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LINEAR RESOLUTION OF POWERS OF IDEALS

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Abstract. Castelnuovo-Mumford regularity is a kind of universal bound for important invariants of graded algebras, such as the maximum degree of the syzygies and the maximum non-vanishing degree of the local cohomology modules. In this paper we give a simple criterion in terms of Rees algebra of a given ideal to show that high enough powers of this ideal have linear resolution.

In this talk we discuss about upper bound for regularity of powers of an ideal and some implementations in CoCoA are also involved.

1. Introduction

Let $S = K[x_1, \cdots, x_r]$ and let I be a graded ideal, $\mathfrak{m} = (x_1, \cdots, x_r)$ the maximal ideal of S, and $n = \dim (S/I)$. Let

$$a_i(S/I) = \max\{t; \mathrm{H}^i_{\mathfrak{m}}(S/I)_t \neq 0\}, 0 \leq i \leq n,$$

where $\mathrm{H}^i_{\mathfrak{m}}(S/I)$ is the ith local cohomology module with the support in $\mathfrak{m}$ (with the convention $\max \emptyset = -\infty$). Then the regularity is the number

$$\mathrm{reg(S/I) = \max\{a_i(S/I) + i; 0 \leq i \leq n\}}.$$

Note that $\mathrm{reg(I) = reg(S/I) + 1}$. We may also compute $\mathrm{reg(I)}$ in terms of Tor by the formula

$$\begin{aligned} \mathrm{reg(I)} &= \max_{i,j}\{j - i : \mathrm{Tor}_i(I, k)_j \neq 0\}, \\ &= \max_{i,j}\{j - i; \beta_{i,j}(I) \neq 0\}. \end{aligned}$$

One has often tried to find upper bounds for the Castelnuovo-Mumford regularity in terms of simpler invariants which reflect the complexity of a graded algebra like dimension and multiplicity. One may expect to have the same inequality for regularity, that is, $\mathrm{reg(I^k) \leq k\,reg(I)}$. Unfortunately this is not

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true in general. However, in [3] Cutkosky, Herzog, and Trung studied the asymptotic behavior of the Castelnuovo-Mumford regularity and showed that the regularity of I^k is a linear function for large k, i.e.,

$$\mathrm{reg}(\mathrm{I^k}) = \mathrm{a(I)k + b(I)}, \quad \forall \mathrm{k} \geq \mathrm{c(I)}. \tag{1}$$

Now assume that I is an equigenerated ideal, that is, generated by forms of the same degree d. Then one has $a(I) = d$ and hence, $\mathrm{reg(I^{k+1}) - reg(I^k) = d}$ for all $k \geq c(I)$. Hence we have

$$\mathrm{reg(I^k) = (k - c(I))d + reg(I^{c(I)})}, \quad \forall \mathrm{k} \geq \mathrm{c(I)}. \tag{2}$$

One says that the regularity of the powers of I jumps at place k if $\mathrm{reg(I^k)} - \mathrm{reg(I^{k-1})} > \mathrm{d}$. In [2] the author gives several examples of ideals generated in degree d ($d = 2, 3$), with linear resolution (i.e., $\mathrm{reg(I) = d}$), and such that the regularity of the powers of I jumps at place 2, i.e., such that $\mathrm{reg(I^2) > 2d}$. As it is indicated in [2], the first example of such an ideal was given by Terai. Throughout this paper we use J for this ideal. Geometrically speaking, this is an example of Reisner which corresponds to the (simplicial complex of a) triangulation of the real projective plane $\mathbb{P}^2$; see [1] for more details. Let $R := K[x_1, \cdots, x_6]$ one has

(3)

$$J = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6).$$

It is known that J is a square-free monomial ideal whose Betti numbers, regularity and projective dimension depend on the characteristic of the base field. Indeed whenever $\mathrm{char(K)} \neq 2$, R/J is Cohen-Macaulay (and otherwise not), moreover one has $\mathrm{reg(J) = 3}$ and $\mathrm{reg(J^2) = 7}$ (which is of course $> 2 \times 3$). If $\mathrm{char(K) = 2}$, then J itself has no linear resolution. So the following natural question arises:

