any two distinct elements x and y in N. If $|N| \geq |M|$ for any other non-commuting subset M in G, then N is said to be a maximal non-commuting subset. In this paper we obtain lower bounds for the cardinality of a maximal non-commuting subset, by n-Singer generator, in a three-dimensional general linear group. Moreover we obtain structural information about every maximal non-commuting subset containing no proper powers.

1. INTRODUCTION

Let G be a non-abelian group and Z(G) be its center. We call a subset N of G non-commuting if $xy \neq yx$ for any distinct elements x, y in N. If $|N| \geq |M|$ for any other non-commuting subset M in G, then N is said to be a maximal non-commuting subset. The cardinality of such a subset is denoted by $\omega(G)$. By a famous result of Neumann [4] answering a question of P. Erdös, we know that the finiteness of $\omega(G)$ in G implies the finiteness of the factor group $\frac{G}{Z(G)}$.

For a prime number p, a finite p-group G is called extra-special if the center, the Frattini subgroup and the derived subgroup of G all coincide and are cyclic of order p. The cardinalities of maximal non-commuting subsets of extra-special pgroups are important as they provide combinatorial information which can be used to calculate their cohomology lengths. (The cohomology length of a non-elementary abelian p-group is a cohomology invariant defined as a result of a theorem of Serre [6]). Y. M. Chin [3] has obtained upper and lower bounds for the cardinality of maximal non-commuting subsets of extra-special p-groups, for odd prime numbers p. For p = 2, it has been shown by Isaacs (see[2, p. 40]) that $\omega(G) = 2n + 1$ for any extra-special group of order 2^{2n+1} . Also in [1, Lemma 4.4], it was proved that $\omega(GL(2, q)) = q^2 + q + 1$. In this paper we consider maximal non-commuting subsets in general linear groups of dimension three over a finite field of order q, and obtain lower bounds for their cardinality.

²⁰⁰⁰ Mathematics Subject Classification: 20D60.

keywords and phrases: Singer cycle subgroup, non-commuting subset, general linear group.

AZIZOLLAH AZAD

2. Main result

Theorem 2.1. Let G = GL(3, q). Then $\omega(G) \ge q^6$ if $q \ge 3$, and if q = 2 then $\omega(G) = 57$.

3. n-Singer generator

Definition 3.1. Let $g \in GL(n, q)$ where $q = p^k$, p a prime, and $|g| = q^n - 1$. Then $\langle g \rangle$ is called a *Singer cycle subgroup* of G.

Definition 3.2. Let V be a vector space over a finite field F with dimension n. We call $V = V_{n_1} \oplus V_{n_2} \oplus \ldots \oplus V_{n_k}$ an (n_1, n_2, \ldots, n_k) -decomposition if (n_1, n_2, \ldots, n_k) is a partition of n and for $i = 1, 2, \ldots, k, V_{n_i}$ is a subspace of V of dimension n_i .

Definition 3.3. Let V be an n-dimensional vector space over a finite field F and $V = V_{n_1} \oplus V_{n_2} \oplus \ldots \oplus V_{n_k}$ be an (n_1, n_2, \ldots, n_k) -decomposition of V. An element g of G is called an (n_1, n_2, \ldots, n_k) -Singer generator if $g = g_{n_1}g_{n_2} \ldots g_{n_k}$ where, for each $i, \langle g_{n_i} \rangle$ is a Singer cycle subgroup of $GL(V_{n_i})$, and if $n_i = n_j$ then $c_{g_{n_i}}(t) \neq c_{g_{n_j}}(t)$ for all $i \neq j$, where $c_{g_{n_i}}(t)$ is the characteristic polynomial for g_{n_i} on V_{n_i} . We call $\prod_{i=1}^k \langle g_{n_i} \rangle$ the (n_1, n_2, \ldots, n_k) -maximal torus corresponding to g.

Note it is not necessary that a group has an (n_1, n_2, \ldots, n_k) -Singer generator for all (n_1, n_2, \ldots, n_k) such that $\sum_{i=1}^k n_i = n$. For example existence of a (1, 1, 1)-Singer generator element, $g_1g_2g_3$ in GL(3, 2) requires three distinct linear polynomials $c_{g_{n_i}}(t) = t - \lambda_i$ and hence requires $q \ge 4$.

