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SEQUENTIALLY PURE INJECTIVITY AND BAER TYPE CRITERIA FOR ACTS OVER SEMIGROUPS

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Abstract. Although the Baer Criterion for injectivity (ideal injectivity implies injectivity) is true for modules over a ring (with an identity), it is an open problem for acts over a semigroup S (with or without identity). In fact, we are not aware of any type of weak injectivity implying injectivity of S-acts, in general, other than Skornjakov-Baer criterion, which says that injectivity with respect to subacts of cyclic acts implies injectivity with respect to all monomorphisms.

Ebrahimi and Mahmoudi in some papers have shown this criterion to be true for some special S.

1. Introduction

Injectivity is one of the central notions in many branches of mathematics. One usually takes a subclass $\mathcal{M}$ of monomorphisms in a category $\mathcal{A}$, members of which may be called $\mathcal{M}$-morphisms, and give the following definition.

An object A of $\mathcal{A}$ is said to be $\mathcal{M}$-*injective* if for any $\mathcal{M}$-morphism $g : B \to C$, any morphism $f : B \to A$ can be lifted to a morphism $\bar{f} : C \to A$ of $\mathcal{A}$. That is, the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ f\downarrow & \swarrow \bar{f} & \\ A & & \end{array}$$

The definitions of $\mathcal{M}$-*retract*, $\mathcal{M}$-*absolute retracts*, and having *enough* $\mathcal{M}$-*injectives* are then defined as usual (see [1]).

In this paper, we use injectivity with respect the classes of, the so called, *s-dense, s-closed, s-pure* monomorphisms.

We use the definitions and ingredients needed in the sequel as given in [4].

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2. SEQUENTIALLY PURE INJECTIVE S-ACTS

In this section we recall the notion of *sequentially pure* (*s-pure*) *monomorphism* mainly from [2, 3]. The behaviour of injectivity with respect to this class of monomorphisms with essential extensions and injective hulls has been studied in [3].

Definition 2.1. We say that A is *s-pure* in an extension B of A if every $\Sigma_A = \{xs = a_s : s \in S, a_s \in A\}$ is solvable in A (i.e., there is some $a \in A$ such that for all $s \in S$, $as = a_s$) whenever it is solvable in B. A monomorphism $f : A \to B$ is *s-pure* if $f(A)$ is s-pure in B.

Note that there is a one to one correspondence between the set of all systems of equations Σ_A of the above form on an S-act A and the set of all functions $k : S \to A$. For any S-act B and $b \in B$, let us define the S-map $\lambda_b : S \to B$ by $\lambda_b(s) := bs$. In these notations, we have

Lemma 2.2. [3] *A map $k : S \to A$ is a homomorphism if and only if there exists an extension B of A and $b \in B$ such that $k = \lambda_b$.*

Remark 2.3. It is easily seen that, for a subact A of B, the following are equivalent:
(i) A is s-pure in B.
(ii) For every $b \in B$ with $bS \subseteq A$ there is an $a \in A$ with $\lambda_b = \lambda_a$.
(iii) Every homomorphism $k : S \to A$ is of the form λ_a for some $a \in A$ whenever it is of the form λ_b for some $b \in B$.

The above remark also shows that if one defines $\tilde{A} := \{b \in B : \exists\, a \in A,\ \lambda_b = \lambda_a\}$ and $\bar{A} := \{b \in B : bS \subseteq A\}$, then A is s-pure in B if and only if $\tilde{A} = \bar{A}$. For more detail, see [2].

Remark 2.4. Denoting the class of s-pure monomorphisms by $\mathcal{M}_p$, one gets the notion of *$\mathcal{M}_p$-injectivity*, or *s-pure injectivity.* Note that every injective S-act is s-pure injective but the converse is not generally true. Let $S = (\mathbb{N}, min)$. We can see that every s-pure monomorphism of $\mathbb{N}$-acts is retractable. So every $\mathbb{N}$-act is s-pure injective, but $\mathbb{N}$ is not an injective $\mathbb{N}$-act; even $id_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ can not be extended to $\mathbb{N}^{\infty}$.

Theorem 2.5. *Every s-pure injective S-act A has a fixed element.*

Lemma 2.6. *Every retract of an s-pure injective is an s-pure injective.*

Proposition 2.7. [2] *For the following pushout diagram in* **Act-S**, *we have*
(i) *If f is a monomorphism then h is a monomorphism.*
(ii) *If f is s-pure then h is s-pure.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g\downarrow & & \downarrow h' \\ C & \xrightarrow{h} & Q \end{array}$$

The following is a part of the so called the First Theorem of the Well-Behaviour of Injectivity (see [1, 3]).