Question 1.1. *How it goes on for the regularity of powers of J?*

By the help of (1) we are able to write $\mathrm{reg(J^k) = 3k + b(J)}$, $\forall \mathrm{k} \geq \mathrm{c(J)}$. But what are $b(J)$ and $c(J)$? In this paper we give an answer to Question 1.1 and prove that J^k has linear resolution (in $\mathrm{char(K) = 0}$) $\forall k \neq 2$, that is, $b(J) = 0$ and $c(J) = 3$. That is

$$\mathrm{reg(J^k) = 3k}, \quad \forall \mathrm{k} \neq 2.$$

2. Main results

Let K be a field, $I = (f_1, \cdots, f_m)$ be a graded ideal of $S = K[x_1, \cdots, x_r]$ generated in a single degree, say d. The Rees algebra of I is known to be

$$R(I) = \bigoplus_{j \geq 0} I^j t^j = S[f_1t, \cdots, f_mt] \subseteq S[t].$$

Let $T = S[t_1, \cdots, t_m]$. Then there is a natural surjective homomorphism of bigraded K-algebras $\varphi : T \longrightarrow R(I)$ with $\varphi(x_i) = x_i$ for $i = 1, \cdots, r$ and $\varphi(y_j) = f_jt$ for $j = 1, \cdots, m$. So one can write $R(I) = T/P$. In this paper we consider T, and so $R(I)$, as a standard bigraded polynomial ring with

$\deg(x_i) = (0,1)$ and $\deg(t_j) = (1,0)$. Indeed if we start with the natural bigraded structure $\deg(x_i) = (0,1)$ and $\deg(f_j t) = (d,1)$ then $R(I)_{(k,vd)} = (I^k)_{vd}$, but the standard bidegree normalizes the bigrading in the following sense:

$$R(I)_{(k,j)} = (I^k)_{kd+j} \tag{4}$$

In [4, Theorem 1.1 and Corollary 1.2] Herzog, Hibi and Zheng showed the following:

Theorem 2.1. *Let $I \subseteq K[x_1, \cdots, x_n] := S$ be an equigenerated graded ideal. Let m be the number of generators of I and let $T := S[t_1, \cdots, t_m]$, and let $R(I) = T/P$ be the Rees algebra associated to I. If for some term order $<$ on T, P has Gröbner basis G whose elements are at most linear in the variables $x_1, \cdots, x_n$, that is $\deg_x(f) \leq 1$ for all $f \in G$, then each power of I has a linear resolution.*

The main motivation for our work is to generalize Herzog, Hibi and Zheng's techniques in order to apply them to a wider class. Furthermore, we will indicate the least exponent k_0 for which I^k has linear resolution for all $k \geq k_0$. Indeed our generalization works for all ideals which admit the following condition:

Theorem 2.2. *Let $Q \subseteq S = K[x_1, \cdots, x_r]$ be a graded ideal which is generated by m polynomials all of the same degree d, and let $I = \mathrm{in(g(P))}$ for some linear bi-transformation $g \in \mathrm{GL_r(K)} \times \mathrm{GL_m(K)}$. Write $I = G + B$ where G is generated by elements of $\deg_x \leq 1$ and B is generated by elements of $\deg_x > 1$. If $I_{(k,j)} = G_{(k,j)}$ for all $k \geq k_0$ and for all $j \in Z$, then Q^k has linear resolution for all $k \geq k_0$. In other words, $\mathrm{reg(Q^k) = kd}$ for all $k \geq k_0$.*

Furthermore since Theorem 2.1 is subject to condition that $\mathrm{in(P) = (u_1, \cdots, u_m)}$ and $\deg_x(u_i) \leq 1$. So the natural way to generalize it is to change the upper bound for x-degree of u_i with some number t. That way, we end up with $\mathrm{reg(I^n) \leq nd + (t-1)\, pd(T/in(P))}$.

Proposition 2.3. *Let $I \subseteq S$ be an equigenerated graded ideal and let $R(I) = T/P$. If $\mathrm{in(P) = (u_1, \cdots, u_m)}$ and $\deg_x(u_i) \leq t$, then $\mathrm{reg(I^n) \leq nd + (t-1)\, pd(T/in(P))}$.*

One can see that now Theorem 2.1 is the special case of Proposition 2.3 with $t = 1$.

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