Lemma 3.4. Let G = GL(3,q), where $q = p^k \ge 4$. Suppose that $g \in G$ is an (n_1, \ldots, n_k) -Singer generator, where (n_1, \ldots, n_k) is a partition of 3. Then $C_G(g) = \prod_{i=1}^k \langle g_{n_i} \rangle$.

4. Proof of Theorem 1.1

Definition 4.1. Let N be a maximal non-commuting subset of finite group G. An element $x \in N$ is called a *proper power* if there exists y in G such that $x = y^k$ and |x| < |y|.

Lemma 4.2. Let G be a group and N a maximal non-commuting subset of G. Suppose $x \in N$ and there exists y such that $x = y^s$ and |x| < |y|. Then $\overline{N} = (N \setminus \{x\}) \cup \{y\}$ is a non-commuting subset of G. So G has a maximal non-commuting subset that contains no proper powers.

Lemma 4.3. Let G = GL(3, q), where $q = p^k > 2$, and let N_3 consist of one (3)-Singer generator of G corresponding to each (3)-maximal torus of G. Then N_3 is a non-commuting subset of size $\frac{|G|}{(q^3-1)^3}$. Moreover, if N is a maximal non-commuting subset containing no proper powers, then N contains (3)-Singer generator of each (3)-maximal torus.

Lemma 4.4. Let G = GL(3, q), where $q = p^k > 2$. Let N_{12} consist of one (1, 2)-Singer generator element of G corresponding to each (1, 2)-maximal torus of G. Then N_{12} is a non-commuting subset of size $\frac{q^3(q^3-1)}{2}$. Moreover, if N is a maximal non-commuting subset containing no proper powers, then N contains a (1, 2)-Singer generator of each (1, 2)-maximal torus.

38

THE PROPERTIES OF n-SINGER GENERATOR AND NON-COMMUTING SUBSETS ... 39

Lemma 4.5. Let G = GL(3, q), where $q = p^k \ge 4$. Let N_{111} consist of one (1, 1, 1)-Singer generator element of G corresponding to each (1, 1, 1)-maximal torus of G. Then N_{111} is a non-commuting subset of size $\frac{|G|}{6(q-1)^3}$. Moreover, if N is a maximal non-commuting subset containing no proper powers, then N contains a (1, 1, 1)-Singer generator of each (1, 1, 1)-maximal torus.

Lemma 4.6. Let G = GL(3, q), where $q = p^k \ge 4$. Let $x, y, z \in G$ be a (3)-Singer generator, (1, 2)-Singer generator and (1, 1, 1)-Singer generator element, respectively. Then $xy \ne yx$, $xz \ne zx$ and $yz \ne zy$.

Proof of Theorem 1.1

By Lemmas 4.3, 4.4, 4.5 and 4.6, $N_3 \cup N_{12} \cup N_{111}$ is a non-commuting subset of size $\frac{|G|}{3(q^3-1)} + \frac{q^3(q^3-1)}{2} + \frac{|G|}{6(q-1)^3} = q^6$.

Moreover we have shown that every maximal non-commuting subset N of G that contains no proper power, must contains a subset of the form $N_3 \cup N_{12} \cup N_{111}$.

References

- A. Abdollahi, A. Akbari and H. R. Maimani, Non-commuting graph of a group, J. Algbera 298 (2006), 468-492.
- [2] E. A. Bertram, Some applications of graph theory to finite groups, Discrete Math. 44 (1983), 31-43.
- [3] A. M. Y. Chin, On non-commuting sets in an extra special p-group, J. Group Theory 8 (2005), 189-194.
- [4] B. H. Neumann, A problem of Paul Erdös on groups, J. Aust. Math. Soc. Ser. A 21 (1976), 467-472.
- [5] L. Pyber, The number of pairwise non-commuting elements and the index of the centre in a finite group, J. London Math. Soc. (2) 35 (1987), 287-295.
- [6] J. P. Serre, Sur la dimension cohomologique des groups profinis, Topology 3 (1965), 413-420.