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Theorem 2.8. [3] *For every semigroup S and S-act A, the following are equivalent:*

(i) *A is s-pure injective.*

(ii) *A is s-pure absolute retract; that is, every s-pure monomorphism from A has a left inverse S-map.*

Remark 2.9. Let for a family $\{A_\alpha : \alpha \in I\}$ of S-acts each A_i have a fixed (zero) element. Then, similar to the case for modules, each A_i is a retract of $\prod A_i$ and $\coprod A_i$. In particular each A_i is s-pure in $\prod A_i$ and $\coprod A_i$. So, *the product $\prod A_i$ of S-acts is s-pure injective if and only if each A_i is s-pure injective.* But the coproducts do not behave as well as products. We clearly have *if a coproduct $\coprod A_i$ of acts is s-pure injective then each A_i is s-pure injective if and only if for each $i \in I$, A_i has a fixed element.*

The following result is a partial answer to the converse of this fact.

Theorem 2.10. *Let S have a zero 0 but has no zero divisors. Consider the following statements:*

(i) *All coproducts of s-pure injective S-acts are s-pure injective.*

(ii) *The S-act $\mathbf{2} = \{x, y\}$, with two zero elements, is s-pure injective.*

(iii) *S is indecomposable.*

Then, we have (i)$\Rightarrow$(ii) *and* (ii)$\Rightarrow$(iii).

Lemma 2.11. *If $\oplus A_i$ (The direct sum of $\{A_i\}$) is s-pure injective, then every A_i is s-pure injective.*

Definition 2.12. A is said to be *f-s-pure injective* if it is s-pure injective with respect to finitely generated s-pure subacts.

Theorem 2.13. *If A_i, $i \in I$, is a family of f-s-pure (s-pure) injective S-acts, then $\oplus A_i$ is f-s-pure injective.*

Theorem 2.14. *Let S be a finitely generated semigroup. Then, every direct sum of s-pure injective S-acts is s-pure injective if and only if each direct sum of s-pure injective S-acts is a retract of their direct product.*

3. SOME BAER CRITERIA FOR INJECTIVITY OF S-ACTS

Recall that, an S-act A is said to be *ideal injective* if every homomorphism $f : I \to A$ for every ideal I of S is of the form λ_a for some $a \in A$. And A is *weakly injective* if every S-map $f : I \to A$ can be extended to $\bar{f} : S \to A$. Clearly, ideal injectivity implies weak injectivity and if S has a left identity, these two notions coincide.

Definition 3.1. For a subact A of B, let $\bar{A} := \{b \in B : bS \subseteq A\}$. Then, A is said to be *s-dense* (*s-closed*) in B if $\bar{A} = B$ ($\bar{A} = A$).

Also, an S-map $f : A \to B$ is said to be s-dense or s-closed if $f(A)$ is such in B.

Lemma 3.2. [2] *Any s-dense, s-pure monomorphism is a retraction.*

Theorem 3.3. *If A is s-pure injective then it is s-closed injective. The converse is true if $S^2 = S$.*

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Theorem 3.4. *An S-act A is injective if and only if it is s-dense injective as well as s-closed injective.*

Definition 3.5. An S-act A is called *s-complete* if every consistent system Σ_A has a solution in A.

Theorem 3.6. [2] *For an S-act A, the following are equivalent*
(i) *A is s-complete.*
(ii) *A is absolutely s-pure; that is, A is s-pure in every extension of it.*
(iii) *Every homomorphism $k : S \to A$ is of the form λ_a for some $a \in A$.*
(iv) *A is injective with respect to s-dense monomorphisms.*

Theorem 3.7. *For a semigroup S, the following are equivalent:*
(i) *Every S-act is s-complete.*
(ii) *Every S-act is principally s-complete (Which is s-complete in every cyclic extension).*
(ii) *S is s-complete.*
(iii) *S has a left identity*

Theorem 3.8. *An S-act A is injective if and only if it is s-complete as well as s-pure injective.*

Theorem 3.9. *If every s-pure monomorphism is a retraction then every s-complete (s-dense injective) S-act is injective. In this case every ideal-injective is injective.*

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